Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 2086–2098
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Identities involving degenerate Euler numbers and polynomials arising from non-linear differential equations

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Communicated by S.-H. Rim

Abstract

The purpose of this paper is to construct some new non-linear differential equations and investigate the solutions of these non-linear differential equations. In addition, we give some new identities involving degenerate Euler numbers and polynomials arising from those non-linear differential equations.

Keywords: Degenerate Euler numbers, degenerate Euler polynomials, non-linear differential equation, degenerate Bernoulli numbers, degenerate Bernoulli polynomials.
2010 MSC: 05A19, 11B83, 34A34.

1. Introduction

As is well known, the Euler polynomials of order $r(\in \mathbb{N})$ are defined by the generating function

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!}, \quad \text{(see [1–15])}. \tag{1.1}$$

When $x = 0$, $E_n^{(r)} = E_n^{(r)}(0)$ are called the higher-order Euler numbers.

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Received 2015-12-29

In particular, $r = 1$, $E_n(x) = E_n^{(1)}(x)$ are called ordinary Euler polynomials.

In [2, 3], L. Carlitz considered the degenerate Euler polynomials which are given by the generating function

$$\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}^{(r)}(x) \frac{t^n}{n!}. \tag{1.2}$$

When $x = 0$, $\mathcal{E}_{n,\lambda}^{(r)} = \mathcal{E}_{n,\lambda}^{(r)}(0)$ are called the higher-order degenerate Euler numbers.

In particular, for $r = 1$, $\mathcal{E}_{n,\lambda} = \mathcal{E}_{n,\lambda}^{(1)}(0)$ and $\mathcal{E}_{n,\lambda}(x) = \mathcal{E}_{n,\lambda}^{(1)}(x)$ are respectively called the degenerate Euler numbers and the degenerate Euler polynomials.

From (1.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}^{(r)}(x) \frac{t^n}{n!} &= \left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(x \mid \lambda)_{n-l}\,\mathcal{E}_l^{(r)}\right)\frac{t^n}{n!},\end{aligned} \tag{1.3}$$

where $(x \mid \lambda)_n = x(x-\lambda)\cdots(x-(n-1)\lambda)$.

Thus, by (1.3), we get

$$\mathcal{E}_{n,\lambda}^{(r)}(x) = \sum_{l=0}^{n}\binom{n}{l}(x \mid \lambda)_{n-l}\,\mathcal{E}_l^{(r)}, \quad (n \geq 0), \quad (\text{see } [3, 10, 12]). \tag{1.4}$$

In [9, 11], Kim and Kim, and Kim developed some new methods for obtaining identities related to Bernoulli numbers of the second kind and Frobenius-Euler polynomials of higher order arising from certain non-linear differential equations.

For example,

$$\begin{aligned}&(-1)^N \sum_{j=0}^{\min\{n,N-1\}} (N-j)!\,(N-1)!H_{N-1,N-1-j}\,(n)_j\, b_{n-j}^{(N+1-j)} \\ &= \begin{cases}(-1)^N N! \prod_{l=0}^{n-1}(N-l), & \text{if } 0 \leq n < N,\\ \sum_{l=0}^{n-N-1}\binom{n}{l}\frac{b_{n-l}}{n-l}\prod_{l=0}^{l+N}(n-l), & \text{if } n \geq N+1,\end{cases}\end{aligned}$$

where $H_{N,0} = 1$, for all $N(\in \mathbb{N})$,

$$\begin{aligned}H_{N,1} &= H_N = 1 + \frac{1}{2} + \cdots + \frac{1}{N},\\ H_{N,j} &= \frac{H_{N-1,j-1}}{N} + \frac{H_{N-2,j-1}}{N-1} + \cdots + \frac{H_{0,j-1}}{1}, \quad H_{0,j-1} = 0 \quad (2 \leq j \leq N),\\ b_n^{(k)} &= \text{the } n\text{th Bernoulli numbers of the second kind with order } k \quad (\text{see } [9, 11]).\end{aligned}$$

The rising factorial sequence is defined as

$$(x)_n = x(x+1)\cdots(x+n-1) = \sum_{l=0}^{n}|S_1(n,l)|\,x^l, \quad (n \geq 0), \tag{1.5}$$

where $|S_1(n,l)|$ are called the unsigned Stirling numbers of the first kind (see [1–9, 11–13]).

The purpose of this paper is to construct some new non-linear differential equations and investigate the solutions of these non-linear differential equations. In addition, we give some new identities involving degenerate Euler numbers and polynomials arising from those non-linear differential equations.

2. Identities of degenerate Euler numbers and polynomials

Now, we construct the non-linear differential equations with the solution $F(t) = \frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}+1}$.

Let

$$F = F(t) = F(t;\lambda) = \frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}+1}, \tag{2.1}$$

and

$$F^N = \underbrace{F \times F \times \cdots \times F}_{N-\text{times}}, \quad \text{where } N \in \mathbb{N}. \tag{2.2}$$

From (2.1), we note that

$$\begin{aligned} F^{(1)} = \frac{dF}{dt} &= \frac{-(1+\lambda t)^{\frac{1}{\lambda}}}{\left((1+\lambda t)^{\frac{1}{\lambda}}+1\right)^2 (1+\lambda t)} \\ &= \frac{(-1)}{1+\lambda t}\left(F - F^2\right). \end{aligned} \tag{2.3}$$

Thus, by (2.3), we get

$$F^{(1)} = \frac{dF}{dt}(t) = \frac{(-1)}{1+\lambda t}\left(F - F^2\right). \tag{2.4}$$

From (2.4), we can derive

$$\begin{aligned} F^{(2)} &= \frac{dF^{(1)}}{dt} \\ &= \frac{(-1)^2 \lambda}{(1+\lambda t)^2}\left(F - F^2\right) + \frac{(-1)}{(1+\lambda t)}\left(F^{(1)} - 2FF^{(1)}\right) \\ &= \frac{(-1)^2 \lambda}{(1+\lambda t)^2}\left(F - F^2\right) + \frac{(-1)}{(1+\lambda t)}\left\{\frac{(-1)}{(1+\lambda t)}\left(F - F^2\right) - 2F\left(\frac{(-1)}{1+\lambda t}\left(F - F^2\right)\right)\right\} \\ &= \frac{(-1)^2 \lambda}{(1+\lambda t)^2}\left(F - F^2\right) + \frac{(-1)^2}{(1+\lambda t)^2}\left(F - F^2\right) + \frac{(-1)^3 2!}{(1+\lambda t)^2}\left(F^2 - F^3\right) \\ &= \frac{(-1)^2 (\lambda+1)}{(1+\lambda t)^2}\left(F - F^2\right) + \frac{(-1)^3 2!}{(1+\lambda t)^2}\left(F^2 - F^3\right), \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} F^{(3)} &= \frac{dF^{(2)}}{dt} \\ &= \frac{\lambda(-1)^3 2!(\lambda+1)}{(1+\lambda t)^3}\left(F - F^2\right) + \frac{(-1)^2(\lambda+1)}{(1+\lambda t)^2}\left(F^{(1)} - 2FF^{(1)}\right) \\ &\quad + \frac{(-1)^4 2!2\lambda}{(1+\lambda t)^3}\left(F^2 - F^3\right) + \frac{(-1)^3 2!}{(1+\lambda t)^2}\left(2FF^{(1)} - 3F^2F^{(1)}\right) \\ &= \frac{(-1)^3(1+\lambda)(2\lambda+1)}{(1+\lambda t)^3}\left(F - F^2\right) + \frac{(-1)^4(\lambda+1)2}{(1+\lambda t)^3}\left(F^2 - F^3\right) + \frac{(-1)^4 2!2\lambda}{(1+\lambda t)^3}\left(F^2 - F^3\right) \\ &\quad + \frac{(-1)^4 2!2}{(1+\lambda t)^3}\left(F^2 - F^3\right) + \frac{(-1)^5 2!3}{(1+\lambda t)^3}\left(F^3 - F^4\right) \\ &= \frac{(-1)^3(1+\lambda)(2\lambda+1)}{(1+\lambda t)^3}F + \frac{(-1)^4(2\lambda+7)(\lambda+1)}{(1+\lambda t)^3}F^2 + \frac{(-1)^5 3!(\lambda+2)}{(1+\lambda t)^3}F^3 \\ &\quad + \frac{(-1)^6 3!}{(1+\lambda t)^3}F^4. \end{aligned} \tag{2.6}$$

Thus we are led to set

$$F^{(N)} = \frac{(-1)^N}{(1+\lambda t)^N} \sum_{i=1}^{N+1} a_i(N,\lambda)(-1)^{i-1} F^i, \quad (N \in \mathbb{N}), \tag{2.7}$$

where

$$F^{(N)} = \frac{d^N F}{dt^N}(t) = \underbrace{\frac{d}{dt} \times \cdots \times \frac{d}{dt}}_{N-\text{times}} F(t).$$

To determine the coefficients $a_i(N,\lambda)$ in (2.7), we take the derivative of (2.7) with respect to t as follows:

$$\begin{aligned} F^{(N+1)} = \frac{dF^{(N)}}{dt} &= \frac{(-1)^{N+1}\lambda N}{(1+\lambda t)^{N+1}} \sum_{i=1}^{N+1} a_i(N,\lambda)(-1)^{i-1} F^i \\ &+ \frac{(-1)^N}{(1+\lambda t)^N} \sum_{i=1}^{N+1} a_i(N,\lambda)(-1)^{i-1} iF^{i-1}F^{(1)}. \end{aligned} \tag{2.8}$$

From (2.4) and (2.8), we have

$$\begin{aligned} F^{(N+1)} &= \frac{(-1)^{N+1}\lambda N}{(1+\lambda t)^{N+1}} \sum_{i=1}^{N+1} a_i(N,\lambda)(-1)^{i-1} F^i \\ &\quad + \frac{(-1)^{N+1}}{(1+\lambda t)^{N+1}} \sum_{i=1}^{N+1} a_i(N,\lambda)(-1)^{i-1} i\left(F^i - F^{i+1}\right) \\ &= \frac{(-1)^{N+1}}{(1+\lambda t)^{N+1}} \left\{ \sum_{i=1}^{N+1} \left(\lambda N a_i(N,\lambda) + i a_i(N,\lambda)\right)(-1)^{i-1} F^i \right. \\ &\quad \left. + \sum_{i=2}^{N+2} a_{i-1}(N,\lambda)(-1)^{i-1}(i-1) F^i \right\} \\ &= \frac{(-1)^{N+1}}{(1+\lambda t)^{N+1}} \left\{ \left(\lambda N a_1(N,\lambda) + a_1(N,\lambda)\right) F \right. \\ &\quad + \sum_{i=2}^{N+1} \left(\lambda N a_i(N,\lambda) + i a_i(N,\lambda) + (i-1) a_{i-1}(N,\lambda)\right)(-1)^{i-1} F^i \\ &\quad \left. + a_{N+1}(N,\lambda)(-1)^{N+1}(N+1) F^{N+2} \right\}. \end{aligned} \tag{2.9}$$

By (2.7) and (2.9), we easily get

$$\begin{aligned} &\frac{(-1)^{N+1}}{(1+\lambda t)^{N+1}} \Big\{ \left(\lambda N a_1(N,\lambda) + a_1(N,\lambda)\right) F + a_{N+1}(N,\lambda)(-1)^{N+1}(N+1) F^{N+2} \\ &\quad + \sum_{i=2}^{N+1} \left(\lambda N a_i(N,\lambda) + i a_i(N,\lambda) + (i-1) a_{i-1}(N,\lambda)\right)(-1)^{i-1} F^i \Big\} \\ &= \frac{(-1)^{N+1}}{(1+\lambda t)^{N+1}} \sum_{i=1}^{N+2} a_i(N+1,\lambda)(-1)^{i-1} F^i. \end{aligned} \tag{2.10}$$

By comparing the coefficients on both sides of (2.10), we get

$$\begin{aligned} a_1(N+1,\lambda) &= \lambda N a_1(N,\lambda) + a_1(N,\lambda) \\ &= (\lambda N + 1) a_1(N,\lambda), \end{aligned} \tag{2.11}$$

$$a_{N+2}\left(N+1,\lambda\right)=\left(N+1\right)a_{N+1}\left(N,\lambda\right), \tag{2.12}$$

and

$$a_{i}\left(N+1,\lambda\right)=\left(\lambda N+i\right)a_{i}\left(N,\lambda\right)+\left(i-1\right)a_{i-1}\left(N,\lambda\right), \tag{2.13}$$

where $2\leq i\leq N+1$.

From (2.4) and (2.7), we have

$$\frac{(-1)}{1+\lambda t}\left(F-F^{2}\right)=F^{(1)}=\frac{(-1)}{1+\lambda t}\left(a_{1}\left(1,\lambda\right)F-a_{2}\left(1,\lambda\right)F^{2}\right). \tag{2.14}$$

Thus, by (2.14), we get

$$a_{1}\left(1,\lambda\right)=1,\quad \text{and}\quad a_{2}\left(1,\lambda\right)=1. \tag{2.15}$$

From (2.11) and (2.15), we can derive the following identities:

$$\begin{aligned}
a_{1}\left(N+1,\lambda\right)&=\left(\lambda N+1\right)a_{1}\left(N,\lambda\right)\\
&=\left(\lambda N+1\right)\left(\lambda\left(N-1\right)+1\right)a_{1}\left(N-1,\lambda\right)\\
&=\left(\lambda N+1\right)\left(\lambda\left(N-1\right)+1\right)\left(\lambda\left(N-2\right)+1\right)a_{1}\left(N-2,\lambda\right)\\
&\vdots\\
&=\left(\lambda N+1\right)\left(\lambda\left(N-1\right)+1\right)\cdots\left(\lambda+1\right)a_{1}\left(1,\lambda\right)\\
&=\left(\lambda N+1\right)\left(\lambda\left(N-1\right)+1\right)\cdots\left(\lambda+1\right)\cdot 1\\
&=\lambda^{N+1}\left(\frac{1}{\lambda}\right)_{N+1},
\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}
a_{N+2}\left(N+1,\lambda\right)&=\left(N+1\right)a_{N+1}\left(N,\lambda\right)\\
&=\left(N+1\right)Na_{N}\left(N-1,\lambda\right)\\
&\vdots\\
&=\left(N+1\right)N\left(N-1\right)\cdots 2a_{2}\left(1,\lambda\right)\\
&=\left(N+1\right)!.
\end{aligned} \tag{2.17}$$

We observe that

$$\begin{aligned}
&a_{1}\left(1,\lambda\right)=1,\ a_{1}\left(2,\lambda\right)=\left(1+\lambda\right),\quad a_{1}\left(3,\lambda\right)=\left(1+\lambda\right)\left(1+2\lambda\right),\quad \cdots\\
&a_{1}\left(N,\lambda\right)=\left(1+\lambda\right)\left(1+2\lambda\right)\cdots\left(1+\left(N-1\right)\lambda\right)=\lambda^{N}\left(\frac{1}{\lambda}\right)_{N},
\end{aligned} \tag{2.18}$$

and

$$a_{2}\left(1,\lambda\right)=1,a_{3}\left(2,\lambda\right)=2!,a_{4}\left(3,\lambda\right)=3!,\ldots,a_{N+1}\left(N,\lambda\right)=N!. \tag{2.19}$$

That is, the matrix $\left(a_{i}\left(j,\lambda\right)\right)_{1\leq i\leq N+1,1\leq j\leq N}$ is given by

$$\begin{array}{cc}
 & \overbrace{\hphantom{1\quad (1+\lambda)\quad (1+\lambda)(1+2\lambda)\quad \cdots\quad \lambda^{N}\left(\frac{1}{\lambda}\right)_{N}}}^{N} \\
N+1\left\{\vphantom{\begin{matrix}1\\1\\1\\1\\1\\1\end{matrix}}\right. & \begin{bmatrix}
1 & (1+\lambda) & (1+\lambda)(1+2\lambda) & \cdots & \lambda^{N}\left(\frac{1}{\lambda}\right)_{N}\\
1! & \times & \times & \cdots & \times\\
 & 2! & \times & \cdots & \times\\
 & & 3! & \cdots & \times\\
 & 0 & & \ddots & \times\\
 & & & & N!
\end{bmatrix}
\end{array}$$

From (2.13), we have

$$
\begin{aligned}
a_2(N+1,\lambda) &= (\lambda N+2)\,a_2(N,\lambda)+a_1(N,\lambda)\\
&= (\lambda N+2)\{(\lambda(N-1)+2)\,a_2(N-1,\lambda)+a_1(N-1,\lambda)\}+a_1(N,\lambda)\\
&= (\lambda N+2)(\lambda(N-1)+2)\,a_2(N-1,\lambda)+(\lambda N+2)\,a_1(N-1,\lambda)+a_1(N,\lambda)\\
&= a_1(N,\lambda)+(\lambda N+2)\,a_1(N-1,\lambda)+(\lambda N+2)(\lambda(N-1)+2)\,a_1(N-2,\lambda)\\
&\quad+(\lambda N+2)(\lambda(N-1)+2)(\lambda(N-2)+2)\,a_2(N-2,\lambda)\\
&\ \vdots\\
&= a_1(N,\lambda)+\sum_{m_1=1}^{N-1}\left(\prod_{l=0}^{m_1-1}(\lambda(N-l)+2)\right)a_1(N-m_1,\lambda)\\
&\quad+(\lambda N+2)(\lambda(N-1)+2)\cdots(\lambda+2)\cdot 1\\
&= \lambda^N\left(\frac{1}{\lambda}\right)_N+\sum_{m_1=1}^{N-1}\lambda^{m_1}\left(\frac{2}{\lambda}+N-m_1+1\right)_{m_1}\lambda^{N-m_1}\left(\frac{1}{\lambda}\right)_{N-m_1}\\
&\quad+\lambda^N\left(\frac{2}{\lambda}+1\right)_N\\
&= \sum_{m_1=0}^{N}\lambda^{m_1}\left(\frac{2}{\lambda}+N-m_1+1\right)_{m_1}\lambda^{N-m_1}\left(\frac{1}{\lambda}\right)_{N-m_1}\\
&= \lambda^N\sum_{m_1=0}^{N}\left(\frac{2}{\lambda}+N-m_1+1\right)_{m_1}\left(\frac{1}{\lambda}\right)_{N-m_1},
\end{aligned}
\tag{2.20}
$$

and

$$
\begin{aligned}
a_3(N+1,\lambda) &= (\lambda N+3)\,a_3(N,\lambda)+2!a_2(N,\lambda)\\
&= 2!a_2(N,\lambda)+(\lambda N+3)\{(\lambda(N-1)+3)\,a_3(N-1,\lambda)+2a_2(N-1,\lambda)\}\\
&= 2!a_2(N,\lambda)+2!(\lambda N+3)\,a_2(N-1,\lambda)\\
&\quad+(\lambda N+3)(\lambda(N-1)+3)\,a_3(N-1,\lambda)\\
&= 2!a_2(N,\lambda)+2!(\lambda N+3)\,a_2(N-1,\lambda)\\
&\quad+2!(\lambda N+3)(\lambda(N-1)+3)\,a_2(N-2,\lambda)\\
&\quad+(\lambda N+3)(\lambda(N-1)+3)(\lambda(N-2)+3)\,a_3(N-2,\lambda)\\
&\ \vdots\\
&= 2!a_2(N,\lambda)+2!\sum_{m_2=1}^{N-2}\left(\prod_{l=0}^{m_2-1}(\lambda(N-l)+3)\right)a_2(N-m_2,\lambda)\\
&\quad+2!(\lambda N+3)(\lambda(N-1)+3)\cdots(3\lambda+3)(2\lambda+3)\\
&= 2!a_2(N,\lambda)+2!\sum_{m_2=1}^{N-2}\lambda^{m_2}\left(N-m_2+1+\frac{3}{\lambda}\right)_{m_2}a_2(N-m_2,\lambda)\\
&\quad+2!(\lambda N+3)(\lambda(N-1)+3)\cdots(3\lambda+3)(2\lambda+3)\\
&= 2!a_2(N,\lambda)+2!\sum_{m_2=1}^{N-2}\lambda^{m_2}\left(N-m_2+1+\frac{3}{\lambda}\right)_{m_2}a_2(N-m_2,\lambda)\\
&\quad+2!\lambda^{N-1}\left(\frac{3}{\lambda}+2\right)_{N-1}
\end{aligned}
\tag{2.21}
$$

$$
\begin{aligned}
&= 2! \sum_{m_2=0}^{N-1} \lambda^{m_2} \left(N - m_2 + 1 + \frac{3}{\lambda}\right)_{m_2} a_2 \left(N - m_2, \lambda\right) \\
&= 2! \lambda^{N-1} \sum_{m_2=0}^{N-1} \sum_{m_1=0}^{N-m_2-1} \left(N - m_2 + 1 + \frac{3}{\lambda}\right)_{m_2} \\
&\quad \times \left(N - m_2 - m_1 + \frac{2}{\lambda}\right)_{m_1} \left(\frac{1}{\lambda}\right)_{N-m_2-m_1-1}.
\end{aligned}
$$

From (2.13), we note that

$$
a_4 \left(N + 1, \lambda\right) = \left(\lambda N + 4\right) a_4 \left(N, \lambda\right) + 3a_3 \left(N, \lambda\right). \tag{2.22}
$$

Thus, by (2.21) and (2.22), we get

$$
\begin{aligned}
a_4 \left(N + 1, \lambda\right) &= 3a_3 \left(N, \lambda\right) + \left(\lambda N + 4\right) \left\{\left(\lambda \left(N - 1\right) + 4\right) a_4 \left(N - 1, \lambda\right) + 3a_3 \left(N - 1, \lambda\right)\right\} \\
&= 3a_3 \left(N, \lambda\right) + 3 \left(\lambda N + 4\right) a_3 \left(N - 1, \lambda\right) \\
&\quad + \left(\lambda N + 4\right) \left(\lambda \left(N - 1\right) + 4\right) a_4 \left(N - 1, \lambda\right) \\
&= 3a_3 \left(N, \lambda\right) + 3 \left(\lambda N + 4\right) a_3 \left(N - 1, \lambda\right) \\
&\quad +3 \left(\lambda N + 4\right) \left(\lambda \left(N - 1\right) + 4\right) a_3 \left(N - 2, \lambda\right) \\
&\quad + \left(\lambda N + 4\right) \left(\lambda \left(N - 1\right) + 4\right) \left(\lambda \left(N - 2\right) + 4\right) a_4 \left(N - 2, \lambda\right) \\
&\vdots \\
&= 3a_3 \left(N, \lambda\right) + 3 \sum_{m_3=1}^{N-3} \left(\prod_{l=0}^{m_3-1} \left(\lambda \left(N - l\right) + 4\right)\right) a_3 \left(N - m_3, \lambda\right) \\
&\quad +3! \left(\lambda N + 4\right) \left(\lambda \left(N - 1\right) + 4\right) \cdots \left(3\lambda + 4\right) \\
&= 3a_3 \left(N, \lambda\right) + 3 \sum_{m_3=1}^{N-3} \lambda^{m_3} \left(\frac{4}{\lambda} + N - m_3 + 1\right)_{m_3} a_3 \left(N - m_3, \lambda\right) \\
&\quad +3! \lambda^{N-2} \left(\frac{4}{\lambda} + 3\right)_{N-2} \\
&= 3 \sum_{m_3=0}^{N-2} \lambda^{m_3} \left(\frac{4}{\lambda} + N - m_3 + 1\right)_{m_3} a_3 \left(N - m_3, \lambda\right) \\
&= 3! \sum_{m_3=0}^{N-2} \lambda^{m_3} \left(\frac{4}{\lambda} + N - m_3 + 1\right)_{m_3} \lambda^{N-m_3-2} \\
&\quad \times \sum_{m_2=0}^{N-m_3-2} \sum_{m_1=0}^{N-m_3-m_2-2} \left(N - m_3 - m_2 + \frac{3}{\lambda}\right)_{m_2} \\
&\quad \times \left(N - m_3 - m_2 - m_1 - 1 + \frac{2}{\lambda}\right)_{m_1} \left(\frac{1}{\lambda}\right)_{N-m_3-m_2-m_1-2}.
\end{aligned} \tag{2.23}
$$

By (2.23), we see that

$$
\begin{aligned}
a_4 \left(N + 1, \lambda\right) = 3! \lambda^{N-2} & \sum_{m_3=0}^{N-2} \sum_{m_2=0}^{N-m_3-2} \sum_{m_1=0}^{N-m_3-m_2-2} \left(\frac{4}{\lambda} + N - m_3 + 1\right)_{m_3} \\
& \times \left(N - m_3 - m_2 + \frac{3}{\lambda}\right)_{m_2} \\
& \times \left(N - m_3 - m_2 - m_1 - 1 + \frac{2}{\lambda}\right)_{m_1} \left(\frac{1}{\lambda}\right)_{N-m_3-m_2-m_1-2}.
\end{aligned} \tag{2.24}
$$

Continuing this process, we get

$$\begin{aligned} a_i(N+1,\lambda) &= (i-1)!\lambda^{N-i+2} \sum_{m_{i-1}=0}^{N-i+2} \sum_{m_{i-2}=0}^{N-m_{i-1}-i+2} \cdots \sum_{m_1=0}^{N-m_{i-1}-\cdots-m_2-i+2} \left(\frac{i}{\lambda}+N-m_{i-1}+1\right)_{m_{i-1}} \\ &\times \left(N-m_{i-1}-m_{i-2}+\frac{i-1}{\lambda}\right)_{m_{i-2}} \cdots \left(N-m_{i-1}-\cdots-m_1-i+3+\frac{2}{\lambda}\right)_{m_1} \\ &\times \left(\frac{1}{\lambda}\right)_{N-m_{i-1}-m_{i-2}-\cdots-m_1-i+2}. \end{aligned} \tag{2.25}$$

Therefore, by (2.7) and (2.25), we obtain the following theorem.

Theorem 2.1. *For $N \in \mathbb{N}$, let us consider the following non-linear differential equation with respect to t:*

$$F^{(N)} = \frac{(-1)^N}{(1+\lambda t)^N} \sum_{i=1}^{N+1} a_i(N,\lambda)(-1)^{i-1} F^i, \tag{2.26}$$

where

$$\begin{aligned} a_i(N,\lambda) &= (i-1)!\lambda^{N-i+1} \sum_{m_{i-1}=0}^{N-i+1} \sum_{m_{i-2}=0}^{N-m_{i-1}-i+1} \cdots \sum_{m_1=0}^{N-m_{i-1}-\cdots-m_2-i+1} \left(N-m_{i-1}+\frac{i}{\lambda}\right)_{m_{i-1}} \\ &\times \left(N-m_{i-1}-m_{i-2}-1+\frac{i-1}{\lambda}\right)_{m_{i-2}} \cdots \left(N-m_{i-1}-\cdots-m_1-i+2+\frac{2}{\lambda}\right)_{m_1} \\ &\times \left(\frac{1}{\lambda}\right)_{N-m_{i-1}-m_{i-2}-\cdots-m_1-i+1}. \end{aligned}$$

Then $F = F(t) = \frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}+1}$ is a solution of (2.26).

Now, we observe that

$$\begin{aligned} F^{(N)} &= \frac{1}{2}\frac{d^N}{dt^N}\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right) \\ &= \frac{1}{2}\frac{d^N}{dt^N}\sum_{m=0}^{\infty}\mathcal{E}_{m,\lambda}\frac{t^m}{m!} \\ &= \frac{1}{2}\sum_{m=N}^{\infty}\mathcal{E}_{m,\lambda}\frac{m(m-1)\cdots(m-N+1)}{m!}t^{m-N} \\ &= \frac{1}{2}\sum_{m=0}^{\infty}\mathcal{E}_{m+N,\lambda}\frac{t^m}{m!}. \end{aligned} \tag{2.27}$$

Thus, by (2.27), we get

$$\begin{aligned} (1+\lambda t)^N F^{(N)} &= \left(\sum_{l=0}^{\infty}\binom{N}{l}\lambda^l t^l\right)\left(\frac{1}{2}\sum_{m=0}^{\infty}\mathcal{E}_{m+N,\lambda}\frac{t^m}{m!}\right) \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{N}{l}(n)_{\underline{l}}\,\lambda^l\mathcal{E}_{n-l+N,\lambda}\right)\frac{t^n}{n!}, \end{aligned} \tag{2.28}$$

where $(x)_{\underline{n}} = x(x-1)\cdots(x-n+1)$, $(n \geq 0)$.

From (1.2), we have

$$\begin{aligned} F^i &= \frac{1}{2^i} \underbrace{\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1} \right) \times \cdots \times \left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1} \right)}_{i-\text{times}} \\ &= \frac{1}{2^i} \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}^{(i)} \frac{t^n}{n!}. \end{aligned} \tag{2.29}$$

Therefore, by Theorem 2.1, (2.28), and (2.29), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, $N \in \mathbb{N}$, we have*

$$\begin{aligned} &\sum_{l=0}^{n} \binom{N}{l} (n)_{\underline{l}} \lambda^l \mathcal{E}_{n-l+N,\lambda} \\ &\quad = \sum_{i=1}^{N+1} (i-1)! \lambda^{N-i+1} \sum_{m_{i-1}=0}^{N-i+1} \sum_{m_{i-2}=0}^{N-m_{i-1}-i+1} \cdots \sum_{m_1=0}^{N-m_{i-1}-\cdots-m_2-i+1} \left(N - m_{i-1} + \frac{i}{\lambda} \right)_{m_{i-1}} \\ &\quad\quad \times \left(N - m_{i-1} - m_{i-2} - 1 + \frac{i-1}{\lambda} \right)_{m_{i-1}} \cdots \\ &\quad\quad \times \left(N - m_{i-1} - m_{i-2} - \cdots - m_1 - i + 2 + \frac{2}{\lambda} \right)_{m_1} \left(\frac{1}{\lambda} \right)_{N-m_{i-1}-\cdots-m_1-i+1} \\ &\quad\quad \times (-1)^{N+i-1} \frac{1}{2^{i-1}} \mathcal{E}_{n,\lambda}^{(i)}, \end{aligned}$$

where $(x)_{\underline{n}} = x(x-1)\cdots(x-n+1)$.

Let

$$F(t) = \frac{1}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}. \tag{2.30}$$

Then, by (2.30), we get

$$\begin{aligned} F^{(1)} = \frac{dF}{dt} &= \frac{(-1)}{(1+\lambda t)} \left\{ \frac{(1+\lambda t)^{\frac{1}{\lambda}}}{\left((1+\lambda t)^{\frac{1}{\lambda}} - 1 \right)^2} \right\} \\ &= \frac{(-1)}{1+\lambda t} \left(F + F^2 \right), \end{aligned} \tag{2.31}$$

$$\begin{aligned} F^{(2)} = \frac{dF^{(1)}}{dt} &= \frac{(-1)^2 \lambda}{(1+\lambda t)^2} \left(F + F^2 \right) + \frac{(-1)}{1+\lambda t} \left(F^{(1)} + 2FF^{(1)} \right) \\ &= \frac{(-1)^2 (\lambda+1)}{(1+\lambda t)^2} F + \frac{(-1)^2 (\lambda+3)}{(1+\lambda t)^2} F^2 + \frac{(-1)^2 2}{(1+\lambda t)^2} F^3 \end{aligned} \tag{2.32}$$

and

$$\begin{aligned} F^{(3)} &= \frac{dF^{(2)}}{dt} \\ &= \frac{(-1)^3 (\lambda+1)(2\lambda+1)}{(1+\lambda t)^3} F + \frac{(-1)^3 (2\lambda+7)(\lambda+1)}{(1+\lambda t)^3} F^2 \\ &\quad + \frac{(-1)^3 3! (\lambda+2)}{(1+\lambda t)^3} F^3 + \frac{(-1)^3 3!}{(1+\lambda t)^3} F^4. \end{aligned} \tag{2.33}$$

So we are led to put

$$F^{(N)}=\frac{(-1)^N}{(1+\lambda t)^N}\sum_{i=1}^{N+1}a_i(N,\lambda)F^i. \tag{2.34}$$

Thus, by (2.34), we get

$$\begin{aligned}F^{(N+1)}&=\frac{dF^{(N)}}{dt}\\&=\frac{(-1)^{N+1}\lambda N}{(1+\lambda t)^{N+1}}\sum_{i=1}^{N+1}a_i(N,\lambda)F^i\\&\quad+\frac{(-1)^N}{(1+\lambda t)^N}\sum_{i=1}^{N+1}a_i(N,\lambda)iF^{i-1}F^{(1)}\\&=\frac{(-1)^{N+1}}{(1+\lambda t)^{N+1}}\sum_{i=1}^{N+1}(\lambda N+i)a_i(N,\lambda)F^i\\&\quad+\frac{(-1)^{N+1}}{(1+\lambda t)^{N+1}}\sum_{i=2}^{N+2}a_{i-1}(N,\lambda)(i-1)F^i.\end{aligned} \tag{2.35}$$

From (2.34) and (2.35), we note that

$$\begin{aligned}F^{(N+1)}=&\frac{(-1)^{N+1}}{(1+\lambda t)^{N+1}}\Big\{(\lambda N+1)a_1(N,\lambda)F+a_{N+1}(N,\lambda)(N+1)F^{N+2}\\&+\sum_{i=2}^{N+1}((\lambda N+i)a_i(N,\lambda)+(i-1)a_{i-1}(N,\lambda))F^i\Big\}\\=&\frac{(-1)^{N+1}}{(1+\lambda t)^{N+1}}\sum_{i=1}^{N+2}a_i(N+1,\lambda)F^i.\end{aligned} \tag{2.36}$$

By comparing the coefficients on the both sides of (2.36), we get

$$a_1(N+1,\lambda)=(\lambda N+1)a_1(N,\lambda),\quad a_{N+2}(N+1,\lambda)=(N+1)a_{N+1}(N,\lambda), \tag{2.37}$$

and

$$(\lambda N+i)a_i(N,\lambda)+(i-1)a_{i-1}(N,\lambda)=a_i(N+1,\lambda),\quad (2\le i\le N+1). \tag{2.38}$$

Also, we observe that

$$\begin{aligned}F^{(1)}&=\frac{(-1)}{1+\lambda t}\left\{a_1(1,\lambda)F+a_2(1,\lambda)F^2\right\}\\&=\frac{(-1)}{1+\lambda t}\left(F+F^2\right).\end{aligned} \tag{2.39}$$

Thus, from (2.39), we get

$$a_1(1,\lambda)=1,\quad \text{and}\quad a_2(1,\lambda)=1. \tag{2.40}$$

Therefore the relations in (2.37), (2.38), and (2.40) are the same as the ones in (2.11), (2.12), (2.13), and (2.15). Hence, from (2.25), we obtain the following theorem.

Theorem 2.3. *For $N\in\mathbb{N}$, the following non-linear differential equation*

$$F^{(N)}=\frac{(-1)^N}{(1+\lambda t)^N}\sum_{i=1}^{N+1}a_i(N,\lambda)F^i \tag{2.41}$$

has the solution $F=F(t)=\frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}-1}$, where

$$a_i(N,\lambda) = (i-1)!\lambda^{N-i+1}\sum_{m_{i-1}=0}^{N-i+1}\sum_{m_{i-2}=0}^{N-m_{i-1}-i+1}\cdots\sum_{m_1=0}^{N-m_{i-1}-\cdots-m_2-i+1}\left(N-m_{i-1}+\frac{i}{\lambda}\right)_{m_{i-1}}$$
$$\times\left(N-m_{i-1}-m_{i-2}-1+\frac{i-1}{\lambda}\right)_{m_{i-2}}\cdots\left(N-m_{i-1}-\cdots-m_1-i+2+\frac{2}{\lambda}\right)_{m_1}$$
$$\times\left(\frac{1}{\lambda}\right)_{N-m_{i-1}-m_{i-2}-\cdots-m_1-i+1}.$$

For $r\in\mathbb{N}$, the degenerate Bernoulli polynomials of order r are defined by Carlitz as

$$\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(r)}(x)\frac{t^n}{n!},\quad(\text{see [3]}).\tag{2.42}$$

When $x=0$, $\beta_{n,\lambda}^{(r)}=\beta_{n,\lambda}^{(r)}(0)$ are called the degenerate higher-order Bernoulli numbers. In particular, $r=1$, $\beta_{n,\lambda}=\beta_{n,\lambda}^{(1)}$ are called the degenerate Bernoulli numbers. Note that $\beta_{0,\lambda}=1$.

We observe that

$$\begin{aligned}F=F(t)&=\frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\\&=\frac{1}{t}\sum_{n=0}^{\infty}\beta_{n,\lambda}\frac{t^n}{n!}=\sum_{n=1}^{\infty}\beta_{n,\lambda}\frac{t^{n-1}}{n!}+\frac{1}{t}\\&=\sum_{n=0}^{\infty}\frac{\beta_{n+1,\lambda}}{n+1}\frac{t^n}{n!}+\frac{1}{t}.\end{aligned}\tag{2.43}$$

Thus, by (2.43), we get

$$\begin{aligned}F^{(N-1)}&=\frac{d^{N-1}}{dt^{N-1}}\left(\frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)\\&=\sum_{n=N-1}^{\infty}\frac{\beta_{n+1,\lambda}}{n+1}\frac{t^{n-N+1}}{(n-N+1)!}+\frac{(-1)^{N-1}}{t^N}(N-1)!\\&=\sum_{n=0}^{\infty}\frac{\beta_{n+N,\lambda}}{n+N}\frac{t^n}{n!}+\frac{1}{t^N}(-1)^{N-1}(N-1)!.\end{aligned}\tag{2.44}$$

From (2.44), we have

$$\begin{aligned}t^NF^{(N-1)}&=\sum_{n=N-1}^{\infty}\frac{\beta_{n+1,\lambda}}{n+1}\frac{t^{n+1}}{(n-N+1)!}+(-1)^{N-1}(N-1)!\\&=\sum_{n=N}^{\infty}\frac{\beta_{n,\lambda}}{n}\frac{t^n}{(n-N)!}+(-1)^{N-1}(N-1)!.\end{aligned}\tag{2.45}$$

Replacing N by $N+1$, we get

$$\begin{aligned}(1+\lambda t)^Nt^{N+1}F^{(N)}&=(1+\lambda t)^N\sum_{n=N+1}^{\infty}\frac{\beta_{n,\lambda}}{n}\frac{t^n}{(n-N-1)!}+(-1)^NN!(1+\lambda t)^N\\&=\sum_{n=N+1}^{\infty}\left(\sum_{l=0}^{n-N-1}\lambda^l\binom{N}{l}\frac{\beta_{n-l,\lambda}}{n-l}n(n-1)\cdots(n-l-N)\right)\frac{t^n}{n!}\\&\quad+(-1)^NN!\sum_{n=0}^{N}(N)_{\underline{n}}\lambda^n\frac{t^n}{n!},\end{aligned}\tag{2.46}$$

where $(x)_{\underline{n}} = x(x-1)\cdots(x-n+1)$.

From Theorem 2.3, we have

$$\begin{aligned}(1+\lambda t)^N t^{N+1} F^{(N)} &= (-1)^N \sum_{j=1}^{N+1} a_j(N,\lambda) F^j t^{N+1} \\ &= (-1)^N \sum_{j=1}^{N+1} a_j(N,\lambda) \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^j t^{N+1-j} \\ &= (-1)^N \sum_{j=0}^{N} a_{N+1-j}(N,\lambda) t^j \sum_{m=0}^{\infty} \beta_{m,\lambda}^{(N+1-j)} \frac{t^m}{m!} \\ &= (-1)^N \sum_{n=0}^{\infty} \left\{ \sum_{j=0}^{\min\{n,N\}} a_{N+1-j}(N,\lambda) \frac{n!}{(n-j)!} \beta_{n-j,\lambda}^{(N+1-j)} \right\} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left\{ (-1)^N \sum_{j=0}^{\min\{n,N\}} a_{N+1-j}(N,\lambda) n(n-1)\cdots(n-j+1) \beta_{n-j,\lambda}^{(N+1-j)} \right\} \frac{t^n}{n!}.\end{aligned} \tag{2.47}$$

Therefore, by (2.46) and (2.47), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$\begin{aligned}&(-1)^N \sum_{j=0}^{\min\{n,N\}} a_{N+1-j}(N,\lambda) n(n-1)\cdots(n-j+1) \beta_{n-j,\lambda}^{(N+1-j)} \\ &\quad = \begin{cases} (-1)^N N! (N)_{\underline{n}} \lambda^n & \text{if } 0 \leq n \leq N, \\ \sum_{l=0}^{n-N-1} \lambda^l \binom{N}{l} \frac{\beta_{n-l,\lambda}}{n-l} n(n-1)\cdots(n-l-N) & \text{if } n \geq N+1, \end{cases}\end{aligned}$$

where

$$\begin{aligned}a_i(N,\lambda) &= (i-1)!\lambda^{N-i+1} \sum_{m_{i-1}=0}^{N-i+1} \sum_{m_{i-2}=0}^{N-m_{i-1}-i+1} \cdots \sum_{m_1=0}^{N-m_{i-1}-\cdots-m_2-i+1} \left(N-m_{i-1}+\frac{i}{\lambda}\right)_{m_{i-1}} \\ &\times \left(N-m_{i-1}-m_{i-2}-1+\frac{i-1}{\lambda}\right)_{m_{i-2}} \cdots \left(N-m_{i-1}-\cdots-m_1-i+2+\frac{2}{\lambda}\right)_{m_1} \\ &\times \left(\frac{1}{\lambda}\right)_{N-m_{i-1}-m_{i-2}-\cdots-m_1-i+1}.\end{aligned}$$

Acknowledgements

The first author is appointed as a chair professor at Tian- jin Polytechnic University by Tianjin City in China from August 2015 to August 2019.

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