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PROOFS OF THREE OPEN INEQUALITIES WITH POWER-EXPONENTIAL FUNCTIONS

VASILE CIRTOAJE

Dedicated to Themistocles M. Rassias on the occasion of his sixtieth birthday
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Abstract. The main aim of this paper is to give a complete proof to the open inequality with power-exponential functions

$$a^{ea} + b^{eb} \geq a^{eb} + b^{ea},$$

which holds for all positive real numbers a and b. Notice that this inequality was proved in [1] for only $a \geq b \geq \frac{1}{e}$ and $\frac{1}{e} \geq a \geq b$. In addition, other two open inequalities with power-exponential functions are proved, and three new conjectures are presented.

1. Introduction

We conjectured in [1] and [3] that e is the greatest possible value of a positive real number r such that the following inequality holds for all positive real numbers a and b:

$$a^{ra} + b^{rb} \geq a^{rb} + b^{ra}. \tag{1.1}$$

In addition, we proved in [1] the following results related to this inequality.

Theorem A. *If (1.1) holds for $r = r_0 > 0$, then it holds for all $0 < r \leq r_0$.*

Theorem B. *If $\max\{a, b\} \geq 1$, then (1.1) holds for any $r > 0$.*

Theorem C. *If $r > e$, then (1.1) does not hold for all positive real numbers a and b.*

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Theorem D. *If a and b are positive real numbers such that either $a \geq b \geq \frac{1}{r}$ or $\frac{1}{r} \geq a \geq b$, then (1.1) holds for all $0 < r \leq e$.*

2. Main result

In order to give a complete answer to our problem, we only need to prove the following theorem.

Theorem 2.1. *If a and b are positive real numbers such that $0 < b \leq \frac{1}{e} \leq a \leq 1$, then*

$$a^{ea} + b^{eb} \geq a^{eb} + b^{ea}.$$

The proof of Theorem 2.1 relies on the following four lemmas.

Lemma 2.1. *If $x > 0$, then*

$$x^x - 1 \geq (x-1)e^{x-1}.$$

Lemma 2.2. *If $0 < y \leq 1$, then*

$$1 - \ln y \geq e^{1-y}.$$

Lemma 2.3. *If $x \geq 1$, then*

$$\ln x \geq (x-1)e^{1-x}.$$

Lemma 2.4. *If $x \geq 1$ and $0 < y \leq 1$, then*

$$x^{y-1} \geq y^{x-1}.$$

Notice that Lemma 2.1 is a particular case of Theorem 2.1, namely the case where $a = \frac{x}{e}$ and $b = \frac{1}{e}$.

On the other hand, from Theorem B and its proof in [1], it follows that $a, b \in (0, 1]$ is the main case of the inequality (1.1). However, we conjecture that the following sharper inequality still holds in the same conditions:

Conjecture 2.1. *If $a, b \in (0, 1]$ and $r \in (0, e]$, then*

$$2\sqrt{a^{ra}b^{rb}} \geq a^{rb} + b^{ra}.$$

In the particular case $r = 2$, we get the elegant inequality

$$2a^a b^b \geq a^{2b} + b^{2a}, \tag{2.1}$$

which is also an open problem. A similar inequality is

$$2a^a b^b \geq (ab)^a + (ab)^b, \tag{2.2}$$

where $a, b \in (0, 1]$. Notice that a proof of (2.2) is given in [2]. It seems that this inequality can be extended to three variables, as follows.

Conjecture 2.2. *If $a, b, c \in (0, 1]$, then*

$$3a^a b^b c^c \geq (abc)^a + (abc)^b + (abc)^c.$$

3. Proof of Lemmas

Proof of Lemma 2.1. Write the desired inequality as $f(x) \geq 0$, where

$$f(x) = x\ln x - \ln[1 + (x-1)e^{x-1}]$$

has the derivatives

$$f'(x) = 1 + \ln x - \frac{xe^{x-1}}{1 + (x-1)e^{x-1}}$$

and

$$f''(x) = \frac{x(x-1)e^{x-1}(e^{x-1}-1) + (e^{x-1}-1)^2}{x[1 + (x-1)e^{x-1}]^2}.$$

Since $(x-1)(e^{x-1}-1) \geq 0$, we have $f''(x) \geq 0$, and hence $f'(x)$ is strictly increasing for $x > 0$. Since $f'(1) = 0$, it follows that $f'(x) < 0$ for $0 < x < 1$, and $f'(x) > 0$ for $x > 1$. Therefore, $f(x)$ is strictly decreasing on $(0, 1]$ and strictly increasing on $[1, \infty)$, and then $f(x) \geq f(1) = 0$. □

Proof of Lemma 2.2. We need to show that $f(y) \geq 0$ for $0 < y \leq 1$, where

$$f(y) = 1 - \ln y - e^{1-y}.$$

Write the derivative in the form

$$f'(y) = \frac{e^{1-y}g(y)}{y},$$

where

$$g(y) = y - e^{y-1}.$$

Since $g'(y) = 1 - e^{y-1} > 0$ for $0 < y < 1$, $g(y)$ is strictly increasing, $g(y) \leq g(1) = 0$, $f'(y) < 0$ for $0 < y < 1$, $f(y)$ is strictly decreasing, and hence $f(y) \geq f(1) = 0$. □

Proof of Lemma 2.3. Since

$$e^{1-x} = \frac{1}{e^{x-1}} \leq \frac{1}{1 + (x-1)} = \frac{1}{x},$$

it suffices to show that $f(x) \geq 0$ for $x \geq 1$, where

$$f(x) = \ln x + \frac{1}{x} - 1.$$

This is true because $f'(x) = \dfrac{x-1}{x^2} \geq 0$, $f(x)$ is strictly increasing, and hence $f(x) \geq f(1) = 0$. □

Proof of Lemma 2.4. Consider the nontrivial case when $0 < y < 1$. For fixed $y \in (0, 1)$, we write the desired inequality as $f(x) \geq 0$ for $x \geq 1$, where

$$f(x) = (y-1)\ln x - (x-1)\ln y.$$

We have

$$f'(x) = \frac{y-1}{x} - \ln y \geq y - 1 - \ln y.$$

Let us denote $g(y) = y - 1 - \ln y$. Since $g'(y) = 1 - \dfrac{1}{y} < 0$, $g(y)$ is strictly decreasing on $(0, 1)$, and then $g(y) > g(1) = 0$. Therefore, $f'(x) > 0$, $f(x)$ is strictly increasing for $x \geq 1$, and hence $f(x) \geq f(1) = 0$. □

4. Proof of Theorem 2.1

Making the substitutions $x = ea$ and $y = eb$, we have to show that

$$(x^x - y^x)e^{-x} + (y^y - x^y)e^{-y} \geq 0 \tag{4.1}$$

for $0 < y \leq 1 \leq x \leq e$. By Lemma 2.1, we have

$$x^x \geq 1 + (x-1)e^{x-1}$$

and

$$y^y \geq 1 + (y-1)e^{y-1}.$$

Therefore, it suffices to show that

$$(1 + (x-1)e^{x-1} - y^x)e^{-x} + (1 + (y-1)e^{y-1} - x^y)e^{-y} \geq 0,$$

which is equivalent to

$$x + y - 2 + (1 - y^x)e^{1-x} + (1 - x^y)e^{1-y} \geq 0.$$

For fixed $y \in (0, 1]$, write this inequality as $f(x) \geq 0$, where

$$f(x) = x + y - 2 + (1 - y^x)e^{1-x} + (1 - x^y)e^{1-y}, \quad 1 \leq x \leq e.$$

If $f'(x) \geq 0$, then $f(x) \geq f(1) = 0$, and the conclusion follows. We have

$$f'(x) = 1 - e^{1-x} - yx^{y-1}e^{1-y} + y^x(1 - \ln y)e^{1-x}$$

and, by Lemma 2.2, it follows that

$$f'(x) \geq 1 - e^{1-x} - yx^{y-1}e^{1-y} + y^x e^{2-x-y}.$$

For fixed $x \in [1, e]$, let us denote

$$g(y) = 1 - e^{1-x} - yx^{y-1}e^{1-y} + y^x e^{2-x-y}, \quad 0 < y \leq 1.$$

We need to show that $g(y) \geq 0$. Since $g(1) = 0$, it suffices to prove that $g'(y) \leq 0$ for $0 < y \leq 1$. We have

$$e^{y-1}g'(y) = (y-1)x^{y-1} - yx^{y-1}\ln x + (xy^{x-1} - y^x)e^{1-x}$$

and, by Lemma 2.3, we get

$$e^{y-1}g'(y) \leq (y-1)x^{y-1} + (yx^{y-1} - yx^y + xy^{x-1} - y^x)e^{1-x}.$$

If $yx^{y-1} - yx^y + xy^{x-1} - y^x \leq 0$, then clearly $g'(y) \leq 0$. Consider now that $yx^{y-1} - yx^y + xy^{x-1} - y^x > 0$. Since $e^{1-x} \leq \dfrac{1}{x}$, we have

$$\begin{aligned} e^{y-1}g'(y) &\leq (y-1)x^{y-1} + \frac{yx^{y-1} - yx^y + xy^{x-1} - y^x}{x} \\ &= \frac{(x-y)(y^{x-1} - x^{y-1})}{x}, \end{aligned}$$

and, by Lemma 2.4, it follows that $g'(y) \leq 0$. Thus, the proof is completed. □

5. Other related inequalities

We posted in [1] the following two open inequalities.

Proposition 5.1. *If a, b are nonnegative real numbers satisfying $a + b = 2$, then*

$$a^{3b} + b^{3a} \le 2,$$

with equality for $a = b = 1$.

Proposition 5.2. *If a, b are nonnegative real numbers satisfying $a + b = 1$, then*

$$a^{2b} + b^{2a} \le 1,$$

with equality for $a = b = \frac{1}{2}$, for $a = 0$ and $b = 1$, and for $a = 1$ and $b = 0$.

A complicated solution of Proposition 5.1 was given by L. Matejicka in [4]. We will give further a much simpler proof of Proposition 5.1, and a proof of Proposition 5.2. However, it seems that the following generalization of Proposition 5.2 holds.

Conjecture 5.1. *Let a, b be nonnegative real numbers satisfying $a + b = 1$. If $k \ge 1$, then*

$$a^{(2b)^k} + b^{(2a)^k} \le 1.$$

6. Proof of Proposition 5.1

Without loss of generality, assume that $a \ge b$. For $a = 2$ and $b = 0$, the desired inequality is obvious. Otherwise, using the substitutions $a = 1+x$ and $b = 1-x$, $0 \le x < 1$, we can write the inequality as

$$e^{3(1-x)\ln(1+x)} + e^{3(1+x)\ln(1-x)} \le 2.$$

Applying Lemma 6.1 below, it suffices to show that $f(x) \le 2$, where

$$f(x) = e^{3(1-x)(x-\frac{x^2}{2}+\frac{x^3}{3})} + e^{-3(1+x)(x+\frac{x^2}{2}+\frac{x^3}{3})}.$$

If $f'(x) \le 0$ for $x \in [0, 1)$, then $f(x)$ is decreasing, and hence $f(x) \le f(0) = 2$. Since

$$\begin{aligned} f'(x) =& (3 - 9x + \frac{15}{2}x^2 - 4x^3)e^{3x-\frac{9x^2}{2}+\frac{5x^3}{2}-x^4} \\ & -(3 + 9x + \frac{15}{2}x^2 + 4x^3)e^{-3x-\frac{9x^2}{2}-\frac{5x^3}{2}-x^4}, \end{aligned}$$

$f'(x) \le 0$ is equivalent to

$$e^{-6x-5x^3} \ge \frac{6 - 18x + 15x^2 - 8x^3}{6 + 18x + 15x^2 + 8x^3}.$$

For the nontrivial case $6-18x+15x^2-8x^3 > 0$, we rewrite the required inequality as $g(x) \ge 0$, where

$$g(x) = -6x - 5x^3 - \ln(6 - 18x + 15x^2 - 8x^3) + \ln(6 + 18x + 15x^2 + 8x^3).$$

If $g'(x) \geq 0$ for $x \in [0, 1)$, then $g(x)$ is increasing, and hence $g(x) \geq g(0) = 0$. From

$$\frac{1}{3}g'(x) = -2 - 5x^2 + \frac{(6+8x^2) - 10x}{6 + 15x^2 - (18x + 8x^3)} + \frac{(6+8x^2) + 10x}{6 + 15x^2 + (18x + 8x^3)},$$

it follows that $g'(x) \geq 0$ is equivalent to

$$2(6+8x^2)(6+15x^2) - 20x(18x+8x^3) \geq (2+5x^2)[(6+15x^2)^2 - (18x+8x^3)^2].$$

Since

$$(6+15x^2)^2 - (18x+8x^3)^2 \leq (6+15x^2)^2 - 324x^2 - 288x^4 \leq 4(9 - 36x^2),$$

it suffices to show that

$$(3+4x^2)(6+15x^2) - 5x(18x+8x^3) \geq (2+5x^2)(9-36x^2).$$

This reduces to $6x^2 + 200x^4 \geq 0$, which is clearly true. □

Lemma 6.1. *If $t > -1$, then*

$$\ln(1+t) \leq t - \frac{t^2}{2} + \frac{t^3}{3}.$$

Proof. We need to prove that $f(t) \geq 0$, where

$$f(t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \ln(1+t).$$

Since

$$f'(t) = \frac{t^3}{t+1},$$

$f(t)$ is decreasing on $(-1, 0]$ and increasing on $[0, \infty)$. Therefore, $f(t) \geq f(0) = 0$. □

7. Proof of Proposition 5.2

Without loss of generality, assume that

$$0 \leq b \leq \frac{1}{2} \leq a \leq 1.$$

Applying Lemma 7.1 below for $c = 2b$, $0 \leq c \leq 1$, we get

$$a^{2b} \leq (1-2b)^2 + 4ab(1-b) - 2ab(1-2b)\ln a,$$

which is equivalent to

$$a^{2b} \leq 1 - 4ab^2 - 2ab(a-b)\ln a. \tag{7.1}$$

Similarly, applying Lemma 7.2 for $d = 2a - 1$, $d \geq 0$, we get

$$b^{2a-1} \leq 4a(1-a) + 2a(2a-1)\ln(2a+b-1),$$

which is equivalent to

$$b^{2a} \leq 4ab^2 + 2ab(a-b)\ln a. \tag{7.2}$$

Adding up (7.1) and (7.2), the desired inequality follows. □

Lemma 7.1. *If* $0 < a \le 1$ *and* $c \ge 0$*, then*

$$a^c \le (1-c)^2 + ac(2-c) - ac(1-c)\ln a,$$

with equality for $a = 1$*, for* $c = 0$*, and for* $c = 1$*.*

Proof. Using the substitution $a = e^{-x}$, $x \ge 0$, we need to prove that $f(x) \ge 0$, where

$$f(x) = (1-c)^2 e^x + c(2-c) + c(1-c)x - e^{(1-c)x},$$

$$f'(x) = (1-c)[(1-c)e^x + c - e^{(1-c)x}].$$

If $f'(x) \ge 0$ for $x \ge 0$, then $f(x)$ is increasing, and $f(x) \ge f(0) = 0$. In order to prove this, we consider two cases. For $0 \le c \le 1$, by the weighted AM-GM inequality, we have

$$(1-c)e^x + c \ge e^{(1-c)x},$$

and hence $f'(x) \ge 0$. For $c \ge 1$, by the weighted AM-GM inequality, we have

$$(c-1)e^x + e^{(1-c)x} \ge c,$$

and hence $f'(x) \ge 0$, too. □

Lemma 7.2. *If* $0 \le b \le 1$ *and* $d \ge 0$*, then*

$$b^d \le 1 - d^2 + d(1+d)\ln(b+d),$$

with equality for $d = 0$*, and for* $b = 0$*,* $d = 1$*.*

Proof. Excepting the equality cases, from

$$1 - d + d\ln(b+d) \ge 1 - d + d\ln d \ge 0,$$

we get $1 - d + d\ln(b+d) > 0$. So, we may write the required inequality as

$$\ln(1+d) + \ln[1 - d + d\ln(b+d)] \ge d\ln b.$$

Using the substitution $b = e^{-x} - d$, $-\ln(1+d) \le x \le -\ln d$, we need to prove that $f(x) \ge 0$, where

$$f(x) = \ln(1+d) + \ln(1-d-dx) + dx - d\ln(1-de^x).$$

Since

$$f'(x) = \frac{d^2(e^x - 1 - x)}{(1-d-dx)(1-de^x)} \ge 0,$$

$f(x)$ is increasing, and hence

$$f(x) \ge f(-\ln(1+d)) = \ln[1 - d^2 + d(1+d)\ln(1+d)].$$

To complete the proof, we only need to show that $-d^2 + d(1+d)\ln(1+d) \ge 0$; that is,

$$(1+d)\ln(1+d) \ge d.$$

This inequality follows from $e^x \ge 1 + x$ for $x = \dfrac{-d}{1+d}$. □

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* Department of Automatic Control and Computers, University of Ploiesti, Romania.

E-mail address: vcirtoaje@upg-ploiesti.ro