Common fixed point theorems of generalized Lipschitz mappings in cone b-metric spaces over Banach algebras and applications

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Abstract

In this paper, we introduce the concept of cone b-metric space over Banach algebra and present some common fixed point theorems in such spaces. Moreover, we support our results by two examples. In addition, some applications in the solutions of several equations are given to illustrate the usability of the obtained results. ©2015 All rights reserved.

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1. Introduction and Preliminaries

The Banach contraction mapping principle is widely recognized as one of the most influential sources in pure and applied mathematics. A mapping $T : X \to X$, where (X, d) is a metric space, is said to be a contraction mapping if, for all $x, y \in X$, there is a contractive constant $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y).$$

According to this principle, any mapping T satisfying the above inequality in a complete metric space will have a unique fixed point. This principle has been generalized in different directions in all kinds of spaces by mathematicians over the years. Also, in the contemporary research, it remains a heavily investigated branch as a consequence of the strong applicability. The concept of cone metric space, as a

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meaningful generalization of metric spaces, was introduced in the work of Huang and Zhang [9] where they also established the Banach contraction mapping principle in such spaces. Subsequently, [10] introduced cone b-metric space which greatly expanded cone metric space. Afterwards, many authors have focused on fixed point problems in cone metric spaces or cone b-metric spaces. A large number of works are noted in [1, 2, 3, 8, 11, 12, 15, 17, 18, 20] and the relevant literature therein. Unfortunately, recently these problems became not attractive since some scholars found the equivalence of fixed point results between cone metric spaces and metric spaces, also between cone b-metric spaces and b-metric spaces (see [4, 5, 6, 7, 13, 14]). However, quite fortunately, very recently, Liu and Xu [16] introduced the notion of cone metric space over Banach algebra and considered fixed point theorems in such spaces in a different way by restricting the contractive constants to be vectors and the relevant multiplications to be vector ones instead of usual real constants and scalar multiplications. And that they provided an example to explain the non-equivalence of fixed point results between the vectorial versions and scalar versions. As a result, there is still both interest and need for research in the field of studying fixed point theorems in the framework of cone metric or cone b-metric spaces. Throughout this paper, we introduce the concept of cone b-metric space over Banach algebra as a further generalization of cone metric space over Banach algebra and prove some common fixed point theorems of generalized Lipschitz mappings in such setting without assumption of normality. The results not only directly improve and expand several well-known comparable assertions in b-metric spaces and cone metric spaces, but also unify and complement some previous results in cone metric spaces over Banach algebras. Furthermore, we give two examples to support our conclusions. Otherwise, we use our results to demonstrate the crucial role of obtaining the existence and uniqueness of the solution for some equations.

For the sake of readers, we shall recall some basic notions and lemmas which are listed as follows.

A Banach algebra $\mathcal{A}$ is a Banach space over $\mathbf{K} \in \{\mathbb{R}, \mathbb{C}\}$ together with an associative and distributive multiplication such that $a(xy) = (ax)y = x(ay), x, y \in \mathcal{A}, a \in \mathbf{K}$ and $\|xy\| \leq \|x\|\|y\|$, $x, y \in \mathcal{A}$, where $\|\cdot\|$ is the norm on $\mathcal{A}$. Let $\mathcal{A}$ be a Banach algebra with a unit e, and θ the zero element of $\mathcal{A}$. A nonempty closed convex subset P of $\mathcal{A}$ is called a cone if $\{\theta, e\} \subset P$, $P^2 = PP \subset P$, $P \cap (-P) = \{\theta\}$ and $\lambda P + \mu P \subset P$ for all $\lambda, \mu \geq 0$. On this basis, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will indicate that $y - x \in \text{int}P$, where $\text{int}P$ stands for the interior of P. If $\text{int}P \neq \emptyset$, then P is called a solid cone. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in \mathcal{A}$, $\theta \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$. The least positive number satisfying above is called the normal constant of P.

In the following we always suppose that $\mathcal{A}$ is a Banach algebra with a unit e, P is a solid cone in $\mathcal{A}$, and $\preceq$ is a partial ordering with respect to P.

Definition 1.1 ([16]). Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathcal{A}$ satisfies:

(i) $\theta \prec d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space over Banach algebra $\mathcal{A}$.

Definition 1.2 ([16]). Let (X, d) be a cone metric space over Banach algebra $\mathcal{A}$, $x \in X$ and $\{x_n\}$ a sequence in X. Then

(i) $\{x_n\}$ converges to x whenever for every $c \gg \theta$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$.

(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \gg \theta$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(iii) (X, d) is complete if every Cauchy sequence is convergent.

Definition 1.3 ([1]). Let $f, g : X \to X$ be mappings on a set X.

(i) If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g;

(ii) The pair (f, g) is called weakly compatible if f and g commute at all of their coincidence points, that is, $fgx = gfx$ for all $x \in C(f, g) = \{x \in X : fx = gx\}$.

Definition 1.4 ([21]). Let P be a solid cone in a Banach space E. A sequence $\{u_n\} \subset P$ is said to be a c-sequence if for each $c \gg \theta$ there exists a natural number N such that $u_n \ll c$ for all $n > N$.

Lemma 1.5 ([21]). *Let P be a solid cone in a Banach algebra $\mathcal{A}$. Suppose that $k \in P$ and $\{u_n\}$ is a c-sequence in P. Then $\{ku_n\}$ is a c-sequence.*

Lemma 1.6 ([19]). *Let $\mathcal{A}$ be a Banach algebra with a unit e, $k \in \mathcal{A}$, then $\lim\limits_{n \to \infty} \|k^n\|^{\frac{1}{n}}$ exists and the spectral radius $\rho(k)$ satisfies*

$$\rho(k) = \lim_{n \to \infty} \|k^n\|^{\frac{1}{n}} = \inf \|k^n\|^{\frac{1}{n}}.$$

If $\rho(k) < |\lambda|$, then $\lambda e - k$ is invertible in $\mathcal{A}$, moreover,

$$(\lambda e - k)^{-1} = \sum_{i=0}^{\infty} \frac{k^i}{\lambda^{i+1}},$$

where λ is a complex constant.

Lemma 1.7 ([19]). *Let $\mathcal{A}$ be a Banach algebra with a unit e, $a, b \in \mathcal{A}$. If a commutes with b, then*

$$\rho(a + b) \leq \rho(a) + \rho(b), \qquad \rho(ab) \leq \rho(a)\rho(b).$$

Lemma 1.8 ([1]). *Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

2. Main results

In this section, firstly, inspired by Definition 1.1, we introduce a new concept called cone b-metric space over Banach algebra and then offer several examples to claim that it is an interesting improvement and increase of Definition 1.1. Secondly, we give some valuable lemmas in Banach algebras which will be used in the sequel. Thirdly, we prove several common fixed point theorems in cone b-metric spaces over Banach algebras instead of the theorems only in cone b-metric spaces with usual Banach spaces, which, to the best of our knowledge, are new. Otherwise, we highlight our results by two nontrivial examples.

Definition 2.1. Let X be a nonempty set and $s \geq 1$ be a constant. Suppose that the mapping $d : X \times X \to \mathcal{A}$ satisfies:
(i) $\theta \prec d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \preceq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.
Then d is called a cone b-metric on X, and (X, d) is called a cone b-metric space over Banach algebra $\mathcal{A}$.

Remark 2.2. Similar to Definition 1.2, we are easy to write the notions of convergent sequence, Cauchy sequence, and complete space in cone b-metric space over Banach algebra and therefore omit them.

Remark 2.3. The class of cone b-metric space over Banach algebra is larger than the class of cone metric space over Banach algebra since the latter must be the former, but the converse is not true. We can present many examples, as follows, which show that introducing a cone b-metric space over Banach algebra instead of a cone metric space over Banach algebra is very meaningful since there exist cone b-metric spaces over Banach algebras which are not cone metric spaces over Banach algebras.

Example 2.4. Let $\mathcal{A} = C[a, b]$ be the set of continuous functions on the interval $[a, b]$ with the supremum norm. Define multiplication in the usual way. Then $\mathcal{A}$ is a Banach algebra with a unit 1.

Set $P = \{x \in \mathcal{A} : x(t) \geq 0, t \in [a, b]\}$ and $X = \mathbb{R}$. Define a mapping $d : X \times X \to \mathcal{A}$ by $d(x, y)(t) = |x - y|^p e^t$ for all $x, y \in X$, where $p > 1$ is a constant. This makes (X, d) into a cone b-metric space over Banach algebra $\mathcal{A}$ with the coefficient $s = 2^{p-1}$, but it is not a cone metric space over Banach algebra since the triangle inequality is not satisfied.

Example 2.5. Let $\mathcal{A} = \{a = (a_{ij})_{3 \times 3} : a_{ij} \in \mathbb{R}, 1 \leq i, j \leq 3\}$ and $\|a\| = \frac{1}{3} \sum\limits_{1 \leq i, j \leq 3} | a_{ij} |$. Take a cone $P = \{a \in \mathcal{A} : a_{ij} \geq 0, 1 \leq i, j \leq 3\}$ in $\mathcal{A}$. Let $X = \{1, 2, 3\}$. Define a mapping $d : X \times X \to \mathcal{A}$ by $d(1, 1) = d(2, 2) = d(3, 3) = \theta$ and

$$d(1, 2) = d(2, 1) = \begin{pmatrix} 1 & 1 & 4 \\ 4 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad d(1, 3) = d(3, 1) = \begin{pmatrix} 4 & 1 & 4 \\ 4 & 3 & 5 \\ 2 & 3 & 1 \end{pmatrix}, \quad d(2, 3) = d(3, 2) = \begin{pmatrix} 9 & 5 & 6 \\ 16 & 4 & 4 \\ 3 & 4 & 2 \end{pmatrix}.$$

It ensures us that (X, d) is a cone b-metric space over Banach algebra $\mathcal{A}$ with the coefficient $s = \frac{5}{2}$, but it is not a cone metric space over Banach algebra since the triangle inequality is lacked.

Example 2.6. Let $X = l_p = \{x = (x_n)_{n \geq 1} : \sum\limits_{n=1}^{\infty} |x_n|^p < \infty\}$ $(0 < p < 1)$. Define a mapping $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \Big(\sum_{n=1}^{\infty} |x_n - y_n|^p \Big)^{\frac{1}{p}},$$

where $x = (x_n)_{n \geq 1}$, $y = (y_n)_{n \geq 1} \in l_p$. Clearly, (X, d) is a b-metric space.

Put $\mathcal{A} = l^1 = \{a = (a_n)_{n \geq 1} : \sum\limits_{n=1}^{\infty} |a_n| < \infty\}$ with convolution as multiplication:

$$ab = (a_n)_{n \geq 1}(b_n)_{n \geq 1} = \Big(\sum_{i+j=n} a_i b_j \Big)_{n \geq 1}.$$

It is valid that $\mathcal{A}$ is a Banach algebra with a unit $e = (1, 0, 0, \ldots)$. Choose a cone $P = \{a = (a_n)_{n \geq 1} \in \mathcal{A} : a_n \geq 0, \text{ for all } n \geq 1\}$. Define $\tilde{d} : X \times X \to \mathcal{A}$ by $\tilde{d}(x, y) = (\frac{d(x,y)}{2^n})_{n \geq 1}$, it may be verified that $(X, \tilde{d})$ is a cone b-metric space over Banach algebra $\mathcal{A}$ with the coefficient $s = 2^{\frac{1}{p}-1} > 1$, but it is not a cone metric space over Banach algebra since the triangle inequality does not hold.

Lemma 2.7. *Let $\mathcal{A}$ be a Banach algebra with a unit e and P be a solid cone in $\mathcal{A}$. Let $h \in \mathcal{A}$ and $u_n = h^n$. If $\rho(h) < 1$, then $\{u_n\}$ is a c-sequence.*

Proof. Since $\rho(h) = \lim\limits_{n \to \infty} \|h^n\|^{\frac{1}{n}} < 1$, then there exists $\alpha > 0$ such that $\lim\limits_{n \to \infty} \|h^n\|^{\frac{1}{n}} < \alpha < 1$. Letting n be big enough, we obtain $\|h^n\|^{\frac{1}{n}} \leq \alpha$, which implies that $\|h^n\| \leq \alpha^n \to 0$ $(n \to \infty)$. So $\|h^n\| \to 0$ $(n \to \infty)$, i.e., $\|u_n\| \to 0$ $(n \to \infty)$. Note that for each $c \gg \theta$, there is $\delta > 0$ such that

$$U(c, \delta) = \{x \in E : \|x - c\| < \delta\} \subset P.$$

In view of $\|u_n\| \to 0$ $(n \to \infty)$, then there exists N such that $\|u_n\| < \delta$ for all $n > N$. Consequently, $\|(c - u_n) - c\| = \|u_n\| < \delta$, this leads to $c - u_n \in U(c, \delta) \subset P$, that is, $c - u_n \in \text{int} P$, thus $u_n \ll c$ for all $n > N$. $\square$

Lemma 2.8. *Let $\mathcal{A}$ be a Banach algebra with a unit e and $k \in \mathcal{A}$. If λ is a complex constant and $\rho(k) < |\lambda|$, then*

$$\rho\big((\lambda e - k)^{-1}\big) \leq \frac{1}{|\lambda| - \rho(k)}.$$

Proof. Since $\rho(k) < |\lambda|$, it follows by Lemma 1.6 that $\lambda e - k$ is invertible and $(\lambda e - k)^{-1} = \sum_{i=0}^{\infty} \frac{k^i}{\lambda^{i+1}}$. Set $s = \sum_{i=0}^{\infty} \frac{k^i}{\lambda^{i+1}}$, $s_n = \sum_{i=0}^{n} \frac{k^i}{\lambda^{i+1}}$, then $s_n \to s$ $(n \to \infty)$ and s_n commutes with s for all n. It follows immediately from Lemma 1.7 that

$$\rho(s_n) = \rho(s_n - s + s) \le \rho(s - s_n) + \rho(s) \Rightarrow \rho(s_n) - \rho(s) \le \rho(s - s_n),$$
$$\rho(s) = \rho(s - s_n + s_n) \le \rho(s - s_n) + \rho(s_n) \Rightarrow \rho(s) - \rho(s_n) \le \rho(s - s_n),$$

which imply that

$$|\rho(s_n) - \rho(s)| \le \rho(s - s_n) \le \|s - s_n\| \Rightarrow \rho(s_n) \to \rho(s)(n \to \infty).$$

Thus again by Lemma 1.7,

$$\begin{aligned}
\rho\big((\lambda e - k)^{-1}\big) = \rho\Big(\sum_{i=0}^{\infty} \frac{k^i}{\lambda^{i+1}}\Big) &= \rho(s) = \lim_{n \to \infty} \rho(s_n) \\
&= \lim_{n \to \infty} \rho\Big(\sum_{i=0}^{n} \frac{k^i}{\lambda^{i+1}}\Big) \le \lim_{n \to \infty} \sum_{i=0}^{n} \frac{[\rho(k)]^i}{|\lambda|^{i+1}} \\
&= \sum_{i=0}^{\infty} \frac{[\rho(k)]^i}{|\lambda|^{i+1}} = \frac{1}{|\lambda| - \rho(k)}.
\end{aligned}$$

$\square$

Theorem 2.9. *Let (X, d) be a cone b-metric space over Banach algebra $\mathcal{A}$ with the coefficient $s \ge 1$ and P be a solid cone in $\mathcal{A}$. Let $k_i \in P$ $(i = 1, \ldots, 5)$ be generalized Lipschitz constants with $2s\rho(k_1) + (s+1)\rho(k_2 + k_3 + sk_4 + sk_5) < 2$. Suppose that k_1 commutes with $k_2 + k_3 + sk_4 + sk_5$ and the mappings $f, g : X \to X$ satisfy that*

$$d(fx, fy) \preceq k_1 d(gx, gy) + k_2 d(fx, gx) + k_3 d(fy, gy) + k_4 d(gx, fy) + k_5 d(fx, gy) \tag{2.1}$$

for all $x, y \in X$. If the range of g contains the range of f and $g(X)$ is a complete subspace, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. Since $f(X) \subset g(X)$, there exists an $x_1 \in X$ such that $fx_0 = gx_1$. By induction, a sequence $\{fx_n\}$ can be chosen such that $fx_n = gx_{n+1}(n = 0, 1, 2, \ldots)$. Thus, by (2.1), for any natural number n, on the one hand, we have

$$\begin{aligned}
d(gx_{n+1}, gx_n) = d(fx_n, fx_{n-1}) \\
\preceq k_1 d(gx_n, gx_{n-1}) + k_2 d(fx_n, gx_n) + k_3 d(fx_{n-1}, gx_{n-1}) \\
+ k_4 d(gx_n, fx_{n-1}) + k_5 d(fx_n, gx_{n-1}) \\
\preceq (k_1 + k_3 + sk_5)d(gx_n, gx_{n-1}) + (k_2 + sk_5)d(gx_{n+1}, gx_n),
\end{aligned}$$

which implies that

$$(e - k_2 - sk_5)d(gx_{n+1}, gx_n) \preceq (k_1 + k_3 + sk_5)d(gx_n, gx_{n-1}). \tag{2.2}$$

On the other hand, we have

$$\begin{aligned}
d(gx_n, gx_{n+1}) = d(fx_{n-1}, fx_n) \\
\preceq k_1 d(gx_{n-1}, gx_n) + k_2 d(fx_{n-1}, gx_{n-1}) + k_3 d(fx_n, gx_n) \\
+ k_4 d(gx_{n-1}, fx_n) + k_5 d(fx_{n-1}, gx_n) \\
\preceq (k_1 + k_2 + sk_4)d(gx_{n-1}, gx_n) + (k_3 + sk_4)d(gx_n, gx_{n+1}),
\end{aligned}$$

which means that
$$(e - k_3 - sk_4)d(gx_{n+1}, gx_n) \preceq (k_1 + k_2 + sk_4)d(gx_n, gx_{n-1}). \tag{2.3}$$

Add up (2.2) and (2.3) yields that
$$(2e - k_2 - k_3 - sk_4 - sk_5)d(gx_{n+1}, gx_n)(2k_1 + k_2 + k_3 + sk_4 + sk_5)d(gx_n, gx_{n-1}). \tag{2.4}$$

Denote $k_2 + k_3 + sk_4 + sk_5 = k$, then (2.4) yields that
$$(2e - k)d(gx_{n+1}, gx_n) \preceq (2k_1 + k)d(gx_n, gx_{n-1}). \tag{2.5}$$

Note that
$$2\rho(k) \leq (s+1)\rho(k) \leq 2s\rho(k_1) + (s+1)\rho(k) < 2$$

leads to $\rho(k) < 1 < 2$, then by Lemma 1.6 it follows that $2e - k$ is invertible. Furthermore,
$$(2e - k)^{-1} = \sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}.$$

By multiplying in both sides of (2.5) by $(2e - k)^{-1}$, we arrive at
$$d(gx_{n+1}, gx_n) \preceq (2e - k)^{-1}(2k_1 + k)d(gx_n, gx_{n-1}). \tag{2.6}$$

Denote $h = (2e - k)^{-1}(2k_1 + k)$, then by (2.6) we get
$$d(gx_{n+1}, gx_n) \preceq hd(gx_n, gx_{n-1}) \preceq \cdots \preceq h^n d(gx_1, gx_0) = h^n d(fx_0, gx_0).$$

Since k_1 commutes with k, it follows that
$$(2e - k)^{-1}(2k_1 + k) = \Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}} \Big)(2k_1 + k) = 2\Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}} \Big)k_1 + \sum_{i=0}^{\infty} \frac{k^{i+1}}{2^{i+1}}$$
$$= 2k_1\Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}} \Big) + k\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}$$
$$= (2k_1 + k)\Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}} \Big) = (2k_1 + k)(2e - k)^{-1},$$

that is to say, $(2e - k)^{-1}$ commutes with $2k_1 + k$. Then by Lemma 1.7 and Lemma 2.8 we gain
$$\rho(h) = \rho\big((2e - k)^{-1}(2k_1 + k)\big)$$
$$\leq \rho\big((2e - k)^{-1}\big)\rho(2k_1 + k)$$
$$\leq \frac{1}{2 - \rho(k)}[2\rho(k_1) + \rho(k)] < \frac{1}{s},$$

which establishes that $e - sh$ is invertible and $\|h^m\| \to 0 \ (m \to \infty)$. Hence, for any $m \geq 1$, $p \geq 1$ and $h \in P$ with $\rho(h) < \frac{1}{s}$, we have that
$$d(gx_m, gx_{m+p}) \preceq s[d(gx_m, gx_{m+1}) + d(gx_{m+1}, gx_{m+p})]$$
$$\preceq sd(gx_m, gx_{m+1}) + s^2[d(gx_{m+1}, gx_{m+2}) + d(gx_{m+2}, gx_{m+p})]$$
$$\preceq sd(gx_m, gx_{m+1}) + s^2 d(gx_{m+1}, gx_{m+2}) + s^3 d(gx_{m+2}, gx_{m+3})$$
$$+ \cdots + s^{p-1}d(gx_{m+p-2}, gx_{m+p-1}) + s^{p-1}d(gx_{m+p-1}, gx_{m+p})$$
$$\preceq sh^m d(fx_0, gx_0) + s^2 h^{m+1} d(fx_0, gx_0) + s^3 h^{m+2} d(fx_0, gx_0)$$
$$+ \cdots + s^{p-1}h^{m+p-2}d(fx_0, gx_0) + s^p h^{m+p-1}d(fx_0, gx_0)$$
$$= sh^m[e + sh + s^2 h^2 + \cdots + (sh)^{p-1}]d(fx_0, gx_0)$$
$$\preceq sh^m(e - sh)^{-1}d(fx_0, gx_0).$$

Taking advantage of Lemma 2.7 and Lemma 1.5, we get $\{gx_n\}$ is a Cauchy sequence. Since $g(X)$ is complete, there is $q \in g(X)$ such that $gx_n \to q$ $(n \to \infty)$. Thus there exists $p \in X$ such that $gp = q$. We shall prove $fp = q$. In order to end this, for one thing,

$$
\begin{aligned}
d(gx_n, fp) &= d(fx_{n-1}, fp) \\
&\preceq k_1 d(gx_{n-1}, gp) + k_2 d(fx_{n-1}, gx_{n-1}) + k_3 d(fp, gp) \\
&\quad + k_4 d(gx_{n-1}, fp) + k_5 d(fx_{n-1}, gp) \\
&\preceq k_1 d(gx_{n-1}, q) + sk_2 d(gx_n, q) + sk_2 d(gx_{n-1}, q) + sk_3 d(gx_n, fp) \\
&\quad + sk_3 d(gx_n, q) + sk_4[d(gx_{n-1}, gx_n) + d(gx_n, fp)] + k_5 d(gx_n, q) \\
&\preceq k_1 d(gx_{n-1}, q) + sk_2 d(gx_n, q) + sk_2 d(gx_{n-1}, q) \\
&\quad + sk_3 d(gx_n, fp) + sk_3 d(gx_n, q) + s^2 k_4 d(gx_{n-1}, q) \\
&\quad + s^2 k_4 d(gx_n, q) + sk_4 d(gx_n, fp) + k_5 d(gx_n, q),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
(e - sk_3 - sk_4)d(gx_n, fp) &\preceq (k_1 + sk_2 + s^2 k_4)d(gx_{n-1}, q) \\
&\quad + (sk_2 + sk_3 + s^2 k_4 + k_5)d(gx_n, q). \tag{2.7}
\end{aligned}
$$

For another thing,

$$
\begin{aligned}
d(gx_n, fp) &= d(fx_{n-1}, fp) = d(fp, fx_{n-1}) \\
&\preceq k_1 d(gp, gx_{n-1}) + k_2 d(fp, gp) + k_3 d(fx_{n-1}, gx_{n-1}) \\
&\quad + k_4 d(gp, fx_{n-1}) + k_5 d(fp, gx_{n-1}) \\
&\preceq k_1 d(gx_{n-1}, q) + sk_2 d(gx_n, fp) + sk_2 d(gx_n, q) \\
&\quad + sk_3 d(gx_n, q) + sk_3 d(gx_{n-1}, q) + k_4 d(gx_n, q) \\
&\quad + sk_5[d(fp, gx_n) + d(gx_n, gx_{n-1})] \\
&\preceq k_1 d(gx_{n-1}, q) + sk_2 d(gx_n, fp) + sk_2 d(gx_n, q) \\
&\quad + sk_3 d(gx_n, q) + sk_3 d(gx_{n-1}, q) + k_4 d(gx_n, q) \\
&\quad + s^2 k_5 d(gx_{n-1}, q) + s^2 k_5 d(gx_n, q) + sk_5 d(gx_n, fp),
\end{aligned}
$$

which means that

$$
\begin{aligned}
(e - sk_2 - sk_5)d(gx_n, fp) &\preceq (k_1 + sk_3 + s^2 k_5)d(gx_{n-1}, q) \\
&\quad + (sk_2 + sk_3 + k_4 + s^2 k_5)d(gx_n, q). \tag{2.8}
\end{aligned}
$$

Combine (2.7) and (2.8), it follows that

$$
\begin{aligned}
(2e - sk)d(gx_n, fp) &\preceq (2e - sk_2 - sk_3 - sk_4 - sk_5)d(gx_n, fp) \\
&\preceq (2k_1 + sk)d(gx_{n-1}, q) \\
&\quad + (sk_2 + sk_3 + k_4 + k_5 + sk)d(gx_n, q). \tag{2.9}
\end{aligned}
$$

Now that

$$
\rho(sk) = s\rho(k) \le (s+1)\rho(k) \le 2s\rho(k_1) + (s+1)\rho(k) < 2,
$$

thus by Lemma 1.6, it concludes that $2e - sk$ is invertible. As a result, it follows immediately from (2.9) that

$$
d(gx_n, fp) \preceq (2e - sk)^{-1}[(2k_1 + sk)d(gx_{n-1}, q) + (sk_2 + sk_3 + k_4 + k_5 + sk)d(gx_n, q)].
$$

Since $\{d(gx_n, q)\}$ and $\{d(gx_{n-1}, q)\}$ are c-sequences, then by Lemma 1.5, we acquire that $\{d(gx_n, fp)\}$ is a c-sequence, thus $gx_n \to fp$ ($n \to \infty$). Hence $fp = gp = q$. In the following we shall prove f and g have a unique point of coincidence.

If there exists $p' \neq p$ such that $fp' = gp'$. Then we get

$$
\begin{aligned}
d(gp', gp) &= d(fp', fp) \\
&\preceq k_1 d(gp', gp) + k_2 d(fp', gp') + k_3 d(fp, gp) \\
&\quad + k_4 d(gp', fp) + k_5 d(fp', gp) \\
&= (k_1 + k_4 + k_5)d(gp', gp)
\end{aligned}
$$

Set $\alpha = k_1 + k_4 + k_5$, then it follows that

$$d(gp', gp) \preceq \alpha d(gp', gp) \preceq \cdots \preceq \alpha^n d(gp', gp). \tag{2.10}$$

Because of

$$2\rho(k_1) + 2\rho(k) \leq 2s\rho(k_1) + (s+1)\rho(k) < 2,$$

it follows that $\rho(k_1) + \rho(k) < 1$. Since k_1 commutes with k, then by Lemma 1.7,

$$\rho(k_1 + k) \leq \rho(k_1) + \rho(k) < 1.$$

Accordingly, by Lemma 2.7, we speculate that $\{(k_1 + k)^n\}$ is a c-sequence. Noticing that $\alpha \preceq k_1 + k$ leads to $\alpha^n \preceq (k_1 + k)^n$, we claim that $\{\alpha^n\}$ is a c-sequence. Consequently, in view of (2.10), it is easy to see $d(gp', gp) = \theta$, that is, $gp' = gp$.

Finally, if (f, g) is weakly compatible, then by using Lemma 1.8, we claim that f and g have a unique common fixed point. $\square$

Corollary 2.10. *Let (X, d) be a cone b-metric space over Banach algebra $\mathcal{A}$ with the coefficient $s \geq 1$ and P be a solid cone in $\mathcal{A}$. Let $k \in P$ be a generalized Lipschitz constant with $\rho(k) < \frac{1}{s}$. Suppose that the mappings $f, g : X \to X$ satisfy that*

$$d(fx, fy) \preceq kd(gx, gy)$$

for all $x, y \in X$. If the range of g contains the range of f and $g(X)$ is a complete subspace, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Choose $k_1 = k$ and $k_2 = k_3 = k_4 = k_5 = \theta$ in Theorem 2.9, the proof is valid. $\square$

Corollary 2.11. *Let (X, d) be a cone b-metric space over Banach algebra $\mathcal{A}$ with the coefficient $s \geq 1$ and P be a solid cone in $\mathcal{A}$. Let $k \in P$ be a generalized Lipschitz constant with $\rho(k) < \frac{1}{s+1}$. Suppose that the mappings $f, g : X \to X$ satisfy that*

$$d(fx, fy) \preceq k[d(fx, gx) + d(fy, gy)]$$

for all $x, y \in X$. If the range of g contains the range of f and $g(X)$ is a complete subspace, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Putting $k_1 = k_4 = k_5 = \theta$ and $k_2 = k_3 = k$ in Theorem 2.9, we complete the proof. $\square$

Corollary 2.12. *Let (X, d) be a cone b-metric space over Banach algebra $\mathcal{A}$ with the coefficient $s \geq 1$ and P be a solid cone in $\mathcal{A}$. Let $k \in P$ be a generalized Lipschitz constant with $\rho(k) < \frac{1}{s^2 + s}$. Suppose that the mappings $f, g : X \to X$ satisfy that*

$$d(fx, fy) \preceq k[d(fx, gy) + d(fy, gx)]$$

for all $x, y \in X$. If the range of g contains the range of f and $g(X)$ is a complete subspace, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Set $k_1 = k_2 = k_3 = \theta$ and $k_4 = k_5 = k$ in Theorem 2.9, the claim holds. $\qquad\square$

Corollary 2.13. *Let (X, d) be a cone b-metric space and P be a solid cone in Banach space E. Let k_i $(i = 1, \ldots, 5)$ be some nonnegative real constants with $2sk_1 + (s + 1)(k_2 + k_3 + sk_4 + sk_5) < 2$. Suppose that the mappings $f, g : X \to X$ satisfy that*

$$d(fx, fy) \preceq k_1 d(gx, gy) + k_2 d(fx, gx) + k_3 d(fy, gy) + k_4 d(gx, fy) + k_5 d(fx, gy)$$

for all $x, y \in X$. If the range of g contains the range of f and $g(X)$ is a complete subspace, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Taking $k_1, k_2, k_3, k_4, k_5 \in \mathbb{R}^+$ in Theorem 2.9, we obtain the desired result. $\qquad\square$

Remark 2.14. Theorem 2.9 and its corollaries dismiss the normality of cones, which may bring us more convenience in applications. This is because the cones in our main results may be not necessarily normal cones. Indeed, there exist numerous non-normal cones besides that the usual normal cones (see [11]).

Remark 2.15. Theorem 2.9 and its corollaries generalize and unify the main results of [21] and [20]. Indeed, obviously, Theorem 3.1, Theorem 3.3 and Theorem 3.2 in [21] are the special cases of these corollaries, respectively with $s = 1$, and $g = I_X$ is the identify mapping on X. Besides these facts, Theorem 2.1 in [20] is the special case of Corollary 2.13 with $s = 1$.

Example 2.16 (the case of a nonnormal cone). Let $X = [0, 1]$ and $\mathcal{A}$ be the set of all real valued functions on X which also have continuous derivatives on X with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and the usual multiplication. Let $P = \{x \in \mathcal{A} : x(t) \geq 0, t \in X\}$. It is clear that P is a nonnormal cone and $\mathcal{A}$ is a Banach algebra with a unit $e = 1$. Define a mapping $d : X \times X \to \mathcal{A}$ by

$$d(x, y)(t) = |x - y|^2 e^t.$$

We make a conclusion that (X, d) is a complete cone b-metric space over Banach algebra $\mathcal{A}$ with the coefficient $s = 2$. Now define the mappings $f, g : X \to X$ by

$$f(x) = \frac{x}{8}, \qquad g(x) = \frac{x}{2}.$$

Choose $k_1 = \frac{1}{8} + \frac{1}{8}t$, $k_2 = \frac{1}{12} + \frac{1}{12}t$ and $k_3 = \frac{1}{16} + \frac{1}{16}t$, $k_4 = k_5 = 0$. Simple calculations show that all conditions of Theorem 2.9 are satisfied. Therefore, 0 is the unique common fixed point of f and g.

Example 2.17 (the case of a normal cone). Let $\mathcal{A} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \middle| \alpha, \beta \in \mathbb{R} \right\}$, $\left\| \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \right\| = |\alpha| + |\beta|$. The multiplication is usual matrix multiplication. Then $\mathcal{A}$ is a Banach algebra with a usual unit. Choose $X = [0, 1]$, $P = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \middle| \alpha, \beta \geq 0 \right\}$. Letting

$$d(x, y) = \begin{pmatrix} |x - y|^2 & 2|x - y|^2 \\ 0 & |x - y|^2 \end{pmatrix}, \quad x, y \in X,$$

we deduce (X, d) is a complete cone b-metric space over $\mathcal{A}$ with the coefficient $s = 2$ and P is a normal solid cone. Define the mappings $f, g : X \to X$ by

$$fx = \frac{1}{4}x, \qquad g(x) = \frac{1}{4}x^2 + \frac{1}{2}x.$$

Set

$$k_1 = \begin{pmatrix} \frac{1}{4} & \frac{1}{3} \\ 0 & \frac{1}{4} \end{pmatrix}, \; k_2 = \begin{pmatrix} \frac{1}{12} & \frac{1}{8} \\ 0 & \frac{1}{12} \end{pmatrix}, \; k_3 = \begin{pmatrix} \frac{1}{12} & \frac{1}{10} \\ 0 & \frac{1}{12} \end{pmatrix}, \; k_4 = \begin{pmatrix} \frac{1}{24} & \frac{1}{12} \\ 0 & \frac{1}{24} \end{pmatrix}, \; k_5 = \begin{pmatrix} \frac{1}{25} & \frac{1}{5} \\ 0 & \frac{1}{25} \end{pmatrix}.$$

It should be noticed that

$$d(fx, fy) \preceq k_1 d(gx, gy) + k_2 d(fx, gx) + k_3 d(fy, gy) + k_4 d(gx, fy) + k_5 d(fx, gy)$$

for all $x, y \in X$. Simple calculations show that all conditions of Theorem 2.9 hold. Accordingly, f and g have a unique common fixed point $x = 0$ in X.

3. Applications

In this section, we shall apply the obtained conclusions to deal with the existence and uniqueness of solution for some equations.

First of all, we refer to the following coupled equations:

$$\begin{cases} F(x, y) = 0, \\ G(x, y) = 0. \end{cases} \tag{3.1}$$

Where $F, G : \mathbb{R}^2 \to \mathbb{R}$ are two mappings.

Theorem 3.1. *For* (3.1), *if there exists* $0 < L < 1$ *such that for all the pairs* $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, *it satisfies that*

$$|F(x_1, y_1) - F(x_2, y_2) + x_1 - x_2| \leq L|x_1 - x_2|,$$
$$|G(x_1, y_1) - G(x_2, y_2) + y_1 - y_2| \leq L|y_1 - y_2|.$$

Then the coupled equation (3.1) *has a unique common solution in* $\mathbb{R}^2$.

Proof. Let $\mathcal{A} = \mathbb{R}^2$ with the norm $\|(u_1, u_2)\| = |u_1| + |u_2|$ and the multiplication by

$$uv = (u_1, u_2)(v_1, v_2) = (u_1 v_1, u_1 v_2 + u_2 v_1).$$

Let $P = \{u = (u_1, u_2) \in \mathcal{A} : u_1, u_2 \geq 0\}$. It is clear that P is a normal cone and $\mathcal{A}$ is a Banach algebra with a unit $e = (1, 0)$. Put $X = \mathbb{R}^2$ and define a mapping $d : X \times X \to \mathcal{A}$ by

$$d((x_1, y_1), (x_2, y_2)) = (|x_1 - x_2|, |y_1 - y_2|).$$

It is easy to see that (X, d) is a complete cone b-metric space over Banach algebra $\mathcal{A}$ with the coefficient $s = 1$. Now define the mappings $S, T : X \to X$ by

$$S(x, y) = (x, y), \qquad T(x, y) = (F(x, y) + x, G(x, y) + y).$$

Then

$$\begin{aligned} d(T(x_1, y_1), T(x_2, y_2)) &= d\big((F(x_1, y_1) + x_1, G(x_1, y_1) + y_1), \\ &\qquad (F(x_2, y_2) + x_2, G(x_2, y_2) + y_2)\big) \\ &= \big(|F(x_1, y_1) - F(x_2, y_2) + x_1 - x_2|, \\ &\qquad |G(x_1, y_1) - G(x_2, y_2) + y_1 - y_2|\big) \\ &\preceq \big(L|x_1 - x_2|, L|y_1 - y_2|\big) \\ &\preceq (L, 1)(|x_1 - x_2|, |y_1 - y_2|) \\ &= (L, 1)d(S(x_1, y_1), S(x_2, y_2)). \end{aligned}$$

Since

$$\|(L,1)^n\|^{\frac{1}{n}} = \|(L^n, nL^{n-1})\|^{\frac{1}{n}} = (L^n + nL^{n-1})^{\frac{1}{n}} \to L < 1 \quad (n \to \infty),$$

then $\rho((L,1)) < 1$. Now choose $k_1 = (L,1)$ and $k_2 = k_3 = k_4 = k_5 = \theta$, then all conditions of Theorem 2.9 are satisfied. Hence, by Theorem 2.9, S and T have a unique common fixed point in X. In other words, the coupled equation (3.1) has a unique common solution in $\mathbb{R}^2$.

Secondly, we shall study the existence of solution to a class of system of nonlinear integral equations. We consider the following system of integral equations

$$\begin{cases} x(t) = \int_a^t f(s, x(s)) \mathrm{d}s, \\ x(t) = \int_a^t x(s) \mathrm{d}s. \end{cases} \tag{3.2}$$

Where $t \in [a,b]$ and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. $\qquad\square$

Theorem 3.2. *Let $L_p[a,b] = \{x = x(t) : \int_a^b |x(t)|^p \mathrm{d}t < \infty\}$ $(0 < p < 1)$. For (3.2), assume that the following hypotheses hold:*
   (i) if $f(s, x(s)) = x(s)$ for all $s \in [a,b]$, then

$$f\left(s, \int_a^s x(w) \mathrm{d}w\right) = \int_a^s f(w, x(w)) \mathrm{d}w$$

for all $s \in [a,b]$.
   (ii) if there exists a constant $M \in (0, 2^{1-\frac{1}{p}}]$ such that the partial derivative f_y of f with respect to y exists and $|f_y(x,y)| \leq M$ for all the pairs $(x,y) \in [a,b] \times \mathbb{R}$.
Then the integral equation (3.2) has a unique common solution in $L_p[a,b]$.

Proof. Let $\mathcal{A} = \mathbb{R}^2$ with the norm $\|(u_1, v_2)\| = |u_1| + |u_2|$ and the multiplication by

$$uv = (u_1, u_2)(v_1, v_2) = (u_1 v_1, u_1 v_2 + u_2 v_1).$$

Let $P = \{u = (u_1, u_2) \in \mathcal{A} : u_1, u_2 \geq 0\}$. It is clear that P is a normal cone and $\mathcal{A}$ is a Banach algebra with a unit $e = (1,0)$. Let $X = L_p[a,b]$. We endow X with the cone b-metric

$$d(x,y) = \left(\left\{ \int_a^b |x(t) - y(t)|^p \mathrm{d}t \right\}^{\frac{1}{p}}, \left\{ \int_a^b |x(t) - y(t)|^p \mathrm{d}t \right\}^{\frac{1}{p}} \right)$$

for all $x, y \in X$. It is clear that (X, d) is a complete cone b-metric space over Banach algebra $\mathcal{A}$ with the coefficient $s = 2^{\frac{1}{p}-1}$. Define the mappings $S, T : X \to X$ by

$$Sx(t) = \int_a^t x(s) \mathrm{d}s, \qquad Tx(t) = \int_a^t f(s, x(s)) \mathrm{d}s$$

for all $t \in [a,b]$. Then the existence of a solution to (3.2) is equivalent to the existence of of common fixed point of S and T. Indeed,

$$d(Tx, Ty) = \left(\left\{ \int_a^b \left| \int_a^t f(s, x(s)) \mathrm{d}s - \int_a^t f(s, y(s)) \mathrm{d}s \right|^p \mathrm{d}t \right\}^{\frac{1}{p}}, \right.$$
$$\left. \left\{ \int_a^b \left| \int_a^t f(s, x(s)) \mathrm{d}s - \int_a^t f(s, y(s)) \mathrm{d}s \right|^p \mathrm{d}t \right\}^{\frac{1}{p}} \right)$$

$$= \left(\left\{ \int_a^b \left| \int_a^t [f(s,x(s)) - f(s,y(s))]\mathrm{d}s \right|^p \mathrm{d}t \right\}^{\frac{1}{p}}, \right.$$

$$\left. \left\{ \int_a^b \left| \int_a^t [f(s,x(s)) - f(s,y(s))]\mathrm{d}s \right|^p \mathrm{d}t \right\}^{\frac{1}{p}} \right)$$

$$\preceq \left(M \left\{ \int_a^b \left| \int_a^t [x(s) - y(s)]\mathrm{d}s \right|^p \mathrm{d}t \right\}^{\frac{1}{p}}, M \left\{ \int_a^b \left| \int_a^t [x(s) - y(s)]\mathrm{d}s \right|^p \mathrm{d}t \right\}^{\frac{1}{p}} \right)$$

$$= (M,0) \left(\left\{ \int_a^b |Sx(t) - Sy(t)|^p \mathrm{d}t \right\}^{\frac{1}{p}}, \left\{ \int_a^b |Sx(t) - Sy(t)|^p \mathrm{d}t \right\}^{\frac{1}{p}} \right)$$

$$= (M,0)d(Sx, Sy).$$

Because

$$\|(M,0)^n\|^{\frac{1}{n}} = \|(M^n,0)\|^{\frac{1}{n}} \to M < 2^{1-\frac{1}{p}} \quad (n \to \infty),$$

which means $\rho((M,0)) < 2^{1-\frac{1}{p}}$. Now choose $k_1 = (M,0)$ and $k_2 = k_3 = k_4 = k_5 = \theta$. Note that by (i), it is easy to see that the mappings S and T are weakly compatible. Therefore, all conditions of Theorem 2.9 are satisfied. As a result, S and T have a unique common fixed point $x^* \in X$. That is, x^* is the unique common solution of the system of integral equation (3.2). $\square$

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