J. Nonlinear Sci. Appl., 3 (2010), no. 4, 302–308

The Journal of Nonlinear Sciences and Applications
http://www.tjnsa.com

ĆIRIĆ'S FIXED POINT THEOREM IN A CONE METRIC SPACE

BESSEM SAMET

Abstract. In this paper, we extend a fixed point theorem due to Ćirić to a cone metric space.

1. Introduction and preliminaries

Many generalizations of the Banach contraction principle [4] have been considered in the literature (see [1]-[3], [5]-[17]).

Huang and Zhang [12] recently have introduced the concept of cone metric space, where the set of real numbers is replaced by an ordered Banach space, and they have established some fixed point theorems for contractive type mappings in a normal cone metric space. The study of fixed point theorems in such spaces is followed by some other mathematicians (see [1]-[3], [5], [13], [14], [16]).

In this paper, we extend a fixed point theorem due to Ćirić ([8]-Theorem 2.5) to a cone metric space. Before presenting our result, we start by recalling some definitions.

Let E be a real Banach space and P a subset of E. P is called a cone if and only if:

(i) P is closed, nonempty, and $P \neq \{0\}$.
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$.
(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by:

$$x \leq y \Leftrightarrow y - x \in P.$$

Date: Received: 7 March 2010.

2000 *Mathematics Subject Classification.* Primary 54H25; Secondary 47H10.
Key words and phrases. Ćirić's theorem; Cone metric space; Fixed point.

We shall write $x < y$ to indicate that $x \le y$ but $x \ne y$, while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P.

The cone P is called normal if there is a number $k > 0$ such that for all $x, y \in E$,

$$0 \le x \le y \Rightarrow \|x\| \le k\|y\|,$$

where $\|\cdot\|$ is the norm in E. In this case, the number k is called the normal constant of P. Rezapour and Hamlbarani [16] proved that there are no normal cones with normal constant $k < 1$ and for each $c > 1$ there are cones with normal constant $k > c$. For this reason, in all this paper, we take $k \ge 1$.

In the following we always suppose E is a Banach space, P is a cone in E with int $P \ne \emptyset$ and $\le$ is partial ordering with respect to P. As it has been defined in [12], a function $d : X \times X \to E$ is called a cone metric on X if it satisfies the following conditions:

(a) $0 < d(x, y)$ for all $x, y \in X$, $x \ne y$ and $d(x, y) = 0$ if and only if $x = y$.
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(c) $d(x, y) \le d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then (X, d) is called a cone metric space.

Let (x_n) be a sequence in X and $x \in X$.

- If for every $c \in E$, $c \gg 0$ there is N such that for all $n > N$, $d(x_n, x) \ll c$, then (x_n) is said to be convergent to x and x is the limit of (x_n). We denote this by $x_n \to x$ as $n \to +\infty$.
- If for any $c \in E$ with $0 \ll c$, there is N such that for all $n, m > N$, $d(x_n, x_m) \ll c$, then (x_n) is called a Cauchy sequence in X.

Let (X, d) be a cone metric space. If every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

The following lemmas will be useful later.

Lemma 1.1. *(Huang and Zhang* [12]*) Let (X, d) be a cone metric space, P be a normal cone. Let (x_n) be a sequence in X. Then (x_n) converges to x if and only if $\|d(x_n, x)\| \to 0$ as $n \to +\infty$.*

Lemma 1.2. *(Huang and Zhang* [12]*) Let (X, d) be a cone metric space, (x_n) be a sequence in X. If (x_n) is convergent, then it is a Cauchy sequence, too.*

Lemma 1.3. *(Huang and Zhang* [12]*) Let (X, d) be a cone metric space, P be a normal cone. Let (x_n) be a sequence in X. Then, (x_n) is a Cauchy sequence if and only if $\|d(x_n, x_m)\| \to 0$ as $n, m \to +\infty$.*

We denote $\mathcal{L}(E)$ the set of linear bounded operators on E, endowed with the following norm:

$$\|S\| = \sup_{x \in E, x \ne 0} \frac{\|Sx\|}{\|x\|}, \ \forall\, S \in \mathcal{L}(E).$$

It is clear that if $S \in \mathcal{L}(E)$, we have:

$$\|Sx\| \le \|S\|\|x\|, \ \forall\, x \in E.$$

We denote by $I : E \to E$ the identity operator, i.e., $Ix = x, \forall\, x \in X$. If $S \in \mathcal{L}(E)$, we denote by $S^{-1} \in \mathcal{L}(E)$ (if such operator exists) the operator defined by:

$$S^{-1}Sx = SS^{-1}x = x, \ \forall\, x \in E.$$

2. Fixed point theorem

The main result of this paper is the following.

Theorem 2.1. *Let (X, d) be a complete cone metric space, P be a normal cone with normal constant k ($k \geq 1$). Suppose the mapping $T : X \to X$ satisfies the following contractive condition:*

$$\begin{aligned} d(Tx, Ty) \ \leq \ & A_1(x,y)d(x,y) + A_2(x,y)d(x,Tx) + A_3(x,y)d(y,Ty) \quad (2.1)\\ & + A_4(x,y)d(x,Ty) + A_4(x,y)d(y,Tx), \end{aligned}$$

for all $x, y \in X$, where $A_i : X \times X \to \mathcal{L}(E)$, $i = 1, \cdots, 4$. Further, assume that for all $x, y \in X$, we have:

$$\exists\, \alpha \in [0, 1/k) \mid \sum_{i=1}^{4} \|A_i(x,y)\| + \|A_4(x,y)\| \leq \alpha \quad (2.2)$$

$$\exists\, \beta \in [0, 1) \mid \|S(x,y)\| \leq \beta \quad (2.3)$$

$$(A_1(x,y) + A_2(x,y))(P) \subseteq P \quad (2.4)$$

$$A_2(x,y)(P) \subseteq P \quad (2.5)$$

$$A_4(x,y)(P) \subseteq P \quad (2.6)$$

$$(I - A_3(x,y) - A_4(x,y))^{-1}(P) \subseteq P. \quad (2.7)$$

Here, $S : X \times X \to \mathcal{L}(E)$ is given by:

$$S(x,y) = (I - A_3(x,y) - A_4(x,y))^{-1}(A_1(x,y) + A_2(x,y) + A_4(x,y)), \ \forall\, x, y \in X.$$

Then, T has a unique fixed point.

Proof. Let $x \in X$ be arbitrary and define the sequence $(x_n)_{n \in \mathbf{N}} \subset X$ by:

$$x_0 = x, x_1 = Tx_0, \cdots, x_n = Tx_{n-1} = T^n x_0, \cdots$$

By (2.1), we get:

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n)\\ &\leq A_1(x_{n-1}, x_n)d(x_{n-1}, x_n) + A_2(x_{n-1}, x_n)d(x_{n-1}, x_n)\\ &\quad + A_3(x_{n-1}, x_n)d(x_n, x_{n+1}) + A_4(x_{n-1}, x_n)d(x_{n-1}, x_{n+1})\\ &\quad + A_4(x_{n-1}, x_n)d(x_n, x_n)\\ &= (A_1(x_{n-1}, x_n) + A_2(x_{n-1}, x_n))d(x_{n-1}, x_n) + A_3(x_{n-1}, x_n)d(x_n, x_{n+1})\\ &\quad + A_4(x_{n-1}, x_n)d(x_{n-1}, x_{n+1}). \end{aligned}$$

Using the triangular inequality, we get:

$$d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1}),$$

i.e.,

$$d(x_{n-1}, x_n) + d(x_n, x_{n+1}) - d(x_{n-1}, x_{n+1}) \in P.$$

From (2.6), it follows that:

$$A_4(x_{n-1},x_n)[d(x_{n-1},x_n)+d(x_n,x_{n+1})-d(x_{n-1},x_{n+1})]\in P,$$

i.e.,

$$A_4(x_{n-1},x_n)d(x_{n-1},x_{n+1})\leq A_4(x_{n-1},x_n)d(x_{n-1},x_n)+A_4(x_{n-1},x_n)d(x_n,x_{n+1}).$$

Then, we have:

$$\begin{aligned} d(x_n,x_{n+1}) &\leq (A_1(x_{n-1},x_n)+A_2(x_{n-1},x_n)+A_4(x_{n-1},x_n))d(x_{n-1},x_n)\\ &\quad +(A_3(x_{n-1},x_n)+A_4(x_{n-1},x_n))d(x_n,x_{n+1}). \end{aligned}$$

Hence,

$$\begin{aligned} (I-A_3(x_{n-1},x_n)-A_4(x_{n-1},x_n))d(x_n,x_{n+1}) &\leq (A_1(x_{n-1},x_n)+A_2(x_{n-1},x_n)\\ &\quad +A_4(x_{n-1},x_n))d(x_{n-1},x_n). \end{aligned}$$

Using (2.7), we get:

$$d(x_n,x_{n+1})\leq S(x_{n-1},x_n)d(x_{n-1},x_n). \tag{2.8}$$

It is not difficult to see that under hypotheses (2.4), (2.6) and (2.7), we have:

$$S(x,y)(P)\subseteq P,\ \forall\, x,y\in X.$$

Using this remark, (2.8) and proceeding by iterations, we get:

$$d(x_n,x_{n+1})\leq S(x_{n-1},x_n)S(x_{n-2},x_{n-1})\cdots S(x_0,x_1)d(x_0,x_1),$$

which implies by (2.3) that:

$$\|d(x_n,x_{n+1})\|\leq k\|S(x_{n-1},x_n)\|\|S(x_{n-2},x_{n-1})\|\cdots\|S(x_0,x_1)\|\|d(x_0,x_1)\|\leq k\beta^n\|d(x_0,x_1)\|.$$

For any positive integer p, we have:

$$d(x_n,x_{n+p})\leq\sum_{i=1}^{p}d(x_{n+i-1},x_{n+i}),$$

which implies that:

$$\begin{aligned} \|d(x_n,x_{n+p})\| &\leq k\sum_{i=1}^{p}\|d(x_{n+i-1},x_{n+i})\|\\ &\leq k^2\sum_{i=1}^{p}\beta^{n+i-1}\|d(x_0,x_1)\|\\ &\leq k^2\frac{\beta^n}{1-\beta}\|d(x_0,x_1)\|. \end{aligned}\tag{2.9}$$

Since $\beta\in[0,1)$, $\beta^n\to 0$ as $n\to+\infty$. So from (2.9) it follows that the sequence $(x_n)_{n\in\mathbf{N}}$ is Cauchy. Since (X,d) is complete, there is a point $u\in X$ such that:

$$\lim_{n\to+\infty}d(Tx_n,u)=\lim_{n\to+\infty}d(x_n,u)=\lim_{n\to+\infty}d(x_n,x_{n+1})=0. \tag{2.10}$$

Now, using the contractive condition (2.1), we get:

$$\begin{aligned} d(Tu,Tx_n) &\leq A_1(u,x_n)d(u,x_n)+A_2(u,x_n)d(u,Tu)\\ &\quad +A_3(u,x_n)d(x_n,x_{n+1})+A_4(u,x_n)d(u,x_{n+1})\\ &\quad +A_4(u,x_n)d(x_n,Tu). \end{aligned}$$

By the triangular inequality, we have:

$$\begin{aligned} d(u,Tu) &\leq d(u,x_{n+1})+d(x_{n+1},Tu)\\ d(x_n,Tu) &\leq d(x_n,Tx_n)+d(Tx_n,Tu). \end{aligned}$$

By (2.5) and (2.6), we get:

$$\begin{aligned} A_2(u,x_n)d(u,Tu) &\leq A_2(u,x_n)(d(u,x_{n+1})+d(x_{n+1},Tu))\\ A_4(u,x_n)d(x_n,Tu) &\leq A_4(u,x_n)d(x_n,Tx_n)+A_4(u,x_n)d(Tx_n,Tu). \end{aligned}$$

Hence,

$$\begin{aligned} d(Tu,Tx_n) &\leq A_1(u,x_n)d(u,x_n)+(A_2(u,x_n)+A_4(u,x_n))d(u,x_{n+1})\\ &\quad +(A_2(u,x_n)+A_4(u,x_n))d(x_{n+1},Tu)\\ &\quad +(A_3(u,x_n)+A_4(u,x_n))d(x_n,x_{n+1}). \end{aligned}$$

Using (2.2), this inequality implies that:

$$\|d(Tu,Tx_n)\| \leq \frac{k\alpha}{1-k\alpha}(\|d(u,x_n)\|+\|d(u,x_{n+1})\|+\|d(x_n,x_{n+1})\|).$$

From (2.10), it follows immediately that:

$$\lim_{n\to+\infty} d(Tu,Tx_n)=0. \tag{2.11}$$

Then, (2.10), (2.11) and the uniqueness of the limit imply that $u=Tu$, i.e., u is a fixed point of T. So we proved that T has least one fixed point $u\in X$.

Now, if $v\in X$ is another fixed point of T, by (2.1), we get:

$$d(u,v)=d(Tu,Tv)\leq A_1(u,v)d(u,v)+2A_4(u,v)d(u,v),$$

which implies that:

$$\|d(u,v)\|\leq k(\|A_1(u,v)\|+2\|A_4(u,v)\|)\|d(u,v)\|\leq k\alpha\|d(u,v)\|,$$

i.e.,

$$(1-k\alpha)\|d(u,v)\|\leq 0.$$

Since $0\leq\alpha<1/k$, we get $d(u,v)=0$, i.e., $u=v$. So the proof of the theorem is complete. □

Now, we will show that Theorem 2.5 of Ćirić [8] is a particular case of Theorem 2.1.

Corollary 2.2. *Let (X,d) be a complete metric space and $T:X\to X$ be a mapping satisfying the following contractive condition:*

$$\begin{aligned} d(Tx,Ty) &\leq a_1(x,y)d(x,y)+a_2(x,y)d(x,Tx)+a_3(x,y)d(y,Ty) \quad (2.12)\\ &\quad +a_4(x,y)(d(x,Ty)+d(y,Tx)), \end{aligned}$$

for all $x, y \in X$*, where* $a_i : X \times X \to [0, +\infty)$*,* $i = 1, \cdots, 4$ *and* $\sum_{i=1}^{4} \alpha_i(x, y) + \alpha_4(x, y) \leq \alpha$ *for each* $x, y \in X$ *and some* $\alpha \in [0, 1)$*. Then,* T *has a unique fixed point.*

Proof. We take $E = \mathbf{R}$ (with the usual norm) and $P = [0, +\infty)$. Then, (X, d) is a complete cone metric space and P is a normal cone with normal constant $k = 1$. For each $i = 1, \cdots, 4$, we define $A_i : X \times X \to \mathcal{L}(E)$ by:

$$A_i(x, y) : t \in \mathbf{R} \mapsto a_i(x, y)t,$$

for all $x, y \in X$. let us check now that all the required hypotheses of Theorem 2.1 are satisfied.

- Condition (2.12) implies that:

$$\begin{aligned} d(Tx, Ty) \quad &\leq \quad A_1(x, y)d(x, y) + A_2(x, y)d(x, Tx) + A_3(x, y)d(y, Ty) \\ &\quad +A_4(x, y)d(x, Ty) + A_4(x, y)d(y, Tx), \end{aligned}$$

  for all $x, y \in X$. Then, condition (2.1) of Theorem 2.1 is satisfied.
- For all $i = 1, \cdots, 4$, we have:

$$\|A_i(x, y)\| = a_i(x, y), \ \forall\, x, y \in X.$$

  Then,

$$\sum_{i=1}^{4} \|A_i(x, y)\| + \|A_4(x, y)\| \leq \alpha, \ \forall\, x, y \in X$$

  and condition (2.2) of Theorem 2.1 is satisfied.
- For all $x, y \in X$, we have:

$$S(x, y)t = \frac{a_1(x, y) + a_2(x, y) + a_4(x, y)}{1 - a_3(x, y) - a_4(x, y)}\, t, \ \forall\, t \in \mathbf{R}.$$

  Then, for all $x, y \in X$, we have:

$$\|S(x, y)\| = \frac{a_1(x, y) + a_2(x, y) + a_4(x, y)}{1 - a_3(x, y) - a_4(x, y)}.$$

  Since $\alpha \in [0, 1)$, we have:

$$a_1(x, y) + a_2(x, y) + a_4(x, y) + \alpha a_3(x, y) + \alpha a_4(x, y) \leq \alpha, \ \forall\, x, y \in X.$$

  Then,

$$\|S(x, y)\| \leq \alpha, \ \forall\, x, y \in X$$

  and condition (2.3) of Theorem 2.1 holds with $\beta = \alpha$.
- Conditions (2.4), (2.5) and (2.6) are easy to check.
- For all $x, y \in X$, we have:

$$(I - A_3(x, y) - A_4(x, y))^{-1}s = \frac{s}{1 - a_3(x, y) - a_4(x, y)}, \ \forall\, s \in \mathbf{R}.$$

  Since $a_3(x, y) + a_4(x, y) < 1$ for all $x, y \in X$, then

$$s \geq 0 \Rightarrow (I - A_3(x, y) - A_4(x, y))^{-1}s \geq 0.$$

  Hence, condition (2.7) of Theorem 2.1 is satisfied.

Now, we are able to apply Theorem 2.1 and then, T has a unique fixed point. $\square$

3. Open problem

We present the following open problem.

In hypothesis (2.2), we assumed that $\alpha \in [0, 1/k)$, where k is the normal constant of the cone P. What can we say about the case when $\alpha \in [1/k, 1)$ with $k > 1$?

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Département de Mathématiques, École Supérieure des Sciences et Techniques de Tunis, 5, Avenue Taha Hussein-Tunis, B.P.:56, Bab Menara-1008, Tunisie.

E-mail address: bessem.samet@gmail.com