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Infinitely many large energy solutions for Schrödinger-Kirchhoff type problem in $\mathbf{R}^N$

Bitao Cheng[a,b,*], Xianhua Tang[b]

[a]School of Mathematics and Information Science, Qujing Normal University, Qujing, Yunnan 655011, P. R. China.
[b]School of Mathematics and Statistics, Central South University, Changsha, Hunan, 410083, P. R. China.

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Abstract

In this paper, we consider the following Schrödinger-Kirchhoff-type problem

$$\begin{cases} -\left(a + b\int_{R^N} |\nabla u|^2 dx\right) \triangle u + V(x)u = g(x,u), & \text{for } x \in R^N, \\ u(x) \to 0, & \text{as } |x| \to \infty, \end{cases} \tag{1.1}$$

where constants $a > 0, b \geq 0$, $N = 1, 2$ or 3, $V \in C(R^N, R)$, $g \in C(R^N \times R, R)$. Under more relaxed assumptions on $g(x,u)$, by using some special techniques, a new existence result of infinitely many energy solutions is obtained via Symmetric Mountain Pass Theorem.

Keywords: Schrödinger-Kirchhoff type problem, critical point, symmetric Mountain Pass Theorem, variational methods.
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1. Introduction and main results

In this paper, we consider the following Schrödinger-Kirchhoff type problem

$$\begin{cases} -\left(a + b\int_{R^N} |\nabla u|^2 dx\right) \triangle u + V(x)u = g(x,u), & \text{for } x \in R^N, \\ u(x) \to 0, & \text{as } |x| \to \infty, \end{cases} \tag{1.1}$$

where constants $a > 0, b \geq 0$, $N = 1, 2$ or 3, $V \in C(R^N, R)$ and $g \in C(R^N \times R, R)$ satisfy some further conditions.

*Corresponding author
Email addresses: chengbitao2006@126.com (Bitao Cheng), tangxh@csu.edu.cn (Xianhua Tang)

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We note that when $a = 1$, $b = 0$, the problem (1.1) reduces to the following semilinear Schrödinger equation

$$\begin{cases} -\triangle u + V(x)u = g(x,u), & \text{for } x \in R^N, \\ u(x) \to 0, & \text{as } |x| \to \infty, \end{cases} \tag{1.2}$$

which has been studied extensively by many authors, and there is a large body of literature on the existence and multiplicity of solutions for the equation (1.2), for example, we refer the reader to [1, 21, 22] and references therein.

When $V \equiv 0$ and R^N is replaced by a bounded domain $\Omega \subset R^N$, the problem (1.1) reduces to the following nonlocal Kirchhoff type problem

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2 dx)\triangle u = g(x,u), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

The problem (1.3) is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \left(a + b\int_{R^N} |\nabla u|^2 dx\right)\triangle u = g(x,t), \tag{1.4}$$

which was proposed by Kirchhoff [13] as a model given by the equation of elastic strings

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0. \tag{1.5}$$

The equation (1.5) is an extension of the classical D'Alembert's wave equation by taking into account the changes in the length of the string during the transverse vibration.

It was pointed out in [9] that Kirchhoff type problem (1.3) models several physical and biological systems, where u describes a process which depends on the average of itself (for example, population density). Moreover, a lot of interesting studies by variational methods can be found in [2, 6, 7, 8, 10, 15, 16, 18, 20, 27] for Kirchhoff type problem (1.3) on bounded domain with several growth conditions on g.

Recently, Kirchhoff type problems setting on the unbounded domain or the whole space R^N have also attracted a lot of attention. Many solvability conditions on the nonlinearity have been given to obtain the existence and multiplicity of solutions for Kirchhoff type problems in R^N, we refer the readers to [3, 4, 11, 12, 14, 17, 23, 24, 25, 26] and references therein. Particularly, Wu obtained four results of the existence of a sequence of high energy solutions for the problem (1.1) by means of symmetric mountain pass theorem in [25]. Those results had been subsequently unified and improved by Y. Ye and C. Tang with the aid of fountain theorem in [26].

Motivated by the works mentioned above, in the present paper, under more relaxed assumptions on the nonlinear term g, we will present a new proof technique to construct infinitely many large energy solutions for the problem (1.1).

In order to reduce the statements of our result, we make the following assumptions.

(V_1) $V \in C(R^N, R)$ satisfies $\inf V(x) \geq V_0 > 0$ and for each $M > 0$, $meas\{x \in R^N : V(x) \leq M\} < +\infty$, where V_0 is a constant and $meas$ denote the Lebesgue measure in R^N.

(g_1) There exist $C_1 > 0$ and $p \in (4, 2^*)$ such that

$$|g(x,t)| \leq C_1(|t| + |t|^{p-1}), \forall (x,t) \in R^N \times R.$$

(g_2) $\frac{G(x,t)}{t^4} \to +\infty$ as $|t| \to +\infty$ uniformly in $x \in R^N$, $G(x,t) = \int_0^t g(x,s)ds$.

(g_3) There exists $L > 0$ such that

$$4G(x,t) - g(x,t)t \leq d|t|^2, \quad \text{for a.e. } x \in R^N \text{ and } \forall |t| \geq L,$$

where $0 \leq d \leq \frac{V_0}{2}$.

(g_4) $g(x,-t) = -g(x,t)$ for all $(x,t) \in R^N \times R$.

Next, we give some notations. Define the function space

$$H^1(R^N) = \{u \in L^2(R^N) : \nabla u \in L^2(R^N)\}$$

with the norm

$$\|u\|_{H^1} = \left(\int_{R^N} \left(|\nabla u|^2 + u^2\right) dx\right)^{\frac{1}{2}}.$$

Denote

$$E = \left\{u \in H^1(R^N) : \int_{R^N} \left(|\nabla u|^2 + V(x)u^2\right) dx < +\infty\right\}$$

with the inner product and the norm

$$\langle u, v\rangle_E = \int_{R^N} (\nabla u \cdot \nabla v + V(x)uv)dx, \quad \|u\|_E = \langle u, u\rangle_E^{\frac{1}{2}}.$$

Obviously, under the assumption (V_1) on $V(x)$, the following embedding

$$E \hookrightarrow L^s(R^N), \quad 2 \le s \le 2^*$$

is continuous. Hence, for any $s \in [2, 2^*]$, there is a constant $a_s > 0$ such that

$$\|u\|_{L^s} \le a_s\|u\|_E. \tag{1.6}$$

It is well known that a weak solution for the problem (1.1) is a critical point of the following functional I defined on E by

$$I(u) = \frac{a}{2}\int_{R^N} |\nabla u|^2dx + \frac{b}{4}\left(\int_{R^N} |\nabla u|^2dx\right)^2 + \frac{1}{2}\int_{R^N} V(x)u^2dx - \int_{R^N} G(x, u)dx \tag{1.7}$$

for all $u \in E$. We say that a weak solutions sequence $\{u_n\} \subset E$ for the problem (1.1) is a large energy solutions sequence if the energy $I(u_n) \to +\infty$ *as* $n \to \infty$.

Now, we can state our result as follows.

Theorem 1.1. *Assume that* (V_1) *and* $(g_1) - (g_4)$ *hold. Then the problem* (1.1) *possesses infinitely many large energy solutions in* E.

Remark 1.2. (i) Since the problem (1.1) is defined in R^N which is unbounded, the lack of compactness of the Sobolev embedding becomes more delicate by using variational techniques. To overcome the lack of compactness, the condition (V_1), which was first introduced by Bartsch and Wang in [5], is always assumed to preserve the compactness of embedding of the working space.

(ii) From Remark 1 in [26], the condition (g_1) is much weaker than the combination of usual subcritical condition and asymptotically linear condition near zero. Furthermore, condition (g_3) is much weaker than the following condition:

(g_3') There exists $L > 0$ such that

$$tg(x, t) - 4G(x, t) \ge 0, \quad \text{for a.e. } x \in R^N \text{ and } \forall |t| \ge L,$$

which was used in Theorem 5 in [26]. Hence, Theorem 1.1 improves Theorem 5 in [26].

2. Some lemmas

In order to apply variational techniques, we first state the key compactness result.

Lemma 2.1 ([28], Lemma 3.4)**.** *Under the assumption* (V_1)*, the embedding*

$$E \hookrightarrow L^s(R^N), \quad 2 \le s < 2^*$$

is compact.

The following lemma has been proved by Lemma 1 in [26].

Lemma 2.2. *Let assumptions* (V_1) *and* (g_1) *hold. Then* I *is well defined on* E, $I \in C^1(E, R)$ *and for any* $u,\ v \in E$,

$$\langle I'(u), v\rangle = \left(a + b\int_{R^N} |\nabla u|^2 dx\right)\int_{R^N} \nabla u\nabla v dx + \int_{R^N} V(x)uvdx - \int_{R^N} g(x,u)vdx. \tag{2.1}$$

Moreover, $\Psi' : E \to E^*$ *is compact, where* $\Psi(u) = \int_{R^N} G(x,u)dx$.

Recall that we say I satisfies the (PS) condition at the level $c \in R$ $\ ((PS)_c$ condition for short) if any sequence $\{u_n\} \subset E$ along with $I(u_n) \to c$ *and* $I'(u_n) \to 0$ *as* $n \to \infty$ possesses a convergent subsequence. If I satisfies $(PS)_c$ condition for each $c \in R$, then we say that I satisfies the (PS) condition.

Lemma 2.3. *Let assumption* (V_1) *and* (g_1) *hold. Then any bounded Palais-Smale sequence of* I *has a strongly convergent subsequence in* E.

Proof. Let $\{u_n\} \subset E$ be any bounded Palais-Smale sequence of I, then, up to a subsequence, there exist $c_1 \in R$ such that

$$I(u_n) \to c_1, \quad I'(u_n) \to 0 \ \text{ and } \ \sup_n \|u_n\|_E < +\infty. \tag{2.2}$$

Since the embedding

$$E \hookrightarrow L^s(R^N), \qquad 2 \le s < 2^*$$

is compact, going if necessary to a subsequence, we can assume that there is a $u \in E$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } E; \\ u_n \to u, & \text{strongly in } L^s(R^N); \\ u_n(x) \to u(x), & \text{a.e.in } R^N. \end{cases} \tag{2.3}$$

In view of (2.1), it has

$$\begin{aligned}
&\langle I'(u_n) - I'(u), u_n - u\rangle \\
&= \left(a + b\int_{R^N} |\nabla u_n|^2 dx\right)\int_{R^N} \nabla u_n \cdot \nabla(u_n - u)dx + \int_{R^N} V(x)|u_n - u|^2 dx \\
&\quad - \left(a + b\int_{R^N} |\nabla u|^2 dx\right)\int_{R^N} \nabla u \cdot \nabla(u_n - u)dx - \int_{R^N}[g(x,u_n) - g(x,u)](u_n - u)dx \\
&= \left(a + b\int_{R^N} |\nabla u_n|^2 dx\right)\int_{R^N} |\nabla(u_n - u)|^2 dx + \int_{R^N} V(x)|u_n - u|^2 dx \\
&\quad - b\left(\int_{R^N} |\nabla u|^2 dx - \int_{R^N} |\nabla u_n|^2 dx\right)\int_{R^N} \nabla u \cdot \nabla(u_n - u)dx - \int_{R^N}[g(x,u_n) - g(x,u)](u_n - u)dx \\
&\ge \min\{a,1\}\|u_n - u\|_E^2 - b\left(\int_{R^N} |\nabla u|^2 dx - \int_{R^N} |\nabla u_n|^2 dx\right)\int_{R^N} \nabla u \cdot \nabla(u_n - u)dx \\
&\quad - \int_{R^N}[g(x,u_n) - g(x,u)](u_n - u)dx.
\end{aligned} \tag{2.4}$$

Then (2.4) implies that

$$\begin{aligned}
\min\{a,1\}\|u_n - u\|_E^2 \ \ &\le \langle I'(u_n) - I'(u_n), u_n - u\rangle \\
&\quad + b\left(\int_{R^N} |\nabla u|^2 dx - \int_{R^N} |\nabla u_n|^2 dx\right)\int_{R^N} \nabla u \cdot \nabla(u_n - u)dx \\
&\quad + \int_{R^N}[g(x,u_n) - g(x,u)](u_n - u)dx.
\end{aligned} \tag{2.5}$$

Define the functional h_u: $E \to R$ by

$$h_u(v) = \int_{R^N} \nabla u \cdot \nabla v dx, \qquad \forall\ v \in E.$$

Obviously, h_u is a linear functional on E. Furthermore,

$$|h_u(v)| \leq \int_{R^N} |\nabla u \cdot \nabla v| dx \leq \|u\|_E \|v\|_E,$$

which implies h_u is bounded on E. Hence $h_u \in E^*$. Since, $u_n \rightharpoonup u$ in E, it has $\lim\limits_{n\to\infty} h_u(u_n) = h_u(u)$, that is, $\int_{R^N} \nabla u \cdot \nabla(u_n - u)dx \to 0$ as $n \to \infty$. Consequently, by (2.3) and the boundedness of $\{u_n\}$, it has

$$b\left(\int_{R^N} |\nabla u|^2 dx - \int_{R^N} |\nabla u_n|^2 dx\right) \int_{R^N} \nabla u \cdot \nabla(u_n - u)dx \to 0\ , \quad n \to +\infty. \tag{2.6}$$

By (g_1), using the Hölder inequality, we can conclude

$$\begin{aligned} \left| \int_{R^N} [g(x,u_n) - g(x,u)](u_n - u)dx \right| &\leq C_1 \int_{R^N} [|u_n| + |u| + |u_n|^{p-1} + |u|^{p-1}]|u_n - u|dx \\ &\leq C_1(\|u_n\|_{L^2} + \|u\|_{L^2})\|u_n - u\|_{L^2} \\ &\quad + C_1(\|u_n\|_{L^p}^{p-1} + \|u\|_{L^p}^{p-1})\|u_n - u\|_{L^p}. \end{aligned}$$

Therefore, it follows from (2.6) that

$$\int_{R^N} [g(x,u_n) - g(x,u)](u_n - u)dx \to 0, \quad \text{as } n \to \infty. \tag{2.7}$$

Moreover, combining (2.5) with (2.6), then

$$< I'(u_n) - I'(u), u_n - u > \to 0, \quad \text{as } n \to \infty. \tag{2.8}$$

Consequently, (2.5)–(2.8) imply that

$$u_n \to u, \quad \text{strongly in } E \text{ as } n \to \infty.$$

This completes the proof. □

Lemma 2.4. *Let assumptions* (V_1), (g_1) *and* (g_3) *hold. Then any Palais-Smale sequence of* I *is bounded.*

Proof. Let $\{u_n\} \subset E$ be any Palais-Smale sequence of I, then, up to a subsequence, there exist $c_1 \in R$ such that

$$I(u_n) \to c_1, \quad \text{and} \quad I'(u_n) \to 0. \tag{2.9}$$

The combination of (1.6), (1.7), (2.1), (2.9), (V_1) with (g_3) implies

$$\begin{aligned} c_1 + 1 + \|u_n\|_E &\geq I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\ &= \frac{a}{4}\int_{R^N} |\nabla u_n|^2 dx + \frac{1}{4}\int_{R^N} V(x)u_n^2 dx + \int_{R^N} \widetilde{G}(x,u_n)dx \\ &\geq \frac{a}{4}\int_{R^N} |\nabla u_n|^2 dx + \frac{1}{4}\int_{R^N} V(x)u_n^2 dx - \frac{d}{4}\int_{R^N} u_n^2 dx + \int_{A_n} \widetilde{G}(x,u_n)dx \\ &\geq \frac{a}{4}\int_{R^N} |\nabla u_n|^2 dx + \frac{1}{4}\int_{R^N} V(x)u_n^2 dx - \frac{1}{8}\int_{R^N} V_0 u_n^2 dx + \int_{A_n} \widetilde{G}(x,u_n)dx \\ &\geq \frac{a}{4}\int_{R^N} |\nabla u_n|^2 dx + \frac{1}{4}\int_{R^N} V(x)u_n^2 dx - \frac{1}{8}\int_{R^N} V(x)u_n^2 dx + \int_{A_n} \widetilde{G}(x,u_n)dx \\ &\geq \frac{1}{16}\min\{a,1\}\|u_n\|_E^2 + \frac{1}{16}\int_{R^N} V(x)u_n^2 dx + \int_{A_n} \widetilde{G}(x,u_n)dx, \end{aligned} \tag{2.10}$$

where $\widetilde{G}(x,u_n)=\frac{1}{4}g(x,u_n)u_n-G(x,u_n)$ and $A_n=\{x\in R^N:\ |u_n|\le L\}$. For $x\in R^N$ and $|u_n|\le L$, by (g_1), it has

$$\begin{aligned}|\widetilde{G}(x,u_n)|&\le \frac{1}{4}|g(x,u_n)||u_n|+|G(x,u_n)|\\&\le \frac{1}{4}C_1(|u_n|^2+|u_n|^p)+C_1\Big(\frac{1}{2}|u_n|^2+\frac{1}{p}|u_n|^p\Big)\\&=C_1\Big(\frac{3}{4}+\frac{p+1}{p}|u_n|^{p-2}\Big)|u_n|^2\\&\le C_1\Big(\frac{3}{4}+\frac{p+1}{p}L^{p-2}\Big)|u_n|^2.\end{aligned}$$

Take $M>\max\Big\{16C_1\Big(\frac{3}{4}+\frac{p+1}{p}L^{p-2}\Big),V_0\Big\}$, then

$$\widetilde{G}(x,u_n)\ge -\frac{M}{16}|u_n|^2,\quad \forall\, x\in\ R^N,\ |u_n|\le L. \tag{2.11}$$

Let $\widetilde{A}=\{x\in R^N:\ V(x)\le M\}$. By (V_1) and (2.11), we can conclude

$$\begin{aligned}\frac{1}{16}\int_{R^N}V(x)u_n^2dx+\int_{A_n}\widetilde{G}(x,u_n)dx&\ge \frac{1}{16}\int_{|u_n|\le L}(V(x)-M)|u_n|^2dx\\&\ge \frac{1}{16}\int_{\widetilde{A}\cap A_n}(V(x)-M)L^2dx\\&\ge \frac{1}{16}(V_0-M)L^2meas(\widetilde{A}\cap A_n)\\&\ge \frac{1}{16}(V_0-M)L^2meas(\widetilde{A}).\end{aligned}\tag{2.12}$$

Note that $meas(\widetilde{A})<+\infty$ due to (V_1), it follows from (2.10) and (2.12) that

$$c_1+1+\|u_n\|_E\ge \frac{1}{16}\min\{a,1\}\|u_n\|_E^2+\frac{1}{16}(V_0-A)L^2meas(\widetilde{A}),$$

which implies $\{u_n\}\subset E$ is bounded in E. Hence the proof is completed. □

Remark 2.5. Comparing with [26], we present a new proof technique to verify the boundedness of Palais-Smale sequences, which is much clearer and simpler than them.

3. Proof of theorem 1.1

In this section we will use the classical Symmetric Mountain Pass Theorem of Rabinowitz instead of Fountain Theorem in [26] to obtain infinitely many large energy solutions for the problem (1.1) and prove Theorem 1.1. First of all, we state some notations.

In view of $E\hookrightarrow L^2(R^N)$ and $L^2(R^N)$ is a separable Hilbert space, then E has a countable orthogonal basis $\{e_j\}_{j=1}^{\infty}$. Let

$$E_k=\bigoplus_{j=1}^{k}X_j,\qquad Z_k=E_k^{\perp},$$

where $X_j=span\{e_j\}$. Thus, $E=E_k\bigoplus Z_k$ and E_k is finite dimensional.

Lemma 3.1. *Let the assumption* (V_1) *hold. Define*

$$\eta(k):=\sup_{u\in Z_k,\|u\|_E=1}\|u\|_{L^2},\qquad k\in\ N,$$

then there exists $k_0\in N$ *such that* $0<\eta(k_0)\le\Big(\frac{\min\{a,1\}}{4C_1}\Big)^{\frac{1}{2}}$.

Proof. Firstly, $\eta(k)$ is convergent science $\eta(k) \geq 0$ and $\eta(k)$ is decreasing in k. Furthermore, for any $k \in N$, by the definition of $\eta(k)$, there exists $u_k \in Z_k$ such that

$$\|u_k\|_E = 1 \quad \text{and} \quad \|u_k\|_{L^2} > \frac{\eta(k)}{2}. \tag{3.1}$$

For any $v \in E$, $v = \sum\limits_{n=1}^{\infty} a_n e_n$, it has

$$| < u_k, v >_E | = \Big| < u_k, \sum_{n=1}^{\infty} a_n e_n >_E \Big| \leq \|u_k\|_E \Big\| \sum_{n=k+1}^{\infty} a_n e_n \Big\|_E \leq \Big\| \sum_{n=k+1}^{\infty} a_n e_n \Big\|_E \to 0$$

as $k \to \infty$, which implies that $u_k \rightharpoonup 0$ weakly in E. By virtue of Lemma 2.1, we can conclude

$$u_k \to 0 \ \text{ strongly in } L^2(R^N). \tag{3.2}$$

The combination (3.1) with (3.2) implies that $\eta(k) \to 0$, as $k \to \infty$. Then there exists $k_0 \in N$ such that $0 < \eta(k_0) \leq \left(\frac{\min\{a,1\}}{4C_1}\right)^{\frac{1}{2}}$. Hence the proof is completed. □

Lemma 3.2. *Let assumptions* (V_1) *and* (g_1) *hold, then there exist some constants* ρ, α *such that* $I(u) \geq \alpha$ *whenever* $u \in Z_{k_0}$ *with* $\|u\|_E = \rho$.

Proof. For any $u \in Z_{k_0}$, by Lemma 3.1, we have

$$\|u\|_{L^2} \leq \eta(k_0)\|u\|_E \quad \text{and} \quad 0 < \eta(k_0) \leq \left(\frac{\min\{a,1\}}{4C_1}\right)^{\frac{1}{2}}. \tag{3.3}$$

By (1.6), (1.7), (g_1), (3.3) and Hölder inequality, it has

$$\begin{aligned}
I(u) &= \frac{a}{2}\int_{R^N} |\nabla u|^2 dx + \frac{b}{4}\left(\int_{R^N} |\nabla u|^2 dx\right)^2 \\
&\quad + \frac{1}{2}\int_{R^N} V(x)u^2 dx - \int_{R^N} G(x,u)dx \\
&\geq \frac{\min\{a,1\}}{2}\|u\|_E^2 - \int_{R^N} G(x,u)dx \\
&\geq \frac{\min\{a,1\}}{2}\|u\|_E^2 - C_1(\|u\|_{L^2}^2 + \|u\|_{L^p}^p) \\
&\geq \frac{\min\{a,1\}}{2}\|u\|_E^2 - C_1\eta^2(k_0)\|u\|_E^2 - C_1 a_p^p \|u\|_E^p \\
&\geq \|u\|_E \Big[\frac{\min\{a,1\}}{4}\|u\|_E - C_1 a_p^p \|u\|_E^{p-1}\Big].
\end{aligned}$$

Set

$$l(t) = \frac{\min\{a,1\}}{4} t - C_1 a_p^p t^{p-1}, \quad \forall t \geq 0.$$

Note that $4 < p < 2^*$, we can conclude that there exists a constant $\rho > 0$ such that

$$l(\rho) = \max_{t \geq 0} l(t) > 0.$$

Therefore,

$$I(u) \geq \rho l(\rho) =: \alpha > 0,$$

whenever $u \in Z_{k_0}$ with $\|u\|_E = \rho$. This completes the proof. □

Lemma 3.3. *Let assumptions $(g_1) - (g_2)$ hold, then for each finite dimensional subspace $\widetilde{E} \subset E$, there is an $r = r(\widetilde{E}) > 0$ such that $I|_{\widetilde{E}\setminus B_r} < 0$.*

Proof. With the aid of assumptions $(g_1) - (g_2)$, for any $K > 0$, there exists $C(K) > 0$ such that

$$G(x,z) \geq K|z|^4 - C(K)|z|^2, \qquad \forall\ (x,z) \in R^N \times R. \tag{3.4}$$

For any finite dimensional subspace $\widetilde{E} \subset E$, by the equivalence of norms in the finite dimensional space, there exists a constant $\beta_s > 0$ such that

$$\|u\|_{L^s} \geq \beta_s \|u\|_E, \qquad \forall\ u \in \widetilde{E}, \tag{3.5}$$

for $2 \leq s < 2^*$. Therefore, Choosing $K > 0$ such that $\frac{b}{4} - K\beta_4^4 < 0$, then the combination of (1.6)–(1.7) with (3.4)–(3.5) implies

$$\begin{aligned}
I(u) &= \frac{a}{2}\int_{R^N} |\nabla u|^2 dx + \frac{b}{4}\left(\int_{R^N} |\nabla u|^2 dx\right)^2 + \frac{1}{2}\int_{R^N} V(x)u^2 dx - \int_{R^N} G(x,u)dx \\
&\leq \frac{1}{2}\max\{a,1\}\|u\|_E^2 + \frac{b}{4}\|u\|_E^4 - K\|u\|_{L^4}^4 + C(K)\|u\|_{L^2}^2 \\
&\leq \frac{1}{2}\max\{a,1\}\|u\|_E^2 + \left(\frac{b}{4} - K\beta_4^4\right)\|u\|_E^4 + C(K)a_2^2\|u\|_E^2 \\
&\leq C_2\|u\|_E^2 - C_3\|u\|_E^4
\end{aligned}$$

for all $u \in \widetilde{E}$, where $C_2 = \frac{1}{2}\max\{a,1\} + C(K)a_2^2 > 0,\ C_3 = K\beta_4^4 - \frac{b}{4} > 0$. Hence there is an $r = r(\widetilde{E}) > 0$ such that $I|_{\widetilde{E}\setminus B_r} < 0$. This completes the proof. □

Next, we shall prove our Theorem 1.1. To begin with, for convenience to quote, we state the classical Symmetric Mountain Pass Theorem as in the following.

Theorem 3.4 ([19], Theorem 9.12)**.** *Let E be an infinite dimensional Banach space, $I \in C^1(E,R)$ be even and satisfy (PS) condition and $I(0) = 0$. If $E = Y \bigoplus Z$, Y is finite dimensional, and I satisfies*
(I_1) There exist constants $\rho,\ \alpha > 0$ such that $I|_{\partial B_\rho \cap Z} \geq \alpha$, and
(I_2) for each finite dimensional subspace $\widetilde{E} \subset E$, there is $r = r(\widetilde{E})$ such that $I \leq 0$ on $\widetilde{E} \setminus B_r$.
Then I possesses an unbounded sequence of critical values.

Proof of Theorem 1.1. The proof is to verify I satisfies all the conditions of Theorem 3.4. Set $Y = E_{k_0},\ Z = Z_{k_0}$, then $E = Y \bigoplus Z$ and Y is finite dimensional. First, I satisfies (I_1) and (I_2) in Theorem 3.4 by Lemma 3.2 and 3.3, respectively. Second, I satisfies (PS) condition by virtue of Lemma 2.3 and 2.4. Finally, $I(0) = 0$, I is even on E due to (g_4) and $I \in C^1(E,R)$ by Lemma 2.2. Hence, the conclusion follows from Theorem 3.4. The proof is completed. □

Remark 3.5. Comparing with Theorem 5 in [26], on one hand, the assumptions imposed on g are much weaker than them. On the other hand, we present a new proof technique to verify the boundedness of Palais-Smale sequences, and apply the classical Symmetric Mountain Pass Theorem of Rabinowitz instead of Fountain Theorem to obtain infinitely many large energy solutions for the problem (1.1). Hence, it is very different from them.

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