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COMMON FIXED POINTS FOR D-MAPS SATISFYING INTEGRAL TYPE CONDITION

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Abstract. In this paper , we obtain two common fixed point theorems one for two pairs of single and set-valued mappings and another for four set-valued mappings satisfying integral type conditions.

1. Introduction and preliminaries

Recently Ali and Imdad [8]obtained some common fixed point theorems for four self maps using implicit relations in a metric space.Branciari [4] introduced integral type contractive conditions and proved a fixed point theorem for a self map on a metric space. Based on this concept, Bouhadjera and Djoudi [3] proved common fixed point theorems for pairs of single and set-valued D-maps satisfying an integral type condition. In this paper, we obtain a theorem different from that of [3] and obtain a generalization of a theorem of [8] . We also obtain common fixed point theorems for four set-valued mappings and obtain a generalization of theorems of [8] and [2] .

In the sequel, we need the following

Let (X, d) be a metric space and $B(X)$, the set of all nonempty bounded subsets of X. For $A, B \in B(X)$, define $\delta(A, B) = sup\{d(a, b) : a \in A, b \in B\}$.

If $A = \{a\}$, then we write $\delta(A, B) = \delta(a, B)$ and also if $B = \{b\}$ then ,we write $\delta(A, B) = d(a, b)$.

From the definition of $\delta(A, B)$, we have $\delta(A, B) = \delta(B, A) \geq 0$,

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$\delta(A,B) = 0$ iff $A = B = \{a\}$, $\delta(A,B) \leq \delta(A,C) + \delta(C,B)$,
$\delta(A,A) = diamA$ for all $A, B, C \in B(X)$.

Definition 1.1. ([6]): A sequence $\{A_n\}$ of nonempty subsets of X is said to be convergent to a subset A of X if
(i) each point a in A is the limit of a convergent sequence $\{a_n\}$, where a_n is in A_n for $n \in N$,
(ii) for arbitrary $\epsilon > 0$, there exists an integer m such that $A_n \subseteq A_\epsilon$ for $n > m$, where A_ϵ denotes the set of all points $x \in X$ for which there exists a point $a \in A$, depending on x, such that $d(x,a) < \epsilon$. A is then said to be the limit of the sequence $\{A_n\}$.

Lemma 1.2. ([6]): *If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ converging to A and B in $B(X)$,respectively, then the sequence$\{\delta(A_n, B_n)\}$ converges to $\delta(A,B)$.*

Lemma 1.3. ([7]):*Let $\{A_n\}$ be a sequence in $B(X)$ and y be a point in X such that $\delta(A_n, y) \longrightarrow 0$. Then the sequence $\{A_n\}$ convrges to the set $\{y\}$ in $B(X)$.*

Definition 1.4. ([9]):The maps $f : X \longrightarrow X$ and $F : X \longrightarrow B(X)$ are weakly compatible or coincidentally commuting (some authors call it as subcompatible) if $\{t \in X/Ft = \{ft\}\} \subseteq \{t \in X/Fft = fFt\}$.

The following definition is an extension of (E.A.)property due to Aamri and Moutawakil [1].

Definition 1.5. ([5]): The maps $f : X \longrightarrow X$ and $F : X \longrightarrow B(X)$ are said to be D- maps if there exists a sequence$\{x_n\}$ in X such that $limfx_n = t$ and $limFx_n = \{t\}$ for some $t \in X$.

Recently in 2008,Bouhadjera and Djoudi [3] proved the following:

Theorem 1.6. *(Theorem 2.1 of [3]): Let f, g be self maps of a metric space(X, d) and let $F, G : X \longrightarrow B(X)$ be two set- valued maps such that*
(1.6.1) $FX \subseteq gX$ and $GX \subseteq fX$,
(1.6.2)

$$\int_0^{\phi\begin{pmatrix} \delta(Fx,Gy), d(fx,gy), \delta(fx,Fx), \delta(gy,Gy), \\ \delta(fx,Gy), \delta(gy,Fx) \end{pmatrix}} \varphi(t)dt \leq 0$$

for all $x, y \in X$, where $\phi : R_+^6 \longrightarrow R$ is continuous function satisfying
(i)$\int_0^{\phi(u,0,0,u,u,0)} \varphi(t)dt \leq 0$ implies $u = 0$,
(ii) $\int_0^{\phi(u,0,u,0,0,0)} \varphi(t)dt \leq 0$ implies $u = 0$,
(iii)$\int_0^{\phi(u,u,0,0,u,u)} \varphi(t)dt > 0$ for all $u > 0$ and
$\varphi : R_+ \longrightarrow R$ is a Lebesgue-integrable map which is summable,
(1.6.3)(a) f and F are subcompatible D-maps; g and G are subcompatible and $FX$is closed, (or)
(1.6.3)(b) g and G are subcompatible D-maps; f and F are subcompatible and $GX$is closed. Then f, g, F and G have a unique common fixed point $t \in X$ such that $Ft = Gt = \{ft\} = \{gt\} = \{t\}$.

In this paper we prove a slight variation theorem of the above theorem using more general contractive condition .

2. Main results

First implicit relation :

Let $\phi : R_+^4 \longrightarrow R$ be a lower semi continuous function satisfying
$\int_0^{\phi(u,u,u,u)} \varphi(t)dt \leq 0$ implies $u = 0$,where $\varphi : R_+ \longrightarrow R$ is a Lebesgue-integrable map which is summable.

Now we give some examples.

(i) Let $\phi(t_1, t_2, t_3, t_4) = t_1 - k\ max\{t_2, t_3, t_4\}$, where $k \in [0, 1)$ and $\varphi(t) = t$ or $\varphi(t) = \frac{3\pi}{4(1+t)^2} Cos(\frac{3\pi t}{4(1+t)})$ for all $t \in R_+$.

Then $\phi(u, u, u, u) = (1 - k)u$.

Case: Suppose $\varphi(t) = t$.

Then $\int_0^{\phi(u,u,u,u)} \varphi(t)dt \leq 0$ implies $\frac{1}{2}(1 - k)^2 u^2 \leq 0$ so that $u \leq 0$.But $u \geq 0$.Hence $u = 0$.

Case : Suppose $\varphi(t) = \frac{3\pi}{4(1+t)^2} Cos(\frac{3\pi t}{4(1+t)})$.

Then $\int_0^{\phi(u,u,u,u)} \varphi(t)dt \leq 0$ implies $Sin(\frac{3\pi(1-k)u}{4(1+(1-k)u)}) \leq 0$ so that $u = 0$ since
$0 \leq \frac{3\pi(1-k)u}{4(1+(1-k)u)} < \pi$.

The following ϕ functions satisfy the first implicit relation with $\varphi(t) = t$ for all $t \in R_+$ or $\varphi(t) = \frac{3\pi}{4(1+t)^2} Cos(\frac{3\pi t}{4(1+t)})$.

(ii)$\phi(t_1, t_2, t_3, t_4) = t_1 - k\ (max\{t_2^2, t_3 t_4\})^{\frac{1}{2}}$, where $k \in [0, 1)$.

(iii)$\phi(t_1, t_2, t_3, t_4) = t_1^2 - \alpha\ max\{t_2^2, t_3^2, t_4^2\} - \beta\ max\{t_2 t_3, t_3 t_4\}$, where $\alpha, \beta \geq 0$ such that $\alpha + \beta < 1$.

(iv)$\phi(t_1, t_2, t_3, t_4) = t_1^3 - \alpha\ max\{t_i t_j t_k / i, j, k \in \{2, 3, 4\}\}$, where $\alpha \in [0, 1)$.

Theorem 2.1. *Let f, g be self maps of a metric space(X, d) and let $F, G : X \longrightarrow B(X)$ be two set- valued maps such that*
(2.1.1)

$$\int_0^{\phi\begin{pmatrix} \delta(Fx,Gy), d(fx,gy)+\delta(fx,Fx)+\delta(gy,Gy) \\ \delta(fx,Fx)+\delta(fx,Gy), \delta(gy,Gy)+\delta(gy,Fx) \end{pmatrix}} \varphi(t)dt \leq 0$$

for all $x, y \in X$, where $\phi : R_+^4 \longrightarrow R$ is a lower semi continuous function satisfying $\int_0^{\phi(u,u,u,u)} \varphi(t)dt \leq 0$ implies $u = 0$ and
$\varphi : R_+ \longrightarrow R$ is a Lebesgue-integrable map which is summable,
(2.1.2) (f, F) and (g, G) are subcompatible pairs,
(2.1.3)(a) (f, F) is a pair of D-maps , $Fx \subseteq g(X) \forall x \in X$ and $f(X)$ is closed
(or)
(2.1.3)(b) (g, G) is a pair of D-maps , $Gx \subseteq f(X) \forall x \in X$ and $g(X)$ is closed.
Then f, g, F and G have a unique common fixed point in X.

Proof. Suppose (2.1.3)(a) holds.

Since(f, F) is a pair of D-maps, there exists a sequence$\{x_n\}$ in X such that $lim fx_n = t$ and $lim Fx_n = \{t\}$ for some $t \in X$.

Since $Fx \subseteq g(X) \forall x \in X$,there exists $\alpha_n \in Fx_n$ and $y_n \in X$ such that $\alpha_n = gy_n \forall n$. Also $d(gy_n, t) = d(\alpha_n, t) \leq \delta(Fx_n, t) \longrightarrow 0$ as $n \longrightarrow \infty$.

Suppose $limGy_n = A$. Now

$$\int_0^{\phi\begin{pmatrix} \delta(Fx_n, Gy_n), d(fx_n, gy_n) + \delta(fx_n, Fx_n) + \delta(gy_n, Gy_n), \\ \delta(fx_n, Fx_n) + \delta(fx_n, Gy_n), \delta(gy_n, Gy_n) + \delta(gy_n, Fx_n) \end{pmatrix}} \varphi(t)dt \leq 0$$

Letting $n \longrightarrow \infty$, we get

$$\int_0^{\phi\begin{pmatrix} \delta(t, A), \delta(t, A), \delta(t, A), \delta(t, A) \end{pmatrix}} \varphi(t)dt \leq 0$$

Hence $\delta(t, A) = 0$ so that $A = \{t\}$.Thus $limGy_n = \{t\}$.
Since $f(X)$ is closed,there exists $u \in X$ such that $t = fu$. Now,

$$\int_0^{\phi\begin{pmatrix} \delta(Fu, Gy_n), d(fu, gy_n) + \delta(fu, Fu) + \delta(gy_n, Gy_n), \\ \delta(fu, Fu) + \delta(fu, Gy_n), \delta(gy_n, Gy_n) + \delta(gy_n, Fu) \end{pmatrix}} \varphi(t)dt \leq 0$$

Letting $n \longrightarrow \infty$, we get

$$\int_0^{\phi\begin{pmatrix} \delta(Fu, t), \delta(Fu, t), \delta(Fu, t), \delta(Fu, t) \end{pmatrix}} \varphi(t)dt \leq 0$$

Hence $\delta(Fu, t) = 0$ so that $Fu = \{t\}$.Thus $Fu = \{t\} = \{fu\}$.
Since$\{t\} = Fu \subseteq g(X)$, there exists $w \in X$ such that $t = gw$. Now,

$$\int_0^{\phi\begin{pmatrix} \delta(Fx_n, Gw), d(fx_n, gw) + \delta(fx_n, Fx_n) + \delta(gw, Gw), \\ \delta(fx_n, Fx_n) + \delta(fx_n, Gw), \delta(gw, Gw) + \delta(gw, Fx_n) \end{pmatrix}} \varphi(t)dt \leq 0$$

Letting $n \longrightarrow \infty$, we get

$$\int_0^{\phi\begin{pmatrix} \delta(t, Gw), \delta(t, Gw), \delta(t, Gw), \delta(t, Gw) \end{pmatrix}} \varphi(t)dt \leq 0$$

Hence $\delta(t, Gw) = 0$ so that $Gw = \{t\}$. Thus $Gw = \{t\} = \{gw\}$.
Since (f, F) is subcompatible,we have $Ft = Ffu = fFu = \{ft\}$.Now,

$$\int_0^{\phi\begin{pmatrix} \delta(Ft, Gw), d(ft, gw) + \delta(ft, Ft) + \delta(gw, Gw), \\ \delta(ft, Ft) + \delta(ft, Gw), \delta(gw, Gw) + \delta(gw, Ft) \end{pmatrix}} \varphi(t)dt \leq 0$$

which implies

$$\int_0^{\phi\begin{pmatrix} \delta(Ft, t), \delta(Ft, t), \delta(Ft, t), \delta(Ft, t) \end{pmatrix}} \varphi(t)dt \leq 0$$

Hence $\delta(Ft, t) = 0$ so that $Ft = \{t\}$.Thus $Ft = \{t\} = \{ft\}$.

Since (g, G) is subcompatible,we have $Gt = Ggw = gGw = \{gt\}$.Now,

$$\int_0^{\phi\begin{pmatrix} \delta(Fu, Gt), d(fu, gt) + \delta(fu, Fu) + \delta(gt, Gt), \\ \delta(fu, Fu) + \delta(fu, Gt), \delta(gt, Gt) + \delta(gt, Fu) \end{pmatrix}} \varphi(t)dt \leq 0$$

which implies

$$\int_0^{\phi\left(\delta(t,Gt),\delta(t,Gt),\delta(t,Gt),\delta(t,Gt) \right)} \varphi(t)dt \leq 0$$

Hence $\delta(t, Gt) = 0$ so that $Gt = \{t\}$.Thus $Gt = \{t\} = \{gt\}$.
Thus t is a common fixed point of F, G, f and g. Uniqueness of common fixed point follows easily from (2.1.1).Similarly, we can prove the theorem if (2,1,3)(b) holds. □

Let Ψ_6 denote the set of all lower semicontinuous functions $\psi : R_+^6 \to R$ satisfying
(i) $\psi(t, 0, t, 0, 0, t) > 0\ \forall t > 0$,
(ii) $\psi(t, 0, 0, t, t, 0) > 0\ \forall t > 0$,
(i) $\psi(t, t, 0, 0, t, t) > 0\ \forall t > 0$.
Clearly the conditions (i),(ii) and (iii) imply $\phi(t, t, t, t) \leq 0 \Rightarrow t = 0$ if we define $\psi(t_1, t_2, t_3, t_4, t_5, t_6) = \phi(t_1, t_2 + t_3 + t_4, t_3 + t_5, t_4 + t_6)$.
We observe that $\phi(t, t, t, t) \leq 0 \Rightarrow t = 0$ need not imply(i),(ii),(iii) if we take $\phi(t_1, t_2, t_3, t_4) = t_1 - k\ max\{t_2, t_3, t_4\}$, where $k \in [0, 1)$ and $\psi(t_1, t_2, t_3, t_4, t_5, t_6) = \phi(t_1t_2, t_2t_3, t_3t_4, t_4t_5)$. Clearly $\psi(t, 0, t, 0, 0, t) = \phi(0, 0, 0, 0) = 0$.

Theorem 2.1 is a generalization of the following

Theorem 2.2. *(Theorem 3.3,*[8]*): Let A, B, S and T be self mappings of a metric space (X, d) satisfying*
(2.2.1)

$$\psi(d(Ax, By), d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax)) \leq 0$$

for all $x, y \in X$, where $\psi \in \Psi_6$.
Suppose that (2.2.2)the pair (A, S) (or (B, T)) has Property(E.A.),
(2.2.3)$A(X) \subseteq T(X)$ (or $B(X) \subseteq S(X)$),
(2.2.4) $S(X)$ (or $T(X)$) is a closed subset of X and
(2.2.5) the pairs (A, S) and (B, T) are weakly compatible.
Then A, B, S and T have a unique common fixed point.

Proof. Let $F = \{A\}, G = \{B\}, f = S, g = T$ be single valued mappings and $\varphi(t) = 1$ for all $t > 0$ in Theorem 2.1. Define $\psi(t_1, t_2, t_3, t_4, t_5, t_6) = \phi(t_1, t_2 + t_3 + t_4, t_3 + t_5, t_4 + t_6)$.Clearly the conditions (i),(ii),(iii) on ψ imply that $\phi(t, t, t, t) \leq 0$ implies that $t = 0$.The rest follows from Theorem 2.1. □

Now ,we prove a common fixed point theorem for four set-valued mappings.

Theorem 2.3. *Let F, G, f and $g : X \longrightarrow B(X)$ be set- valued mappings satisfying*
(2.3.1)

$$\int_0^{\phi\left(\begin{array}{c} \delta(Fx,Gy),\delta(fx,gy)+\delta(fx,Fx)+\delta(gy,Gy) \\ \delta(fx,Fx)+\delta(fx,Gy),\delta(gy,Gy)+\delta(gy,Fx) \end{array} \right)} \varphi(t)dt \leq 0$$

for all $x, y \in X$, where ϕ and φ are as in Theorem 2.1,
(2.3.2)(a) Suppose that there exists a sequence$\{x_n\}$ in X such that $\{Fx_n\}$ and $\{fx_n\}$ converge to the same limit $\{z\}$ for some $z \in X$. (or)

(2.3.2)(b) Suppose that there exists a sequence$\{y_n\}$ *in* X *such that* $\{Gy_n\}$ *and* $\{gy_n\}$ *converge to the same limit* $\{z\}$ *for some* $z \in X$.
(2.3.3)Suppose that the pairs (f, F) *and* (g, G) *are coincidentally commuting,*
(2.3.4)Suppose $fu = \{z\} = gv$ *for some* $u, v \in X$.
(2.3.5) Suppose that Fz *or* fz *is a singleton and* Gz *or* gz *is a singleton.*
Then z *is the unique common fixed point of* F, G, f *and* g, *Also* z *is the unique common fixed point of* F *and* f *as well as of* G *and* g.

Proof. Suppose (2.3.2) (a) holds.

$$\int_0^{\phi\left(\begin{array}{c} \delta(Fx_n, Gv), \delta(fx_n, gv) + \delta(fx_n, Fx_n) + \delta(gv, Gv), \\ \delta(fx_n, Fx_n) + \delta(fx_n, Gv), \delta(gv, Gv) + \delta(gv, Fx_n) \end{array}\right)} \varphi(t)dt \leq 0$$

Letting $n \longrightarrow \infty$, we get

$$\int_0^{\phi\left(\delta(z, Gv), \delta(z, Gv), \delta(z, Gv), \delta(z, Gv) \right)} \varphi(t)dt \leq 0$$

Hence $\delta(z, Gv) = 0$ so that $Gv = \{z\}$. Thus $Gv = \{z\} = gv$.
Since (g, G) is coincidentally commuting, we have $Gz = Ggv = gGv = gz =$ singleton from(2.3.5).Now,

$$\int_0^{\phi\left(\begin{array}{c} \delta(Fx_n, Gz), \delta(fx_n, gz) + \delta(fx_n, Fx_n) + \delta(gz, Gz), \\ \delta(fx_n, Fx_n) + \delta(fx_n, Gz), \delta(gz, Gz) + \delta(gz, Fx_n) \end{array}\right)} \varphi(t)dt \leq 0$$

Letting $n \longrightarrow \infty$, we get

$$\int_0^{\phi\left(\delta(z, Gz), \delta(z, Gz), \delta(z, Gz), \delta(z, Gz) \right)} \varphi(t)dt \leq 0$$

Hence $\delta(z, Gz) = 0$ so that $Gz = \{z\}$. Thus $Gz = \{z\} = gz$.

$$\int_0^{\phi\left(\begin{array}{c} \delta(Fu, Gz), \delta(fu, gz) + \delta(fu, Fu) + \delta(gz, Gz), \\ \delta(fu, Fu) + \delta(fu, Gz), \delta(gz, Gz) + \delta(gz, Fu) \end{array}\right)} \varphi(t)dt \leq 0$$

which implies

$$\int_0^{\phi\left(\delta(Fu, z), \delta(Fu, z), \delta(Fu, z), \delta(Fu, z) \right)} \varphi(t)dt \leq 0$$

Hence $\delta(Fu, z) = 0$ so that $Fu = \{z\}$. Thus $Fu = \{z\} = fu$.
Since (f, F) is coincidentally commuting, we have $Fz = Ffu = fFu = fz =$ singleton from (2.3.5). Now,

$$\int_0^{\phi\left(\begin{array}{c} \delta(Fz, Gz), \delta(fz, gz) + \delta(fz, Fz) + \delta(gz, Gz), \\ \delta(fz, Fz) + \delta(fz, Gz), \delta(gz, Gz) + \delta(gz, Fz) \end{array}\right)} \varphi(t)dt \leq 0$$

which implies
$$\int_0^{\phi\left(\ \delta(Fz,z),\delta(Fz,z),\delta(Fz,z),\delta(Fz,z)\ \right)}\varphi(t)dt \le 0$$
Hence $\delta(Fz,z)=0$ so that $Fz=\{z\}$. Thus $Fz=\{z\}=fz$. Thus z is a common fixed point of F,G,f and g. Uniqueness of common fixed point follows easily from (2.3.1).
Suppose $fw=\{w\}=Fw$ for some $w\in X$.
$$\int_0^{\phi\left(\begin{array}{c}\delta(Fw,Gz),\delta(fw,gz)+\delta(fw,Fw)+\delta(gz,Gz),\\ \delta(fw,Fw)+\delta(fw,Gz),\delta(gz,Gz)+\delta(gz,Fw)\end{array}\right)}\varphi(t)dt \le 0$$
which implies
$$\int_0^{\phi\left(\ d(w,z),d(w,z),d(w,z),d(w,z)\ \right)}\varphi(t)dt \le 0$$
Hence $d(w,z)=0$ so that $w=z$. Thus z is the unique common fixed point of f and F. Similarly we can show that z is the unique common fixed point of g and G. Similarly, we can prove the theorem when (2.3.2)(b) holds. □

Theorem 2.3 is a generalization of the following

Theorem 2.4. *(Theorem 3.1,[8]):Let A,B,S and T be self mappings of a metric space(X,d) satisfying (2.2.1) of Corollary (2.2).Suppose that*
(2.4.1) the pairs (A,S) and (B,T) enjoy the common property(E.A.),
(2.4.2)$S(X)$ and $T(X)$ are closed subsets of X,
(2.4.3) the pairs $((A,S)$ and (B,T) are weakly compatible.
Then A,B,S and T have a unique common fixed point in X.

Proof. Let $F=\{A\},G=\{B\},f=\{S\},g=\{T\}$ be single valued mappings and $\varphi(t)=1$ for all $t>0$ in Theorem 2.3. Define $\psi(t_1,t_2,t_3,t_4,t_5,t_6)=\phi(t_1,t_2+t_3+t_4,t_3+t_5,t_4+t_6)$.
From (2.4.1), there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that
$limAx_n=limSx_n=limBy_n=limTy_n=z$ for some $z\in X$.
From (2.4.2), there exist $u,v\in X$ such that $z=Su=Tv$. The rest follows from Theorem 2.3. □

Second implicit relation :
Let $\phi:R_+^5\longrightarrow R$ be an upper semi continuous function satisfying
$\int_0^{\phi(0,u,u,u,u)}\varphi(t)dt\ge 0$ or $\int_0^{\phi(u,u,u,u,u)}\varphi(t)dt\ge 0$ implies $u=0$,where $\varphi:R_+\longrightarrow R$ is a Lebesgue-integrable map which is summable.
Now , we give some examples .
(i) Let $\phi(t_1,t_2,t_3,t_4,t_5)=t_1-k\ min\{t_2,t_3,t_4,t_5\}$, where $k>1$ and $\varphi(t)=t^2$ or $\varphi(t)=\frac{3\pi}{4(1-t)^2}Cos(\frac{3\pi t}{4(1-t)})$ for all $t\in R_+$.
Case : Suppose $\varphi(t)=t^2$.
Then $\int_0^{\phi(0,u,u,u,u)}\varphi(t)dt\ge 0\Rightarrow -\frac{1}{3}k^3u^3\ge 0\Rightarrow u\le 0$. But $u\ge 0$. Hence $u=0$.
Also $\int_0^{\phi(u,u,u,u,u)}\varphi(t)dt\ge 0\Rightarrow \frac{1}{3}(1-k)^3u^3\ge 0\Rightarrow u\le 0$. But $u\ge 0$. Hence $u=0$.
Case : $\varphi(t)=\frac{3\pi}{4(1-t)^2}Cos(\frac{3\pi t}{4(1-t)})$.

Then $\int_0^{\phi(0,u,u,u,u)}\varphi(t)dt \geq 0 \Rightarrow Sin(\frac{-3\pi ku}{4(1+ku)}) \geq 0 \Rightarrow Sin(\frac{3\pi ku}{4(1+ku)}) \leq 0 \Rightarrow u = 0$ since $0 \leq \frac{3\pi ku}{4(1+ku)} < \pi$.
$\int_0^{\phi(u,u,u,u,u)}\varphi(t)dt \geq 0 \Rightarrow Sin(\frac{3\pi(1-k)u}{4(1-(1-k)u)}) \geq 0 \Rightarrow Sin(\frac{3\pi(k-1)u}{4(1+(k-1)u)}) \leq 0 \Rightarrow u = 0.$
The following ϕ functions satisfy the second implicit relation with $\varphi(t) = t^2$ or $\varphi(t) = \frac{3\pi}{4(1-t)^2}Cos(\frac{3\pi t}{4(1-t)})$ for all $t \in R_+$.
(ii)$\phi(t_1, t_2, t_3, t_4, t_5) = t_1 - at_2 - b\frac{(t_2t_3+t_4t_5)}{(t_3+t_4)}$, where $a \geq 0, b \geq 0$ with $a + b > 1$.
(iii)$\phi(t_1, t_2, t_3, t_4, t_5) = t_1 - \alpha t_2 - \beta\ min\{t_3, t_4\} - \gamma\ min\{t_2 + t_3, t_4 + t_5\}$, where $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + 2\gamma > 1$.

Finally, we state the following theorem with expansive condition for four set - valued mappings.

Theorem 2.5. *Theorem 2.3 holds if the inequality(2.3.1) is replaced by (2.5.1)*

$$\int_0^{\phi\begin{pmatrix}\delta(fx,gy),\delta(Fx,Gy),\delta(fx,Fx)+\delta(gy,Gy)\\ \delta(fx,Fx)+\delta(fx,Gy),\delta(gy,Gy)+\delta(gy,Fx)\end{pmatrix}}\varphi(t)dt \geq 0$$

for all $x, y \in X$, where $\phi : R_+^5 \longrightarrow R$ is an upper semi continuous function satisfying $\int_0^{\phi(0,u,u,u,u)}\varphi(t)dt \geq 0$ or $\int_0^{\phi(u,u,u,u,u)}\varphi(t)dt \geq 0$ implies $u = 0$ and φ is as in Theorem 2.1.

Remark 2.6: Theorem 2.5 with f and g as single valued mappings is a generalization of Theorem 3.1 of [2].

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