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Convergence criteria of modified Noor iterations with errors for three asymptotically nonexpansive nonself-mappings

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Abstract

The aim of this paper, is to introduce and study the modified Noor iterative algorithm with errors for approximating common fixed points of three asymptotically nonexpansive nonself-mappings. Several strong and weak convergence results on this algorithm are established under certain conditions in a uniformly convex Banach space. The results obtained in this paper improve and generalize the recent ones announced by Khan and Hussain [S. H. Khan, N. Hussain, Comput. Math. Appl. **55** (2008), 2544–2553.], Nammanee, et. al., [K. Nammanee, M.A. Noor and S. Suantai, J. Math. Anal. Appl. **314** (2006), 320–334.], Suantai [S. Suantai, J. Math. Anal. Appl. **311** (2005), 506–517.], Cho et. al., [Y. J. Cho, H. Y. Zhou and G. Guo, Comput. Math. Appl. **47** (2004), 707–717.], Xu and Noor [B. L. Xu and M.A. Noor, J. Math. Anal. Appl. **267** (2002), 444–453.] and many others.

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1. Introduction

Let C be a nonempty closed convex subset of real normed linear space X. A self-mapping $T : C \to C$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n(x) - T^n(y)\| \quad \leq \quad k_n\|x - y\| \tag{1.1}$$

for all $x, y \in C$ and $n \geq 1$.

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If $k_n \equiv 1$, then T is known as a nonexpansive mapping. A self-mapping T is called *uniformly $L-$ Lipschitzian* if there exists a constant $L > 0$ such that

$$\|T^n(x) - T^n(y)\| \leq L\|x - y\| \tag{1.2}$$

for all $x, y \in C$ and $n \geq 1$.

It is easy to see that if T is an asymptotically nonexpansive, then it is uniformly $L-$Lipschitzian with the uniform Lipschitz constant $L = \sup\{k_n : n \geq 1\}$.

Iterative methods for approximating fixed points of certain mappings have been studied by various authors, using the Mann iterative (a one-step) and the Ishikawa iterative (a two-step) processes. For example, see [3], [4], [9], [11], [15-19]. Goebel and Kirk [7] introduced the class of asymptotically nonexpansive self-mappings, who proved that if C is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping on C, then T has a fixed point.

Glowinski and Le Tallec [6] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. It has been shown in [6] that the three-step iterative scheme gives better numerical results than the two-step and one-step approximate iterations. In 1998, Haubruge, Nguyen and Strodiot [8] studied the convergence analysis of three-step schemes of Glowinski and Le Tallec [6] and applied these schemes to obtain new splitting-type algorithms for solving variation inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iterations lead to highly parallelized algorithms under certain conditions. Thus we conclude that three-step scheme plays an important and significant part in solving various problems, which arise in pure and applied sciences.

The concept of asymptotically nonexpansive nonself-mappings was introduced in [1] in 2003 as the generalization of asymptotically nonexpansive self-mappings. The asymptotically nonexpansive nonself-mapping is defined as follows:

Definition 1.1 ([1]). Let C be a nonempty subset of real normed linear space X. Let $P : X \to C$ be a nonexpansive retraction of X onto C. A nonself-mapping $T : C \to X$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x - y\| \tag{1.3}$$

for all $x, y \in C$ and $n \geq 1$. T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\| \tag{1.4}$$

for all $x, y \in C$ and $n \geq 1$.

By studying the following iteration process:

$$x_1 \in C, \quad x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n),$$

Chidume, Ofoedu and Zegeye [1] gave some strong and weak convergence theorems for asymptotically nonexpansive nonself-mapping in a uniformly convex Banach space.

If T is a self-mapping, then P becomes the identity mapping so that (1.3) and (1.4) reduce to (1.1) and (1.2), respectively.

Recently, Khan and Hussain [10] introduced the following three-step iterative process and used it for the weak and strong convergence of fixed points of asymptotically nonexpansive nonself-mappings in a uniformly convex Banach space. For an arbitrary $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative scheme

$$\begin{aligned} z_n &= P(a_n T(PT)^{n-1}x_n + (1 - a_n)x_n), \\ y_n &= P(b_n T(PT)^{n-1}z_n + c_n T(PT)^{n-1}x_n + (1 - b_n - c_n)x_n), \\ x_{n+1} &= P(\alpha_n T(PT)^{n-1}y_n + \beta_n T(PT)^{n-1}z_n + (1 - \alpha_n - \beta_n)x_n), \quad n \geq 1, \end{aligned} \tag{1.5}$$

where $\{a_n\},\{b_n\},\{c_n\},\{\alpha_n\},\{\beta_n\}$ are appropriate sequences in $[0,1]$ satisfy certain conditions.

Obviously the above process deals with one self mapping only. Note that approximating the common fixed points, has its own importance as it has a direct link with the minimization problem, see for example Takahashi [21].

Inspired and motivated by these facts, a three-step iterative scheme with errors for approximating common fixed points of three asymptotically nonexpansive nonself-mappings is introduced and studied in this paper. The scheme is defined as follows.

Let X be a normed space, C a nonempty convex subset of X, $P : X \to C$ a nonexpansive retraction of X onto C and $T_1, T_2, T_3 : C \to X$ given mappings. Then for an arbitrary $x_1 \in C$, the following iteration scheme is studied:

$$\begin{aligned} z_n &= P(a_n T_1(PT_1)^{n-1}x_n + (1-a_n-\gamma_n)x_n + \gamma_n u_n), \\ y_n &= P(b_n T_2(PT_2)^{n-1}z_n + c_n T_1(PT_1)^{n-1}x_n + (1-b_n-c_n-\mu_n)x_n + \mu_n v_n), \\ x_{n+1} &= P(\alpha_n T_3(PT_3)^{n-1}y_n + \beta_n T_2(PT_2)^{n-1}z_n + (1-\alpha_n-\beta_n-\lambda_n)x_n + \lambda_n w_n), \\ & \quad n \geq 1, \end{aligned} \tag{1.6}$$

where $\{a_n\},\{b_n\},\{c_n\},\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\mu_n\},\{\lambda_n\}$ are appropriate sequences in $[0,1]$ and $\{u_n\},\{v_n\},\{w_n\}$ are bounded sequences in C.

It reduces to the Khan and Hussain iterative process (1.5) for $T_1 = T_2 = T_3 \equiv T : C \to X$ and $\gamma_n = \mu_n = \lambda_n \equiv 0$.

If $T_1 = T_2 = T_3 \equiv T : C \to C$, then the iterative schemes (1.6) reduces to the modified Noor iterations with errors defined by Nammanee, Noor and Suantai [12]

$$\begin{aligned} z_n &= a_n T^n x_n + (1-a_n-\gamma_n)x_n + \gamma_n u_n, \\ y_n &= b_n T^n z_n + c_n T^n x_n + (1-b_n-c_n-\mu_n)x_n + \mu_n v_n, \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n z_n + (1-\alpha_n-\beta_n-\lambda_n)x_n + \lambda_n w_n, \quad n \geq 1, \end{aligned} \tag{1.7}$$

where $\{a_n\},\{b_n\},\{c_n\},\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\mu_n\},\{\lambda_n\}$ are appropriate sequences in $[0,1]$ and $\{u_n\},\{v_n\},\{w_n\}$ are bounded sequences in C.

If $T_1 = T_2 = T_3 \equiv T : C \to C$ and $\gamma_n = \mu_n = \lambda_n \equiv 0$, then (1.6) reduces to the modified Noor iterations defined by Suantai [20]

$$\begin{aligned} z_n &= a_n T^n x_n + (1-a_n)x_n, \\ y_n &= b_n T^n z_n + c_n T^n x_n + (1-b_n-c_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n z_n + (1-\alpha_n-\beta_n)x_n, \quad n \geq 1, \end{aligned} \tag{1.8}$$

where $\{a_n\},\{b_n\},\{c_n\},\{\alpha_n\},\{\beta_n\}$ are appropriate sequences in $[0,1]$.

If $T_1 = T_2 = T_3 \equiv T : C \to C$ and $c_n = \beta_n \equiv 0$, then (1.6) reduces to the three-step iterations with errors defined by Cho, Zhou and Guo [2]

$$\begin{aligned} z_n &= a_n T^n x_n + (1-a_n-\gamma_n)x_n + \gamma_n u_n, \\ y_n &= b_n T^n z_n + (1-b_n-\mu_n)x_n + \mu_n v_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1-\alpha_n-\lambda_n)x_n + \lambda_n w_n, \quad n \geq 1, \end{aligned} \tag{1.9}$$

where $\{a_n\},\{b_n\},\{\alpha_n\},\{\gamma_n\},\{\mu_n\},\{\lambda_n\}$ are appropriate sequences in $[0,1]$ and $\{u_n\},\{v_n\},\{w_n\}$ are bounded sequences in C.

If $T_1 = T_2 = T_3 \equiv T : C \to C$ and $c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, then (1.6) reduces to the Noor iterations defined by Xu and Noor [23]

$$\begin{aligned} z_n &= a_n T^n x_n + (1-a_n)x_n, \\ y_n &= b_n T^n z_n + (1-b_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1-\alpha_n)x_n, \quad n \geq 1, \end{aligned} \tag{1.10}$$

where $\{a_n\}, \{b_n\}, \{\alpha_n\}$ are appropriate sequences in $[0,1]$.

We note that the usual Ishikawa and Mann iterations are special cases of (1.6). The convexity of C then ensures that the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by (1.7)–(1.10) are well defined. If, however, C is a proper subset of the real Banach space X and T maps C into X (as is the case in many applications), then the sequences given by (1.7)–(1.10) may not be well defined. Clearly, we can obtain the corresponding nonself versions of (1.7)–(1.10) . We shall obtain the strong and weak convergence theorems using (1.5)–(1.10) for three asymptotically nonexpansive nonself-mappings in a uniformly convex Banach space. Our results will thus improve and generalize corresponding results of Khan and Hussain [10], Nammanee, Noor and Suantai [12], Suantai [20], Cho, Zhou and Guo [2], Xu and Noor [23] and many others.

2. Preliminaries

Let X be a Banach space with dimension $X \geq 2$. The modulus of X is the function $\delta_X : (0,2] \to [0,1]$ defined by

$$\delta_X(\epsilon) = \inf\{1 - \|\frac{1}{2}(x+y)\| : \|x\| = 1, \|y\| = 1, \epsilon = \|x-y\|\}.$$

Banach space X is uniformly convex if and only if $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0,2]$.

A subset C of X is said to be retract if there exists continuous mapping $P : X \to C$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : X \to X$ is said to be a retraction if $P^2 = P$. It follows that if a mapping P is a retraction, then $Pz = z$ for every $z \in R(P)$, the range of P. A set C is optimal if each point outside C can be moved to be closer to all points of C. It is well known (see [5]) that

(1) If X is a separable, strictly convex, smooth, reflexive Banach space, and if $C \subset$X is an optimal set with interior, then C is a nonexpansive retract of X.

(2) A subset of l^p, with $1 < p < \infty$, is a nonexpansive retract if and only if it is optimal.

Note that every nonexpansive retract is optimal. In strictly convex Banach spaces, optimal sets are closed and convex. Moreover, every closed convex subset of a Hilbert space is optimal and also a nonexpansive retract.

Recall that a Banach space X is said to satisfy *Opial's condition* [14] if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ implying that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

A mapping $T : C \to X$ is said to be demicompact if, for any sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $x^* \in C$.

In the sequel, the following lemmas are needed to prove our main results.

Lemma 2.1 ([22], Lemma 1). *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, \ \ \forall n = 1, 2, \ldots.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

(1) $\lim_{n\to\infty} a_n$ exists .

(2) $\lim_{n\to\infty} a_n = 0$ whenever $\liminf_{n\to\infty} a_n = 0$.

Lemma 2.2 ([13], Lemma 4). *Let X be a uniformly convex Banach space and $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0,1) \to [0,1)$ with $g(0) = 0$ such that*

$$\begin{aligned}\|\lambda x + \mu y + \xi z + \vartheta w\|^2 \ \ \leq \ \ & \lambda\|x\|^2 + \mu\|y\|^2 + \xi\|z\|^2 + \vartheta\|w\|^2 \\ & -\frac{1}{3}\vartheta(\lambda g(\|x-w\|) + \mu g(\|y-w\|) + \xi g(\|z-w\|)),\end{aligned}$$

for all $x, y, z, w \in B_r = \{x \in X : \|x\| \leq r\}$ and $\lambda, \mu, \xi, \vartheta \in [0,1]$ with $\lambda + \mu + \xi + \vartheta = 1$.

Lemma 2.3 ([1], Theorem 3.4). *Let X be a uniformly convex Banach space and C a nonempty closed convex subset of X. Let $T : C \to X$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$. Then $I - T$ is demiclosed at zero, i.e., if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$, where $F(T)$ is the set of fixed points of T.*

Lemma 2.4 ([20], Lemma 2.7). *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X . Let $u, v \in X$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

3. Main results

In this section, we prove strong and weak convergence theorems for the three-step iterative scheme with errors given in (1.6) to a common fixed point for three asymptotically nonexpansive nonself-mappings in a uniformly convex Banach space. In order to prove our main results, the following lemmas are needed.

Lemma 3.1. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to X$ be three asymptotically nonexpansive nonself-mappings of C with sequences $\{k_n\}, \{l_n\}, \{m_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(m_n - 1) < \infty$, $k_n \to 1$, $l_n \to 1$, $m_n \to 1$ as $n \to \infty$, respectively and $F := F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$ be real sequences in $[0, 1]$ such that $a_n + \gamma_n$, $b_n + c_n + \mu_n$ and $\alpha_n + \beta_n + \lambda_n$ are in $[0, 1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$, and let $\{u_n\}, \{v_n\}$ and $\{w_n\}$ be the bounded sequences in C. From an arbitrary $x_1 \in C$, define the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ using (1.6).*

(i) If q is a fixed point of T_1, T_2 and T_3 , then $\lim_{n\to\infty} \|x_n - q\|$ exists.

(ii) If $\liminf_{n\to\infty} \alpha_n > 0$, $\liminf_{n\to\infty} b_n > 0$ and $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} (a_n + \gamma_n) < 1$, then $\lim_{n\to\infty} \|T_1(PT_1)^{n-1}x_n - x_n\| = 0$.

(iii) If $\liminf_{n\to\infty} \alpha_n > 0$ and $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, then $\lim_{n\to\infty} \|T_1(PT_1)^{n-1}x_n - x_n\| = 0$.

(iv) If $\liminf_{n\to\infty} \beta_n > 0$ and $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n + \gamma_n) < 1$, then $\lim_{n\to\infty} \|T_1(PT_1)^{n-1}x_n - x_n\| = 0$.

(v) If $\liminf_{n\to\infty} \alpha_n > 0$ and $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, then $\lim_{n\to\infty} \|T_2(PT_2)^{n-1}z_n - x_n\| = 0$.

(vi) If $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, then $\lim_{n\to\infty} \|T_2(PT_2)^{n-1}z_n - x_n\| = 0$.

(vii) If $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, then $\lim_{n\to\infty} \|T_3(PT_3)^{n-1}y_n - x_n\| = 0$.

Proof. Let $q \in F$, by boundedness of the sequences $\{u_n\}, \{v_n\}$ and $\{w_n\}$, we can put

$$M = \max\{\sup_{n\geq 1} \|u_n - q\|, \sup_{n\geq 1} \|v_n - q\|, \sup_{n\geq 1} \|w_n - q\|\}.$$

(i) Using (1.6), we have

$$\begin{aligned}
\|x_{n+1} - q\| &= \|P(\alpha_n T_3(PT_3)^{n-1}y_n + \beta_n T_2(PT_2)^{n-1}z_n \\
&\quad + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_n w_n) - P(q)\| \\
&\leq \|\alpha_n(T_3(PT_3)^{n-1}y_n - q) + \beta_n(T_2(PT_2)^{n-1}z_n - q) \\
&\quad + (1 - \alpha_n - \beta_n - \lambda_n)(x_n - q) + \lambda_n(w_n - q)\| \\
&\leq \alpha_n\|T_3(PT_3)^{n-1}y_n - q\| + \beta_n\|T_2(PT_2)^{n-1}z_n - q\| \\
&\quad + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\| + \lambda_n\|w_n - q\| \\
&\leq \alpha_n m_n\|y_n - q\| + \beta_n l_n\|z_n - q\| + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\| + M\lambda_n \qquad (3.1)
\end{aligned}$$

and

$$\begin{aligned}\|z_n - q\| &= \|P(a_nT_1(PT_1)^{n-1}x_n + (1 - a_n - \gamma_n)x_n + \gamma_n u_n) - P(q)\|\\ &\leq \|a_n(T_1(PT_1)^{n-1}x_n - q) + (1 - a_n - \gamma_n)(x_n - q) + \gamma_n(u_n - q)\|\\ &\leq a_n\|T_1(PT_1)^{n-1}x_n - q\| + (1 - a_n - \gamma_n)\|x_n - q\| + \gamma_n\|u_n - q\|\\ &\leq a_nk_n\|x_n - q\| + (1 - a_n - \gamma_n)\|x_n - q\| + M\gamma_n\\ &\leq (a_nk_n + (1 - a_n))\|x_n - q\| + M\gamma_n\\ &= (a_n(k_n - 1) + 1)\|x_n - q\| + M\gamma_n\\ &\leq k_n\|x_n - q\| + M\gamma_n. \end{aligned} \tag{3.2}$$

From (3.2), we have

$$\begin{aligned}\|y_n - q\| &= \|P(b_nT_2(PT_2)^{n-1}z_n + c_nT_1(PT_1)^{n-1}x_n\\ &\quad + (1 - b_n - c_n - \mu_n)x_n + \mu_nv_n) - P(q)\|\\ &\leq \|b_n(T_2(PT_2)^{n-1}z_n - q) + c_n(T_1(PT_1)^{n-1}x_n - q)\\ &\quad + (1 - b_n - c_n - \mu_n)(x_n - q) + \mu_n(v_n - q)\|\\ &\leq b_n\|T_2(PT_2)^{n-1}z_n - q\| + c_n\|T_1(PT_1)^{n-1}x_n - q\|\\ &\quad + (1 - b_n - c_n - \mu_n)\|x_n - q\| + \mu_n\|v_n - q\|\\ &\leq b_nl_n\|z_n - q\| + c_nk_n\|x_n - q\| + (1 - b_n - c_n - \mu_n)\|x_n - q\| + M\mu_n\\ &\leq b_nl_n(k_n\|x_n - q\| + M\gamma_n) + c_nk_n\|x_n - q\|\\ &\quad + (1 - b_n - c_n - \mu_n)\|x_n - q\| + M\mu_n\\ &\leq b_nl_nk_n\|x_n - q\| + Mb_nl_n\gamma_n + c_nk_n\|x_n - q\|\\ &\quad + (1 - b_n - c_n)\|x_n - q\| + M\mu_n\\ &= (b_nl_nk_n + c_nk_n + (1 - b_n - c_n))\|x_n - q\| + \epsilon^n_{(1)}\\ &= (1 + (c_n + b_n)(k_n - 1) + b_nk_n(l_n - 1))\|x_n - q\| + \epsilon^n_{(1)}, \end{aligned} \tag{3.3}$$

where $\epsilon^n_{(1)} = Mb_nl_n\gamma_n + M\mu_n$. Since $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\mu_n < \infty$ and $\{l_n\}$ is bounded, we have $\sum_{n=1}^{\infty}\epsilon^n_{(1)} < \infty$.

By using (3.1), (3.2) and (3.3), we have

$$\begin{aligned}\|x_{n+1} - q\| &\leq \alpha_nm_n((1 + (c_n + b_n)(k_n - 1) + b_nk_n(l_n - 1))\|x_n - q\| + \epsilon^n_{(1)})\\ &\quad + \beta_nl_n(k_n\|x_n - q\| + M\gamma_n) + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\| + M\lambda_n\\ &= \alpha_nm_n(1 + (c_n + b_n)(k_n - 1) + b_nk_n(l_n - 1))\|x_n - q\| + \alpha_nm_n\epsilon^n_{(1)}\\ &\quad + \beta_nl_nk_n\|x_n - q\| + M\beta_nl_n\gamma_n + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\| + M\lambda_n\\ &\leq \alpha_nm_n(1 + (c_n + b_n)(k_n - 1) + b_nk_n(l_n - 1))\|x_n - q\|\\ &\quad + \beta_nl_nk_n\|x_n - q\| + (1 - \alpha_n - \beta_n)\|x_n - q\| + \epsilon^n_{(2)}\\ &= (\alpha_nm_n + \alpha_nm_n(c_n + b_n)(k_n - 1) + \alpha_nm_nb_nk_n(l_n - 1) + \beta_nl_nk_n\\ &\quad + 1 - \alpha_n - \beta_n)\|x_n - q\| + \epsilon^n_{(2)}\\ &= (\alpha_n(m_n - 1) + \alpha_nm_n(c_n + b_n)(k_n - 1) + \alpha_nm_nb_nk_n(l_n - 1)\\ &\quad + \beta_n(k_n - 1) + \beta_nk_n(l_n - 1) + 1)\|x_n - q\| + \epsilon^n_{(2)}\\ &\leq (1 + (m_n - 1) + (m_n + 1)(k_n - 1)\\ &\quad + (m_nk_n + k_n)(l_n - 1))\|x_n - q\| + \epsilon^n_{(2)}, \end{aligned} \tag{3.4}$$

where $\epsilon^n_{(2)} = \alpha_nm_n\epsilon^n_{(1)} + M\beta_nl_n\gamma_n + M\lambda_n$ and we note here that $\sum_{n=1}^{\infty}\epsilon^n_{(2)} < \infty$ since $\sum_{n=1}^{\infty}\epsilon^n_{(1)} < \infty$, $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$ and $\{m_n\}$, $\{l_n\}$ are bounded. Since $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$,

$\sum_{n=1}^{\infty}(m_n - 1) < \infty$ and $\sum_{n=1}^{\infty} \epsilon_{(2)}^n < \infty$ we obtained from (3.4) and Lemma 2.1(i) that $\lim_{n\to\infty} \|x_n - q\|$ exists.

(ii) First, we assume that $\liminf_{n\to\infty} \alpha_n > 0$, $\liminf_{n\to\infty} b_n > 0$ and $0 < \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} (a_n+\gamma_n) < 1$. By (i), we have $\lim_{n\to\infty} \|x_n-q\|$ exists for any $q \in F$. It follows that $\{x_n-q\}, \{T_1(PT_1)^{n-1}x_n-q\}, \{z_n-q\}, \{T_2(PT_2)^{n-1}z_n-q\}, \{y_n-q\}$ and $\{T_3(PT_3)^{n-1}y_n-q\}$ are bounded sequences. We may assume that such sequences belong to B_r where $r > 0$. By using Lemma 2.2, we have

$$\begin{aligned}
\|z_n - q\|^2 &= \|P(a_nT_1(PT_1)^{n-1}x_n + (1 - a_n - \gamma_n)x_n + \gamma_n u_n) - P(q)\|^2 \\
&\le \|a_n(T_1(PT_1)^{n-1}x_n - q) + (1 - a_n - \gamma_n)(x_n - q) + \gamma_n(u_n - q)\|^2 \\
&\le a_n\|T_1(PT_1)^{n-1}x_n - q\|^2 + (1 - a_n - \gamma_n)\|x_n - q\|^2 + \gamma_n\|u_n - q\|^2 \\
&\quad - \frac{1}{3}(1 - a_n - \gamma_n)(a_ng(\|T_1(PT_1)^{n-1}x_n - x_n\|) + \gamma_ng(\|u_n - x_n\|)) \\
&\le a_nk_n^2\|x_n - q\|^2 + (1 - a_n - \gamma_n)\|x_n - q\|^2 + \gamma_nM^2 \\
&\quad - \frac{1}{3}a_n(1 - a_n - \gamma_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
&\le a_nk_n^2\|x_n - q\|^2 + (1 - a_n)\|x_n - q\|^2 + \gamma_nM^2 \\
&\quad - \frac{1}{3}a_n(1 - a_n - \gamma_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
&= (1 + a_n(k_n^2 - 1))\|x_n - q\|^2 + \gamma_nM^2 \\
&\quad - \frac{1}{3}a_n(1 - a_n - \gamma_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|)
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
\|y_n - q\|^2 &= \|P(b_nT_2(PT_2)^{n-1}z_n + c_nT_1(PT_1)^{n-1}x_n \\
&\quad + (1 - b_n - c_n - \mu_n)x_n + \mu_nv_n) - P(q)\|^2 \\
&\le \|b_n(T_2(PT_2)^{n-1}z_n - q) + c_n(T_1(PT_1)^{n-1}x_n - q) \\
&\quad + (1 - b_n - c_n - \mu_n)(x_n - q) + \mu_n(v_n - q)\|^2 \\
&\le b_n\|T_2(PT_2)^{n-1}z_n - q\|^2 + c_n\|T_1(PT_1)^{n-1}x_n - q\|^2 \\
&\quad + (1 - b_n - c_n - \mu_n)\|x_n - q\|^2 + \mu_n\|v_n - q\|^2 \\
&\quad - \frac{1}{3}(1 - b_n - c_n - \mu_n)(b_ng(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
&\quad + c_ng(\|T_1(PT_1)^{n-1}x_n - x_n\|) + \mu_ng(\|v_n - x_n\|)) \\
&\le b_nl_n^2\|z_n - q\|^2 + c_nk_n^2\|x_n - q\|^2 + (1 - b_n - c_n)\|x_n - q\|^2 + \mu_nM^2 \\
&\quad - \frac{1}{3}b_n(1 - b_n - c_n - \mu_n)g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
&\quad - \frac{1}{3}c_n(1 - b_n - c_n - \mu_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
&\quad - \frac{1}{3}\mu_n(1 - b_n - c_n - \mu_n)g(\|v_n - x_n\|)) \\
&\le b_nl_n^2\|z_n - q\|^2 + c_nk_n^2\|x_n - q\|^2 + (1 - b_n - c_n)\|x_n - q\|^2 + \mu_nM^2 \\
&\quad - \frac{1}{3}b_n(1 - b_n - c_n - \mu_n)g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
&\quad - \frac{1}{3}c_n(1 - b_n - c_n - \mu_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|).
\end{aligned} \tag{3.6}$$

By (3.5) and (3.6), we also have

$$
\begin{aligned}
\|x_{n+1}-q\|^2 &= \|P(\alpha_n T_3(PT_3)^{n-1}y_n + \beta_n T_2(PT_2)^{n-1}z_n \\
&\quad + (1-\alpha_n-\beta_n-\lambda_n)x_n + \lambda_n w_n) - P(q)\|^2 \\
&\le \|\alpha_n(T_3(PT_3)^{n-1}y_n - q) + \beta_n(T_2(PT_2)^{n-1}z_n - q) \\
&\quad + (1-\alpha_n-\beta_n-\lambda_n)(x_n-q) + \lambda_n(w_n-q)\|^2 \\
&\le \alpha_n\|T_3(PT_3)^{n-1}y_n - q\|^2 + \beta_n\|T_2(PT_2)^{n-1}z_n - q\|^2 \\
&\quad + (1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|^2 + \lambda_n\|w_n-q\|^2 \\
&\quad - \frac{1}{3}(1-\alpha_n-\beta_n-\lambda_n)(\alpha_n g(\|T_3(PT_3)^{n-1}y_n - x_n\|) \\
&\quad + \beta_n g(\|T_2(PT_2)^{n-1}z_n - x_n\|) + \lambda_n g(\|w_n-x_n\|)) \\
&\le \alpha_n m_n^2\|y_n-q\|^2 + \beta_n l_n^2\|z_n-q\|^2 \\
&\quad + (1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|^2 + \lambda_n M^2 \\
&\quad - \frac{1}{3}\alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(\|T_3(PT_3)^{n-1}y_n - x_n\|) \\
&\quad - \frac{1}{3}\beta_n(1-\alpha_n-\beta_n-\lambda_n)g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
&\quad - \frac{1}{3}\lambda_n(1-\alpha_n-\beta_n-\lambda_n)g(\|w_n-x_n\|) \\
&\le \alpha_n m_n^2(b_n l_n^2\|z_n-q\|^2 + c_n k_n^2\|x_n-q\|^2 + (1-b_n-c_n)\|x_n-q\|^2 \\
&\quad + \mu_n M^2 - \frac{1}{3}b_n(1-b_n-c_n-\mu_n)g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
&\quad - \frac{1}{3}c_n(1-b_n-c_n-\mu_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|)) + \beta_n l_n^2\|z_n-q\|^2 \\
&\quad + (1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|^2 + \lambda_n M^2 \\
&\quad - \frac{1}{3}\alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(\|T_3(PT_3)^{n-1}y_n - x_n\|) \\
&\quad - \frac{1}{3}\beta_n(1-\alpha_n-\beta_n-\lambda_n)g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
&\le \alpha_n m_n^2 b_n l_n^2\|z_n-q\|^2 + \alpha_n m_n^2 c_n k_n^2\|x_n-q\|^2 + \alpha_n m_n^2(1-b_n-c_n)\|x_n-q\|^2 \\
&\quad + \alpha_n m_n^2\mu_n M^2 - \frac{1}{3}\alpha_n b_n m_n^2(1-b_n-c_n-\mu_n)g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
&\quad - \frac{1}{3}\alpha_n c_n m_n^2(1-b_n-c_n-\mu_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|) + \beta_n l_n^2\|z_n-q\|^2 \\
&\quad + (1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|^2 + \lambda_n M^2 \\
&\quad - \frac{1}{3}\alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(\|T_3(PT_3)^{n-1}y_n - x_n\|) \\
&\quad - \frac{1}{3}\beta_n(1-\alpha_n-\beta_n-\lambda_n)g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
&= (\alpha_n b_n m_n^2 l_n^2 + \beta_n l_n^2)\|z_n-q\|^2 + \alpha_n m_n^2 c_n k_n^2\|x_n-q\|^2 \\
&\quad + \alpha_n m_n^2(1-b_n-c_n)\|x_n-q\|^2 + \alpha_n m_n^2\mu_n M^2 \\
&\quad + (1-\alpha_n-\beta_n-\lambda_n)\|x_n-q\|^2 + \lambda_n M^2 \\
&\quad - \frac{1}{3}\alpha_n b_n m_n^2(1-b_n-c_n-\mu_n)g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
&\quad - \frac{1}{3}\alpha_n c_n m_n^2(1-b_n-c_n-\mu_n)g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
&\quad - \frac{1}{3}\alpha_n(1-\alpha_n-\beta_n-\lambda_n)g(\|T_3(PT_3)^{n-1}y_n - x_n\|) \\
&\quad - \frac{1}{3}\beta_n(1-\alpha_n-\beta_n-\lambda_n)g(\|T_2(PT_2)^{n-1}z_n - x_n\|)
\end{aligned}
$$

$$
\begin{aligned}
&\leq (\alpha_n b_n m_n^2 l_n^2 + \beta_n l_n^2)((1 + a_n(k_n^2 - 1))\|x_n - q\|^2 + \gamma_n M^2 \\
&\quad - \frac{1}{3} a_n(1 - a_n - \gamma_n) g(\|T_1(PT_1)^{n-1}x_n - x_n\|)) + \alpha_n m_n^2 c_n k_n^2 \|x_n - q\|^2 \\
&\quad + \alpha_n m_n^2 (1 - b_n - c_n)\|x_n - q\|^2 + \alpha_n m_n^2 \mu_n M^2 \\
&\quad + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\|^2 + \lambda_n M^2 \\
&\quad - \frac{1}{3}\alpha_n b_n m_n^2 (1 - b_n - c_n - \mu_n) g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
&\quad - \frac{1}{3}\alpha_n c_n m_n^2 (1 - b_n - c_n - \mu_n) g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
&\quad - \frac{1}{3}\alpha_n (1 - \alpha_n - \beta_n - \lambda_n) g(\|T_3(PT_3)^{n-1}y_n - x_n\|) \\
&\quad - \frac{1}{3}\beta_n (1 - \alpha_n - \beta_n - \lambda_n) g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
&\leq (\alpha_n b_n m_n^2 l_n^2 + \beta_n l_n^2 + (\alpha_n b_n m_n^2 l_n^2 + \beta_n l_n^2) a_n(k_n^2 - 1) + \alpha_n m_n^2 c_n k_n^2 \\
&\quad + \alpha_n m_n^2 (1 - b_n - c_n) + 1 - \alpha_n - \beta_n)\|x_n - q\|^2 \\
&\quad + (\alpha_n b_n m_n^2 l_n^2 + \beta_n l_n^2)\gamma_n M^2 + \alpha_n m_n^2 \mu_n M^2 + \lambda_n M^2 \\
&\quad - \frac{1}{3}(\alpha_n b_n m_n^2 l_n^2 + \beta_n l_n^2) a_n (1 - a_n - \gamma_n) g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
&\quad - \frac{1}{3}\alpha_n b_n m_n^2 (1 - b_n - c_n - \mu_n) g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
&\quad - \frac{1}{3}\alpha_n c_n m_n^2 (1 - b_n - c_n - \mu_n) g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
&\quad - \frac{1}{3}\alpha_n (1 - \alpha_n - \beta_n - \lambda_n) g(\|T_3(PT_3)^{n-1}y_n - x_n\|) \\
&\quad - \frac{1}{3}\beta_n (1 - \alpha_n - \beta_n - \lambda_n) g(\|T_2(PT_2)^{n-1}z_n - x_n\|).
\end{aligned} \tag{3.7}
$$

We note that

$$
\begin{aligned}
&\alpha_n b_n m_n^2 l_n^2 + \beta_n l_n^2 + (\alpha_n b_n m_n^2 l_n^2 + \beta_n l_n^2) a_n (k_n^2 - 1) + \alpha_n m_n^2 c_n k_n^2 + \alpha_n m_n^2 (1 - b_n - c_n) + 1 - \alpha_n - \beta_n \\
&\quad = (\alpha_n b_n m_n^2 l_n^2 - \alpha_n b_n m_n^2) + (\beta_n l_n^2 - \beta_n) + (\alpha_n c_n m_n^2 k_n^2 - \alpha_n c_n m_n^2) + (\alpha_n m_n^2 - \alpha_n) \\
&\qquad + 1 + (\alpha_n b_n m_n^2 l_n^2 + \beta_n l_n^2) a_n (k_n^2 - 1) \\
&\quad = \alpha_n b_n m_n^2 (l_n^2 - 1) + \beta_n (l_n^2 - 1) + \alpha_n c_n m_n^2 (k_n^2 - 1) + \alpha_n (m_n^2 - 1) \\
&\qquad + (\alpha_n b_n m_n^2 l_n^2 + \beta_n l_n^2) a_n (k_n^2 - 1) + 1 \\
&\quad = (\alpha_n b_n m_n^2 + \beta_n)(l_n^2 - 1) + \alpha_n (m_n^2 - 1) \\
&\qquad + (\alpha_n c_n m_n^2 + a_n(\alpha_n b_n m_n^2 l_n^2 + \beta_n l_n^2))(k_n^2 - 1) + 1.
\end{aligned} \tag{3.8}
$$

Since $\{m_n\}$ and $\{l_n\}$ are bounded, there exists a constant $K > 0$ such that

$$
\begin{aligned}
&(\alpha_n b_n m_n^2 + \beta_n)(l_n^2 - 1) + \alpha_n (m_n^2 - 1) + (\alpha_n c_n m_n^2 + a_n(\alpha_n b_n m_n^2 l_n^2 + \beta_n l_n^2))(k_n^2 - 1)\|x_n - q\|^2 \\
&\qquad \leq K((k_n^2 - 1) + (l_n^2 - 1) + (m_n^2 - 1))
\end{aligned} \tag{3.9}
$$

for all $n \geq 1$. By using (3.7), (3.8) and (3.9), we have

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &\leq ((\alpha_n b_n m_n^2 + \beta_n)(l_n^2 - 1) + \alpha_n (m_n^2 - 1) + (\alpha_n c_n m_n^2 \\
&\quad + a_n(\alpha_n b_n m_n^2 l_n^2 + \beta_n l_n^2))(k_n^2 - 1) + 1)\|x_n - q\|^2 \\
&\quad + \alpha_n b_n m_n^2 l_n^2 \gamma_n M^2 + \beta_n l_n^2 \gamma_n M^2 + \alpha_n m_n^2 \mu_n M^2 + \lambda_n M^2
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{3}(\alpha_n b_n m_n^2 l_n^2 + \beta_n l_n^2) a_n (1 - a_n - \gamma_n) g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
&- \frac{1}{3}\alpha_n b_n m_n^2 (1 - b_n - c_n - \mu_n) g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
&- \frac{1}{3}\alpha_n c_n m_n^2 (1 - b_n - c_n - \mu_n) g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
&- \frac{1}{3}\alpha_n (1 - \alpha_n - \beta_n - \lambda_n) g(\|T_3(PT_3)^{n-1}y_n - x_n\|) \\
&- \frac{1}{3}\beta_n (1 - \alpha_n - \beta_n - \lambda_n) g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
\le\; & \|x_n - q\|^2 + K((k_n^2 - 1) + (l_n^2 - 1) + (m_n^2 - 1)) \\
&+ (\alpha_n b_n m_n^2 l_n^2 \gamma_n + \beta_n l_n^2 \gamma_n + \alpha_n m_n^2 \mu_n + \lambda_n) M^2 \\
&- \frac{1}{3}\alpha_n b_n a_n (1 - a_n - \gamma_n) g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
&- \frac{1}{3}\beta_n a_n (1 - a_n - \gamma_n) g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
&- \frac{1}{3}\alpha_n b_n (1 - b_n - c_n - \mu_n) g(\|T_2(PT_2)^{n-1}z_n - x_n\|) \\
&- \frac{1}{3}\alpha_n c_n (1 - b_n - c_n - \mu_n) g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
&- \frac{1}{3}\alpha_n (1 - \alpha_n - \beta_n - \lambda_n) g(\|T_3(PT_3)^{n-1}y_n - x_n\|) \\
&- \frac{1}{3}\beta_n (1 - \alpha_n - \beta_n - \lambda_n) g(\|T_2(PT_2)^{n-1}z_n - x_n\|). \qquad (3.10)
\end{aligned}
$$

From (3.10), we obtain the following six important inequalities:

$$
\begin{aligned}
\frac{1}{3}\alpha_n b_n a_n (1 - a_n - \gamma_n) g(\|T_1(PT_1)^{n-1}x_n - x_n\|) &\le \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\
+K((k_n^2 - 1) + (l_n^2 - 1) + (m_n^2 - 1)) &+ (\alpha_n b_n m_n^2 l_n^2 \gamma_n + \beta_n l_n^2 \gamma_n + \alpha_n m_n^2 \mu_n + \lambda_n) M^2, \qquad (3.11)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{3}\alpha_n c_n (1 - b_n - c_n - \mu_n) g(\|T_1(PT_1)^{n-1}x_n - x_n\|) &\le \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\
+K((k_n^2 - 1) + (l_n^2 - 1) + (m_n^2 - 1)) &+ (\alpha_n b_n m_n^2 l_n^2 \gamma_n + \beta_n l_n^2 \gamma_n + \alpha_n m_n^2 \mu_n + \lambda_n) M^2, \qquad (3.12)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{3}\beta_n a_n (1 - a_n - \gamma_n) g(\|T_1(PT_1)^{n-1}x_n - x_n\|) &\le \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\
+K((k_n^2 - 1) + (l_n^2 - 1) + (m_n^2 - 1)) &+ (\alpha_n b_n m_n^2 l_n^2 \gamma_n + \beta_n l_n^2 \gamma_n + \alpha_n m_n^2 \mu_n + \lambda_n) M^2, \qquad (3.13)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{3}\alpha_n b_n (1 - b_n - c_n - \mu_n) g(\|T_2(PT_2)^{n-1}z_n - x_n\|) &\le \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\
+K((k_n^2 - 1) + (l_n^2 - 1) + (m_n^2 - 1)) &+ (\alpha_n b_n m_n^2 l_n^2 \gamma_n + \beta_n l_n^2 \gamma_n + \alpha_n m_n^2 \mu_n + \lambda_n) M^2, \qquad (3.14)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{3}\beta_n (1 - \alpha_n - \beta_n - \lambda_n) g(\|T_2(PT_2)^{n-1}z_n - x_n\|) &\le \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\
+K((k_n^2 - 1) + (l_n^2 - 1) + (m_n^2 - 1)) &+ (\alpha_n b_n m_n^2 l_n^2 \gamma_n + \beta_n l_n^2 \gamma_n + \alpha_n m_n^2 \mu_n + \lambda_n) M^2 \qquad (3.15)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{3}\alpha_n (1 - \alpha_n - \beta_n - \lambda_n) g(\|T_3(PT_3)^{n-1}y_n - x_n\|) &\le \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\
+K((k_n^2 - 1) + (l_n^2 - 1) + (m_n^2 - 1)) &+ (\alpha_n b_n m_n^2 l_n^2 \gamma_n + \beta_n l_n^2 \gamma_n + \alpha_n m_n^2 \mu_n + \lambda_n) M^2. \qquad (3.16)
\end{aligned}
$$

By our assumption $\liminf_{n\to\infty}\alpha_n>0$, $\liminf_{n\to\infty}b_n>0$ and $0<\liminf_{n\to\infty}a_n\le\limsup_{n\to\infty}(a_n+\gamma_n)<1$, there exists $n_0\in\mathbb{N}$ and $\delta_1,\delta_2,\delta_3,\delta_4\in(0,1)$ such that $0<\delta_1<\alpha_n$, $0<\delta_2<b_n$, $0<\delta_3<a_n$ and $\alpha_n+\gamma_n<\delta_4<1$ for all $n\ge n_0$. Hence, by (3.11), we have

$$\begin{aligned}\frac{1}{3}\delta_1\delta_2\delta_3(1-\delta_4)\sum_{n=n_0}^{m}g(\|T_1(PT_1)^{n-1}x_n-x_n\|&\le\sum_{n=n_0}^{m}(\|x_n-q\|^2-\|x_{n+1}-q\|^2)\\&\quad+K\sum_{n=n_0}^{m}((k_n^2-1)+(l_n^2-1)+(m_n^2-1))\\&\quad+M^2\sum_{n=n_0}^{m}(\alpha_nb_nm_n^2l_n^2\gamma_n+\beta_nl_n^2\gamma_n+\alpha_nm_n^2\mu_n+\lambda_n)\\&=\|x_{n_0}-q\|^2+K\sum_{n=n_0}^{m}((k_n^2-1)+(l_n^2-1)+(m_n^2-1))\\&\quad+M^2\sum_{n=n_0}^{m}(\alpha_nb_nm_n^2l_n^2\gamma_n+\beta_nl_n^2\gamma_n+\alpha_nm_n^2\mu_n+\lambda_n).\end{aligned}\tag{3.17}$$

Since $\sum_{n=1}^{\infty}\gamma_n<\infty$, $\sum_{n=1}^{\infty}\mu_n<\infty$, $\sum_{n=1}^{\infty}\lambda_n<\infty$, $\sum_{n=1}^{\infty}(k_n^2-1)<\infty$, $\sum_{n=1}^{\infty}(l_n^2-1)<\infty$, $\sum_{n=1}^{\infty}(m_n^2-1)<\infty$ and $\{m_n\}$, $\{l_n\}$ are bounded sequences, we have $K\sum_{n=n_0}^{m}((k_n^2-1)+(l_n^2-1)+(m_n^2-1))<\infty$ and $M^2\sum_{n=n_0}^{m}(\alpha_nb_nm_n^2\gamma_n+\beta_nl_n^2\gamma_n+\alpha_nm_n^2\mu_n+\lambda_n)<\infty$.

By letting $m\to\infty$ in (3.17) we get $\sum_{n=n_0}^{\infty}g(\|T_1(PT_1)^{n-1}x_n-x_n\|)<\infty$, and therefore

$$\lim_{n\to\infty}g(\|T_1(PT_1)^{n-1}x_n-x_n\|)=0.$$

Since g is strictly increasing and continuous at 0 with $g(0)=0$, it follows that $\lim_{n\to\infty}\|T_1(PT_1)^{n-1}x_n-x_n\|=0$. Thus (ii) is proved. By using a similar method as in (ii), together with inequalities (3.12), (3.13), (3.14), (3.15) and (3.16), one can show that $(iii)-(vii)$ are satisfied, respectively. □

Lemma 3.2. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P a nonexpansive retraction. Let $T_1,T_2,T_3:C\to X$ be three asymptotically nonexpansive nonself-mappings of C with sequences $\{k_n\},\{l_n\},\{m_n\}\subset[1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n-1)<\infty$, $\sum_{n=1}^{\infty}(l_n-1)<\infty$, $\sum_{n=1}^{\infty}(m_n-1)<\infty$, $k_n\to1$, $l_n\to1$, $m_n\to1$ as $n\to\infty$, respectively and $F:=F(T_1)\cap F(T_2)\cap F(T_3)\neq\emptyset$. Let $\{a_n\},\{b_n\},\{c_n\},\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$ be real sequences in $[0,1]$ such that $a_n+\gamma_n$, $b_n+c_n+\mu_n$ and $\alpha_n+\beta_n+\lambda_n$ are in $[0,1]$ for all $n\ge1$, and $\sum_{n=1}^{\infty}\gamma_n<\infty$, $\sum_{n=1}^{\infty}\mu_n<\infty$, $\sum_{n=1}^{\infty}\lambda_n<\infty$, and let $\{u_n\},\{v_n\}$ and $\{w_n\}$ be the bounded sequences in C. From an arbitrary $x_1\in C$, define the sequences $\{x_n\},\{y_n\}$ and $\{z_n\}$ using (1.6). If $\lim_{n\to\infty}\|T_1(PT_1)^{n-1}x_n-x_n\|=0$, $\lim_{n\to\infty}\|T_2(PT_2)^{n-1}z_n-x_n\|=0$ and $\lim_{n\to\infty}\|T_3(PT_3)^{n-1}y_n-x_n\|=0$, then $\lim_{n\to\infty}\|T_1x_n-x_n\|=\lim_{n\to\infty}\|T_2x_n-x_n\|=\lim_{n\to\infty}\|T_3x_n-x_n\|=0$.*

Proof. Suppose that

$$\begin{aligned}&\lim_{n\to\infty}\|T_1(PT_1)^{n-1}x_n-x_n\|\\&=\lim_{n\to\infty}\|T_2(PT_2)^{n-1}z_n-x_n\|\\&=\lim_{n\to\infty}\|T_3(PT_3)^{n-1}y_n-x_n\|=0.\end{aligned}\tag{3.18}$$

Using (1.6), we have

$$\begin{aligned}\|z_n-x_n\|&=\|P(a_nT_1(PT_1)^{n-1}x_n+(1-a_n-\gamma_n)x_n+\gamma_nu_n)-P(x_n)\|\\&\le\|a_n(T_1(PT_1)^{n-1}x_n-x_n)+\gamma_n(u_n-x_n)\|\\&\le a_n\|T_1(PT_1)^{n-1}x_n-x_n\|+\gamma_n\|u_n-x_n\|\\&\le a_n\|T_1(PT_1)^{n-1}x_n-x_n\|+\gamma_n\|u_n\|+\gamma_n\|x_n\|,\end{aligned}\tag{3.19}$$

$$\begin{aligned}\|y_n - x_n\| &= \|P(b_n T_2(PT_2)^{n-1} z_n + c_n T_1(PT_1)^{n-1} x_n \\ &\quad + (1 - b_n - c_n - \mu_n) x_n + \mu_n v_n) - P(x_n)\| \\ &\le \|b_n(T_2(PT_2)^{n-1} z_n - x_n) + c_n(T_1(PT_1)^{n-1} x_n - x_n) + \mu_n(v_n - x_n)\| \\ &\le b_n\|T_2(PT_2)^{n-1} z_n - x_n\| + c_n\|T_1(PT_1)^{n-1} x_n - x_n\| + \mu_n\|v_n - x_n\| \\ &\le b_n\|T_2(PT_2)^{n-1} z_n - x_n\| + c_n\|T_1(PT_1)^{n-1} x_n - x_n\| + \mu_n\|v_n\| + \mu_n\|x_n\| \end{aligned} \tag{3.20}$$

and

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|P(\alpha_n T_3(PT_3)^{n-1} y_n + \beta_n T_2(PT_2)^{n-1} z_n \\ &\quad + (1 - \alpha_n - \beta_n - \lambda_n) x_n + \lambda_n w_n) - P(x_n)\| \\ &\le \|\alpha_n(T_3(PT_3)^{n-1} y_n - x_n) + \beta_n(T_2(PT_2)^{n-1} z_n - x_n) + \lambda_n(w_n - x_n)\| \\ &\le \alpha_n\|T_3(PT_3)^{n-1} y_n - x_n\| + \beta_n\|T_2(PT_2)^{n-1} z_n - x_n\| + \lambda_n\|w_n - x_n\| \\ &\le \alpha_n\|T_3(PT_3)^{n-1} y_n - x_n\| + \beta_n\|T_2(PT_2)^{n-1} z_n - x_n\| + \lambda_n\|w_n\| + \lambda_n\|x_n\|. \end{aligned} \tag{3.21}$$

Since $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$, and $\{u_n\}, \{v_n\}, \{w_n\}$ and $\{x_n\}$ are all bounded. It follows from (3.18), (3.19), (3.20) and (3.21) that

$$\lim_{n\to\infty} \|z_n - x_n\| = \lim_{n\to\infty} \|y_n - x_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.22}$$

Using (3.18) and (3.22), we have

$$\begin{aligned}\|T_2(PT_2)^{n-1} x_n - x_n\| &= \|T_2(PT_2)^{n-1} x_n - T_2(PT_2)^{n-1} z_n + T_2(PT_2)^{n-1} z_n - x_n\| \\ &\le \|T_2(PT_2)^{n-1} z_n - T_2(PT_2)^{n-1} x_n\| + \|T_2(PT_2)^{n-1} z_n - x_n\| \\ &\le l_n\|z_n - x_n\| + \|T_2(PT_2)^{n-1} z_n - x_n\| \to 0, \quad \text{as } n \to \infty, \end{aligned} \tag{3.23}$$

$$\begin{aligned}\|T_3(PT_3)^{n-1} x_n - x_n\| &= \|T_3(PT_3)^{n-1} x_n - T_3(PT_3)^{n-1} y_n + T_3(PT_3)^{n-1} y_n - x_n\| \\ &\le \|T_3(PT_3)^{n-1} y_n - T_3(PT_3)^{n-1} x_n\| + \|T_3(PT_3)^{n-1} y_n - x_n\| \\ &\le m_n\|y_n - x_n\| + \|T_3(PT_3)^{n-1} y_n - x_n\| \to 0, \quad \text{as } n \to \infty \end{aligned} \tag{3.24}$$

and

$$\begin{aligned}\|x_{n+1} - T_1(PT_1)^{n-1} x_{n+1}\| &= \|x_{n+1} - x_n + x_n - T_1(PT_1)^{n-1} x_n \\ &\quad + T_1(PT_1)^{n-1} x_n - T_1(PT_1)^{n-1} x_{n+1}\| \\ &\le \|x_{n+1} - x_n\| + \|T_1(PT_1)^{n-1} x_{n+1} - T_1(PT_1)^{n-1} x_n\| \\ &\quad + \|T_1(PT_1)^{n-1} x_n - x_n\| \\ &\le \|x_{n+1} - x_n\| + k_n\|x_{n+1} - x_n\| + \|T_1(PT_1)^{n-1} x_n - x_n\| \\ &\to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{3.25}$$

In addition,

$$\begin{aligned}\|x_{n+1} - T_1(PT_1)^{n-2} x_{n+1}\| &= \|x_{n+1} - x_n + x_n - T_1(PT_1)^{n-2} x_n \\ &\quad + T_1(PT_1)^{n-2} x_n - T_1(PT_1)^{n-2} x_{n+1}\| \\ &\le \|x_{n+1} - x_n\| + \|T_1(PT_1)^{n-2} x_n - x_n\| \\ &\quad + \|T_1(PT_1)^{n-2} x_{n+1} - T_1(PT_1)^{n-2} x_n\| \\ &\le \|x_{n+1} - x_n\| + \|T_1(PT_1)^{n-2} x_n - x_n\| + L\|x_{n+1} - x_n\|, \end{aligned}$$

where $L = \sup\{k_n : n \geq 1\}$. It follows from (3.22) and (3.25) that

$$\lim_{n\to\infty} \|x_{n+1} - T_1(PT_1)^{n-2}x_{n+1}\| \quad = \quad 0. \tag{3.26}$$

We denote as $(PT_1)^{1-1}$ the identity maps from C onto itself. Thus by the inequality (3.25) and (3.26), we have

$$\begin{aligned}
\|x_{n+1} - T_1x_{n+1}\| &= \|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1} + T_1(PT_1)^{n-1}x_{n+1} - T_1x_{n+1}\| \\
&\leq \|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1}\| + \|T_1(PT_1)^{n-1}x_{n+1} - T_1x_{n+1}\| \\
&= \|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1}\| + \|T_1(PT_1)^{1-1}(PT_1)^{n-1}x_{n+1} \\
&\quad -T_1(PT_1)^{1-1}x_{n+1}\| \\
&\leq \|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1}\| + L\|(PT_1)^{n-1}x_{n+1} - x_{n+1}\| \\
&= \|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1}\| + L\|(PT_1)(PT_1)^{n-2}x_{n+1} - P(x_{n+1})\| \\
&\leq \|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1}\| + L\|T_1(PT_1)^{n-2}x_{n+1} - x_{n+1}\| \\
&\to 0, \quad \text{as } n \to \infty,
\end{aligned}$$

which implies that $\lim_{n\to\infty} \|T_1x_n - x_n\| = 0$. Similarly, by using (3.23) and (3.24), we may show that $\lim_{n\to\infty} \|T_2x_n - x_n\| = 0$ and $\lim_{n\to\infty} \|T_3x_n - x_n\| = 0$. The proof is completed. □

Theorem 3.3. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to X$ be three asymptotically nonexpansive nonself-mappings of C with sequences $\{k_n\}, \{l_n\}, \{m_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(m_n - 1) < \infty$, $k_n \to 1$, $l_n \to 1$, $m_n \to 1$ as $n \to \infty$, respectively and $F := F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$ be real sequences in $[0,1]$ such that $a_n + \gamma_n$, $b_n + c_n + \mu_n$ and $\alpha_n + \beta_n + \lambda_n$ are in $[0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$, and let $\{u_n\}, \{v_n\}$ and $\{w_n\}$ be the bounded sequences in C. From an arbitrary $x_1 \in C$, define the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ using (1.6) and the parameters satisfy one of the following control conditions:*

(C1) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$,
$0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, *and*
$0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n + \gamma_n) < 1$,

(C2) $0 < \liminf_{n\to\infty} \alpha_n$, $\liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$,
$0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, *and*
$0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n + \gamma_n) < 1$,

(C3) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, *and*
$0 < \liminf_{n\to\infty} b_n$, $\liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$,

(C4) $0 < \liminf_{n\to\infty} \alpha_n$, $\liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$,
$0 < \liminf_{n\to\infty} b_n$, *and* $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n + \gamma_n) < 1$,

(C5) $0 < \liminf_{n\to\infty} \alpha_n$, $\liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, *and*
$0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n + \gamma_n) < 1$,

(C6) $0 < \liminf_{n\to\infty} \alpha_n$, $\liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, *and*
$0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$.

If one of T_1, T_2 and T_3 is either completely continuous or demicompact, then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a common fixed point of T_1, T_2 and T_3.

Proof. Suppose that one of the conditions $(C1)-(C6)$ is satisfied. By Lemma 3.1(i), $\{x_n\}$ is bounded. In addition, by Lemma 3.2, $\lim_{n\to\infty}\|T_1x_n - x_n\| = 0$, $\lim_{n\to\infty}\|T_2x_n - x_n\| = 0$ and $\lim_{n\to\infty}\|T_3x_n - x_n\| = 0$, and then the sequences $\{T_1x_n\}$, $\{T_2x_n\}$ and $\{T_3x_n\}$ are also bounded. If T_1 is completely continuous, there exists a subsequence $\{T_1x_{n_j}\}$ of $\{T_1x_n\}$ such that $T_1x_{n_j} \to q$ as $j \to \infty$. It follows from Lemma 3.2, that $\lim_{j\to\infty}\|T_1x_{n_j} - x_{n_j}\| = \lim_{j\to\infty}\|T_2x_{n_j} - x_{n_j}\| = \lim_{j\to\infty}\|T_3x_{n_j} - x_{n_j}\| = 0$. So by the continuity of T_1 and Lemma 2.3, we have $\lim_{j\to\infty}\|x_{n_j} - q\| = 0$ and $q \in F$. Furthermore, by Lemma 3.1 (i), we get that $\lim_{n\to\infty}\|x_n - q\|$ exists. Thus $\lim_{n\to\infty}\|x_n - q\| = 0$. From (3.22), we have $\lim_{n\to\infty}\|z_n - x_n\| = \lim_{n\to\infty}\|y_n - x_n\| = 0$ and it follows that

$$\lim_{n\to\infty}\|z_n - q\| = \lim_{n\to\infty}\|y_n - q\| = 0.$$

Next, assume that one of T_1, T_2 and T_3 is demicompact, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that x_{n_j} converges strongly to p. It follows from Lemma 2.3 that $p \in F$. Thus $\lim_{n\to\infty}\|x_n - p\|$ exists by Lemma 3.1. Since the subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to p, then $\{x_n\}$ converges strongly to the common fixed point $p \in F$. That is, $\lim_{n\to\infty}\|x_n - p\| = 0$. From (3.22), we have

$$\lim_{n\to\infty}\|z_n - x_n\| = \lim_{n\to\infty}\|y_n - x_n\| = 0,$$

it follows that $\lim_{n\to\infty}\|z_n - p\| = 0$ and $\lim_{n\to\infty}\|y_n - p\| = 0$. This completes the proof. □

For $\gamma_n = \mu_n = \lambda_n \equiv 0$ with the control conditions $(C1)-(C6)$ in Theorem 3.3, we obtain the following result.

Theorem 3.4. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to X$ be three asymptotically nonexpansive nonself-mappings of C with sequences $\{k_n\}, \{l_n\}, \{m_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(m_n - 1) < \infty$, $k_n \to 1$, $l_n \to 1$, $m_n \to 1$ as $n \to \infty$, respectively and $F := F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}$, and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $b_n + c_n$ and $\alpha_n + \beta_n$ are in $[0,1]$ for all $n \geq 1$, and the control conditions $(C1)-(C6)$ in Theorem 3.3 are satisfied. For a given $x_1 \in C$, define*

$$\begin{aligned}
z_n &= P(a_nT_1(PT_1)^{n-1}x_n + (1-a_n)x_n),\\
y_n &= P(b_nT_2(PT_2)^{n-1}z_n + c_nT_1(PT_1)^{n-1}x_n + (1-b_n-c_n)x_n),\\
x_{n+1} &= P(\alpha_nT_3(PT_3)^{n-1}y_n + \beta_nT_2(PT_2)^{n-1}z_n + (1-\alpha_n-\beta_n)x_n), \quad n \geq 1.
\end{aligned}$$

If one of T_1, T_2 and T_3 is either completely continuous or demicompact, then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a common fixed point of T_1, T_2 and T_3.

For $c_n = \beta_n \equiv 0$ with the control condition $(C1)$ in Theorem 3.3, we obtain the following result.

Theorem 3.5. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to X$ be three asymptotically nonexpansive nonself-mappings of C with sequences $\{k_n\}, \{l_n\}, \{m_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(m_n - 1) < \infty$, $k_n \to 1$, $l_n \to 1$, $m_n \to 1$ as $n \to \infty$, respectively and $F := F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{\alpha_n\}, \{\gamma_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$ be real sequences in $[0,1]$ such that $a_n + \gamma_n$, $b_n + \mu_n$ and $\alpha_n + \lambda_n$ are in $[0,1]$ for all $n \geq 1$, and the control condition $(C1)$ in Theorem 3.3 are satisfied. Let $\{u_n\}, \{v_n\}$ and $\{w_n\}$ be the bounded sequences in C. From an arbitrary $x_1 \in C$, define*

$$\begin{aligned} z_n &= P(a_nT_1(PT_1)^{n-1}x_n + (1-a_n-\gamma_n)x_n + \gamma_n u_n), \\ y_n &= P(b_nT_2(PT_2)^{n-1}z_n + (1-b_n-\mu_n)x_n + \mu_n v_n), \\ x_{n+1} &= P(\alpha_nT_3(PT_3)^{n-1}y_n + (1-\alpha_n-\lambda_n)x_n + \lambda_n w_n), \quad n \geq 1. \end{aligned}$$

If one of T_1, T_2 and T_3 is either completely continuous or demicompact, then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a common fixed point of T_1, T_2 and T_3.

For $c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ with the control condition $(C1)$ in Theorem 3.3, we obtain the following result.

Theorem 3.6. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to X$ be three asymptotically nonexpansive nonself-mappings of C with sequences $\{k_n\}, \{l_n\}, \{m_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(m_n - 1) < \infty$, $k_n \to 1$, $l_n \to 1$, $m_n \to 1$ as $n \to \infty$, respectively and $F := F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. Let $\{a_n\}, \{b_n\}$ and $\{\alpha_n\}$ be real sequences in $[0, 1]$ satisfying the control conditions $(C1)$ in Theorem 3.3. From an arbitrary $x_1 \in C$, define*

$$\begin{aligned} z_n &= P(a_nT_1(PT_1)^{n-1}x_n + (1-a_n)x_n), \\ y_n &= P(b_nT_2(PT_2)^{n-1}z_n + (1-b_n)x_n), \\ x_{n+1} &= P(\alpha_nT_3(PT_3)^{n-1}y_n + (1-\alpha_n)x_n), \quad n \geq 1. \end{aligned}$$

If one of T_1, T_2 and T_3 is either completely continuous or demicompact, then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a common fixed point of T_1, T_2 and T_3.

For $T_1 = T_2 = T_3 = T : C \to X$ and $\gamma_n = \mu_n = \lambda_n \equiv 0$, then the iterative scheme (1.6) reduces to that of (1.5) and the following result is directly obtained by Theorem 3.3.

Theorem 3.7. *([10], Theorem 1). Let X be a uniformly convex Banach space and let C be its closed and convex subset. Let $T : C \to X$ be an asymptotically nonexpansive nonself-mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose that the set $F(T)$ of fixed points of T is nonempty. Define a sequence $\{x_n\}$ in C as in (1.5) where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ in $[0, 1]$ are such that $b_n + c_n$ and $\alpha_n + \beta_n$ remain in $[0, 1]$, and $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n) < 1$, and $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$. If T is either completely continuous or demicompact, then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.*

Remark 3.8. If T is a self-mapping and $T_1 = T_2 = T_3 \equiv T$, then Theorem 3.3 generalizes Theorem 2.3 of Nammanee, Noor and Suantai [12]. By the same argument, Theorem 3.4, Theorem 3.5 and Theorem 3.6 are generalization of Theorem 2.3 of Suantai [20], Corollary 2.5 of Cho, Zhou and Guo [2] and Theorem 2.1 of Xu and Noor [23], respectively.

In the remainder of this section, we deal with the weak convergence of the iterative scheme with errors (1.6) for three asymptotically nonexpansive nonself-mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 3.9. *Let X be a uniformly convex Banach space which satisfies Opial's condition and C a nonempty closed convex nonexpansive retract of X with P a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to X$ be three asymptotically nonexpansive nonself-mappings of C with sequences $\{k_n\}, \{l_n\}, \{m_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(m_n - 1) < \infty$, $k_n \to 1$, $l_n \to 1$, $m_n \to 1$ as $n \to \infty$, respectively and $F := F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$ be real sequences in $[0, 1]$ such that $a_n + \gamma_n$, $b_n + c_n + \mu_n$ and $\alpha_n + \beta_n + \lambda_n$ are in $[0, 1]$ for all $n \geq 1$, and*

$\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$, *and let* $\{u_n\}, \{v_n\}$ *and* $\{w_n\}$ *be the bounded sequences in* C. *From an arbitrary* $x_1 \in C$, *define the sequences* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *using (1.6) and the parameters satisfy one of the control conditions* $(C1) - (C6)$. *Then* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge weakly to a common fixed point of* T_1, T_2 *and* T_3.

Proof. Suppose that one of the conditions $(C1) - (C6)$ is satisfied. It follows from Lemma 3.2 that $\lim_{n\to\infty} \|T_1x_n - x_n\| = \lim_{n\to\infty} \|T_2x_n - x_n\| = \lim_{n\to\infty} \|T_3x_n - x_n\| = 0$. Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to u$ weakly as $n \to \infty$, without loss of generality. By Lemma 2.3, we have $u \in F$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. From Lemma 2.3, we have $u, v \in F$. By Lemma 3.1 (i), $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. It follows from Lemma 2.4 that $u = v$. Therefore $\{x_n\}$ converges weakly to a common fixed point of T_1, T_2 and T_3. From (3.22), we have $\lim_{n\to\infty} \|z_n - x_n\| = \lim_{n\to\infty} \|y_n - x_n\| = 0$. Since $x_n \to u$ weakly as $n \to \infty$, it follows that the sequences $\{y_n\}$ and $\{z_n\}$ converge weakly to a common fixed point u of T_1, T_2 and T_3. This completes the proof. □

For $T_1 = T_2 = T_3 \equiv T : C \to X$ and $\gamma_n = \mu_n = \lambda_n \equiv 0$, then the iterative scheme (1.6) reduces to that of (1.5) and the following result is directly obtained by Theorem 3.9.

Theorem 3.10. *([10], Theorem 6). Let* X *be a uniformly convex Banach space satisfying Opial's condition and let* C *be its closed and convex subset. Let* $T : C \to X$ *be an asymptotically nonexpansive nonself-mapping with a sequence* $\{k_n\} \subset [1, \infty)$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Let* $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}$ *and* $\{\beta_n\}$ *be in* $[0, 1]$ *such that* $b_n + c_n$ *and* $\alpha_n + \beta_n$ *are in* $[0, 1]$, *and* $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n) < 1$, *and* $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$. *Define a sequence* $\{x_n\}$ *in* C *as in (1.5). If* $F(T) \neq \emptyset$, *then* $\{x_n\}$ *converges weakly to a fixed point of* T.

Remark 3.11. (1) For $\gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 3.9 with one of the control conditions $(C1) - (C6)$ in Theorem 3.3, we obtain the sequence $\{x_n\}$ defined as in Theorem 3.4 converges weakly to a common fixed point of T_1, T_2 and T_3.
(2) For $c_n = \beta_n \equiv 0$ in Theorem 3.9 with the control condition $(C1)$ in Theorem 3.3, we obtain the sequence $\{x_n\}$ defined as in Theorem 3.5 converges weakly to a common fixed point of T_1, T_2 and T_3.
(3) For $c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 3.9 with the control condition $(C1)$ in Theorem 3.3, we obtain the sequence $\{x_n\}$ defined as in Theorem 3.6 converges weakly to a common fixed point of T_1, T_2 and T_3.
(4) Theorem 3.9 contains Theorem 2.8, Corollaries 2.9 and 2.10 of Nammanee, Noor and Suantai [12] and Theorem 2.1 of Cho, Zhou and Guo [2] as special cases when $T_1 = T_2 = T_3 \equiv T$ is a self-mapping.

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