



Fixed point theorems for Ćirić type generalized contractions defined on cyclic representations

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Abstract

The purpose of this paper is to investigate the properties of some Ćirić type generalized contractions defined on cyclic representations in a metric space. ©2015 All rights reserved.

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1. Introduction

One of the consistent generalizations of the Banach Contraction Principle was given in 2003 by Kirk, Srinivasan and Veeramani [2], using the concept of cyclic operator. More precisely, they proved the following result.

Theorem 1.1 ([2]). *Let $\{A_i\}_{i=1}^m$ be nonempty subsets of a complete metric space and suppose $f : \bigcup_{i=1}^m A_i \rightarrow$*

$\bigcup_{i=1}^m A_i$ satisfies the following conditions (where $A_{m+1} = A_1$):

- (i) $f(A_i) \subseteq A_{i+1}$ for $1 \leq i \leq m$;
- (ii) $d(f(x), f(y)) \leq \psi(d(x, y))$, $\forall x \in A_i, \forall y \in A_{i+1}$, for $1 \leq i \leq m$, where $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is upper semi-continuous from the right and satisfies $0 \leq \varphi(t) < t$ for all $t > 0$.

Then f has a unique fixed point.

Our results generalize some similar theorems regarding Banach, Kannan, Bianchini, Reich, Sehgal, Chatterjea and Zamfirescu type operators (see [5], [6]), in the case of a cyclic condition (see [4]). Also, the main result is a generalization of Theorem 2.1 given by Păcurar and Rus in [3].

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2. Preliminaries

We present here some notions and results which will be used in our main section.

There are several conditions upon comparison functions that have been considered in literature. In order to study the convergence of the Picard iteration $\{x_n\}_{n \geq 0}$ defined by

$$x_n = f(x_{n-1}), \quad n \geq 1 \tag{2.1}$$

in this paper we shall refer to:

Definition 2.1 ([10]). A function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is called a comparison function if it satisfies:

- (i) $_{\varphi}$ φ is increasing;
- (ii) $_{\varphi}$ $\{\varphi^n(t)\}_{n \in \mathbb{N}}$ converges to 0 as $n \rightarrow \infty$, for all $t \in \mathbb{R}_+$.

If the condition (ii) $_{\varphi}$ is replaced by:

$$(iii)_{\varphi} \sum_{k=0}^{\infty} \varphi^k(t) < \infty, \text{ for any } t > 0,$$

then φ is called a strong comparison function.

Lemma 2.2 ([9]). *If $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a comparison function, then the following hold:*

- (i) $\varphi(t) < t$, for any $t > 0$;
- (ii) $\varphi(0) = 0$;
- (iii) φ is continuous at 0.

Lemma 2.3 ([3], [10]). *If $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a strong comparison function, then the following hold:*

- (i) φ is a comparison function;
- (ii) the function $s : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, defined by

$$s(t) = \sum_{k=0}^{\infty} \varphi^k(t), \quad t \in \mathbb{R}_+, \tag{2.2}$$

is increasing and continuous at 0;

- (iii) *there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that*

$$\varphi^{k+1}(t) \leq a\varphi^k(t) + v_k, \text{ for } k \geq k_0 \text{ and any } t \in \mathbb{R}_+.$$

Remark 2.4. Some authors use the notion of (c)-comparison function defined by the conditions (i) and (iii) from Lemma 2.3. Actually, the concept of (c)-comparison function coincides with that of strong comparison function.

Example 2.5. (1) $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\varphi(t) = at$, where $a \in [0, 1[$, is a strong comparison function.

(2) $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\varphi(t) = \frac{t}{1+t}$ is a comparison function, but is not a strong comparison function.

(3) $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\varphi(t) = \begin{cases} \frac{t}{2}, & t \in [0, 1] \\ t - \frac{1}{2}, & t > 1 \end{cases}$ is a strong comparison function.

(4) $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\varphi(t) = at + \frac{[t]}{2}$, where $a \in]0, \frac{1}{2}[$ is a strong comparison function.

For more considerations on comparison functions see [7], [9] and the references therein.

Now, let (X, d) be a metric space. $P(X)$ denotes the collection of nonempty subsets of X , and $P_{cl}(X)$ denotes the collection of nonempty and closed subsets of X . We recall the following notion, introduced in [8], suggested by the considerations in [2].

Definition 2.6. Let X be a nonempty set, m a positive integer and $f : X \rightarrow X$ an operator. By definition,

$\bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to f if

- (i) $X = \bigcup_{i=1}^m A_i$, with $A_i \in P(X)$, for $1 \leq i \leq m$;
- (ii) $f(A_i) \subseteq A_{i+1}$, for $1 \leq i \leq m$, where $A_{m+1} = A_1$.

3. Main results

We start this section by presenting the notion of cyclic φ -contraction of Ćirić type.

Definition 3.1. Let (X, d) be a metric space, m a positive integer, $A_1, \dots, A_m \in P_{cl}(X)$, $Y \in P(X)$ and $f : Y \rightarrow Y$ an operator. If

- (i) $\bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to f ;
- (ii) there exists a strong comparison function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$d(f(x), f(y)) \leq \varphi \left\{ \max\{d(x, y), d(x, f(x)), d(y, f(y)), \frac{1}{2}[d(x, f(y)) + d(y, f(x))]\} \right\},$$

for any $x \in A_i, y \in A_{i+1}$, where $A_{m+1} = A_1$,

then f is said to be a cyclic φ -contraction of Ćirić type.

The main result of this paper is the following.

Theorem 3.2. Let (X, d) be a complete metric space, m be a positive integer, $A_1, \dots, A_m \in P_{cl}(X)$, $Y \in P(X)$, $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a strong comparison function, and $f : Y \rightarrow Y$ be an operator. Assume that f is a cyclic φ -contraction of Ćirić type.

Then:

- (1) f has a unique fixed point $x^* \in \bigcap_{i=1}^m A_i$ and the Picard iteration $\{x_n\}_{n \geq 0}$ given by (2.1) converges to x^*

for any starting point $x_0 \in Y$;

- (2) the following estimates hold:

$$d(x_n, x^*) \leq s(\varphi^n(d(x_0, x_1))), \quad n \geq 1;$$

$$d(x_n, x^*) \leq s(d(x_n, x_{n+1})), \quad n \geq 1;$$

- (3) for any $x \in Y, d(x, x^*) \leq s(d(x, f(x)))$, where s is given by (2.2) in Lemma 2.3;

- (4) $\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty$, i.e. f is a good Picard operator;

- (5) $\sum_{n=0}^{\infty} d(x_n, x^*) < \infty$, i.e. f is a special Picard operator.

Proof. (1) Let $x_0 \in Y$, and $x_n = f(x_{n-1})$, for $n \geq 1$. Then we have

$$d(f(x_{n-1}), f(x_n)) \leq \varphi \left(\max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2}d(x_{n-1}, x_{n+1}) \right\} \right). \tag{3.1}$$

Using the triangle inequality,

$$\frac{1}{2}d(x_{n-1}, x_{n+1}) \leq \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$$

so (3.1) becomes

$$d(x_n, x_{n+1}) \leq \varphi(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}).$$

Supposing that there exists $p \in \mathbb{N}$, $p \geq 1$, such that $d(x_{p-1}, x_p) \leq d(x_p, x_{p+1})$, and taking into consideration that φ is a comparison function, from (3.1) we obtain

$$d(x_p, x_{p+1}) \leq \varphi(d(x_p, x_{p+1})) < d(x_p, x_{p+1}),$$

which is a contradiction.

It follows that $d(x_{n-1}, x_n) > d(x_n, x_{n+1})$, for any $n \geq 1$, thus (3.1) becomes

$$d(x_n, x_{n+1}) \leq \varphi(d(x_{n-1}, x_n)). \tag{3.2}$$

Using the monotonicity of φ , we get

$$d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, x_1)), \tag{3.3}$$

whence, for $p \geq 1$,

$$d(x_n, x_{n+p}) \leq \varphi^n(d(x_0, x_1)) + \varphi^{n+1}(d(x_0, x_1)) + \dots + \varphi^{n+p-1}(d(x_0, x_1)). \tag{3.4}$$

Denoting $S_n := \sum_{k=0}^n \varphi^k(d(x_0, x_1))$, we obtain

$$d(x_n, x_{n+p}) \leq S_{n+p-1} - S_{n-1}. \tag{3.5}$$

As φ is a strong comparison function,

$$\sum_{k=0}^{\infty} \varphi^k(d(x_0, x_1)) < \infty,$$

so there exists $S \in \mathbb{R}_+$ such that $\lim_{n \rightarrow \infty} S_n = S$.

By (3.5), $d(x_n, x_{n+p}) \rightarrow 0$ as $n \rightarrow \infty$, which means that $\{x_n\}_{n \geq 0}$ is a Cauchy sequence in the complete subspace Y , so it is convergent to some $p \in Y$.

The sequence $\{x_n\}_{n \geq 0}$ has an infinite number of terms in each A_i , $i = \overline{1, m}$, therefore from each A_i we can extract a subsequence of $\{x_n\}_{n \geq 0}$ which converges to $p = \lim_{n \rightarrow \infty} x_n$.

Because A_i are closed, it follows $p \in \bigcap_{i=1}^m A_i$, so $\bigcap_{i=1}^m A_i \neq \emptyset$. We consider the restriction

$$f \Big|_{\bigcap_{i=1}^m A_i} : \bigcap_{i=1}^m A_i \rightarrow \bigcap_{i=1}^m A_i.$$

$\bigcap_{i=1}^m A_i$ is also complete. Using Theorem 1.5.1 from [1], $f \Big|_{\bigcap_{i=1}^m A_i}$ has a unique fixed point x^* , which can be obtained by means of the Picard iteration starting from any initial point.

We still have to prove that the Picard iteration converges to x^* for any initial guess $x \in Y$. Note that

$$d(x_{n+1}, x^*) = d(f(x_n), f(x^*)) \leq \varphi \left(\max \left\{ d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, f(x^*)), \frac{1}{2} [d(x_n, f(x^*)) + d(x^*, x_{n+1})] \right\} \right).$$

If we denote $a_n = d(x_n, x^*)$ for all $n \in \mathbb{N}$, the above relation becomes

$$a_{n+1} \leq \varphi \left(\max \left\{ a_n, d(x_n, x_{n+1}), 0, \frac{1}{2} (a_n + a_{n+1}) \right\} \right).$$

Using the fact that $\frac{1}{2}(a_n + a_{n+1}) \leq \max\{a_n, a_{n+1}\}$, we get

$$a_{n+1} \leq \varphi(\max\{a_n, a_{n+1}, d(x_n, x_{n+1})\}).$$

But $\max\{a_n, a_{n+1}, d(x_n, x_{n+1})\} \neq a_{n+1}$, otherwise we would have $a_{n+1} \leq \varphi(a_{n+1})$, contradicting the assumption that $\varphi(t) < t$, for any $t > 0$. Consequently,

$$a_{n+1} \leq \varphi(\max\{a_n, d(x_n, x_{n+1})\}), \text{ for any } n \in \mathbb{N}. \tag{3.6}$$

The following cases need to be analysed:

a) There exists a positive integer k such that $a_k < d(x_k, x_{k+1})$.

For $n = k$, inequality (3.6) becomes

$$a_{k+1} \leq \varphi(d(x_k, x_{k+1})).$$

For $n = k + 1$, using (3.2), inequality (3.6) becomes

$$a_{k+2} \leq \varphi(\max\{a_{k+1}, d(x_{k+1}, x_{k+2})\}) \leq \varphi(\max\{a_{k+1}, \varphi(d(x_k, x_{k+1}))\}) \leq \varphi^2(d(x_k, x_{k+1})).$$

By induction, we obtain

$$a_{k+p} \leq \varphi^p(d(x_k, x_{k+1})) \tag{3.7}$$

and by letting $p \rightarrow \infty$, the sequence $\{a_n\}_{n \geq 0}$ converges to 0.

b) For any $n \in \mathbb{N}^*$, $a_n \geq d(x_n, x_{n+1})$. The inequality (3.6) becomes

$$a_{n+1} \leq \varphi(a_n), \text{ for any } n \in \mathbb{N}^*,$$

so $a_n \leq \varphi^n(a_0)$, which implies again that $a_n \rightarrow 0$, as $n \rightarrow \infty$.

(2) By letting $p \rightarrow \infty$ in (3.4), we obtain the a priori estimate

$$d(x_n, x^*) \leq s(\varphi^n(d(x_0, x_1))), \text{ for any } n \geq 1.$$

Using (3.2) and the monotonicity of φ , we have

$$d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{n+p-1}, x_{n+p}) \leq \sum_{k=0}^{p-1} \varphi^k(d(x_n, x_{n+1})),$$

and letting $p \rightarrow \infty$,

$$d(x_n, x^*) \leq \sum_{k=0}^{\infty} \varphi^k(d(x_n, x_{n+1})), \text{ } n \geq 0. \tag{3.8}$$

Considering the definition of s , this yields the a posteriori estimate

$$d(x_n, x^*) \leq s(d(x_n, x_{n+1})), \text{ for any } n \geq 1.$$

(3) Let $x \in Y$. From (3.8), for $x_0 := x$ we have:

$$d(x, x^*) \leq \sum_{k=0}^{\infty} \varphi^k(d(x, f(x))).$$

(4) Using the inequality (3.3),

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) \leq \sum_{n=0}^{\infty} \varphi^n(d(x_0, x_1)) = s(d(x_0, x_1)) < \infty.$$

(5) We use the inequality (3.6), i.e.

$$a_{n+1} \leq \varphi(\max\{a_n, d(x_n, x_{n+1})\}),$$

for any $n \in \mathbb{N}$, where $a_n := d(x_n, x^*)$. We need to discuss two cases.

a) If there exists $k \in \mathbb{N}$ such that $a_k < d(x_k, x_{k+1})$, then the inequality (3.7), i.e.

$$a_{k+p} \leq \varphi^p(d(x_k, x_{k+1}))$$

holds for any $p \in \mathbb{N}$. Then

$$\sum_{n=k+1}^{\infty} a_n \leq \sum_{n=1}^{\infty} \varphi^n(d(x_k, x_{k+1})) < \infty,$$

so

$$\sum_{n=0}^{\infty} d(x_n, x^*) < \infty.$$

b) If $a_n \geq d(x_n, x_{n+1})$ for any $n \in \mathbb{N}$, then (3.6) becomes

$$a_{n+1} \leq \varphi(a_n), \text{ for any } n \in \mathbb{N}$$

which implies $a_n \leq \varphi^n(a_0)$. Then

$$\sum_{n=0}^{\infty} a_n \leq \sum_{n=0}^{\infty} \varphi^n(a_0) < \infty,$$

so again

$$\sum_{n=0}^{\infty} d(x_n, x^*) < \infty.$$

□

Theorem 3.3. *Let $f : Y \rightarrow Y$ be as in Theorem 3.2. Then the fixed point problem for f is well posed, that is, assuming there exist $z_n \in Y$, $n \in \mathbb{N}$ such that*

$$d(z_n, f(z_n)) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

this implies that

$$z_n \rightarrow x^* \text{ as } n \rightarrow \infty,$$

where $F_f = \{x^\}$.*

Proof. Using the inequality $d(x, x^*) \leq s(d(x, f(x)))$ from Theorem 3.2, for $x := z_n$, we have:

$$d(z_n, x^*) \leq s(d(z_n, f(z_n))), \quad n \in \mathbb{N},$$

and letting $n \rightarrow \infty$ we obtain

$$d(z_n, x^*) \rightarrow 0, \quad n \rightarrow \infty.$$

□

Theorem 3.4. *Let $f : Y \rightarrow Y$ be as in Theorem 3.2, and $g : Y \rightarrow Y$ be such that:*

- (i) *g has at least one fixed point $x_g^* \in F_g$;*
- (ii) *there exists $\eta > 0$ such that*

$$d(f(x), g(x)) \leq \eta, \text{ for any } x \in Y.$$

Then $d(x_f^, x_g^*) \leq s(\eta)$, where $F_f = \{x_f^*\}$ and s is defined in Lemma 2.3.*

Proof. By letting $x := x_g^*$ in the inequality $d(x, x^*) \leq s(d(x, f(x)))$, we have

$$d(x_f^*, x_g^*) \leq s(d(x_g^*, f(x_g^*))) = s(d(g(x_g^*), f(x_g^*))).$$

Using the monotonicity of s we obtain $d(x_f^*, x_g^*) \leq s(\eta)$. □

Theorem 3.5. *Let $f : Y \rightarrow Y$ be as in Theorem 3.2 and $f_n : Y \rightarrow Y, n \in \mathbb{N}$, be such that:*

- (i) *for each $n \in \mathbb{N}$ there exists $x_n^* \in F_{f_n}$;*
- (ii) *$\{f_n\}_{n \geq 0}$ converges uniformly to f .*

Then $x_n^ \rightarrow x^*$ as $n \rightarrow \infty$, where $F_f = \{x^*\}$.*

Proof. As $\{f_n\}_{n \geq 0}$ converges uniformly to f , there exists $\eta_n \in \mathbb{R}_+, n \in \mathbb{N}$, such that $\eta_n \rightarrow 0$ as $n \rightarrow \infty$, and $d(f_n(x), f(x)) \leq \eta_n$, for any $x \in Y$.

Using Theorem 3.4 for $g := f_n, n \in \mathbb{N}$, we have

$$d(x_n^*, x^*) \leq s(\eta_n), n \in \mathbb{N}.$$

By letting $n \rightarrow \infty$ above, we get $d(x_n, x^*) \rightarrow 0$. □

The following theorem is a Maia type result regarding Ćirić type generalized contractions defined on cyclic representations.

Theorem 3.6. *Let X be a nonempty set, d and ρ be two metrics on X , m be a positive integer, $A_1, \dots, A_m \in P_{cl}(X), Y \in P(X)$ and $f : Y \rightarrow Y$ be an operator. Suppose that:*

- (i) *there exists $c > 0$ such that $d(x, y) \leq c \cdot \rho(x, y)$, for any $x, y \in Y$;*
- (ii) *(Y, d) is a complete metric space;*
- (iii) *$f : (Y, d) \rightarrow (Y, d)$ is continuous;*
- (iv) *$f : (Y, \rho) \rightarrow (Y, \rho)$ is a cyclic φ -contraction of Ćirić type.*

Then f has a unique fixed point $x^ \in \bigcap_{i=1}^m A_i$ and the Picard iteration $\{x_n\}_{n \geq 0}$ given by (2.1) converges to x^* for any starting point $x_0 \in Y$.*

Proof. By the same reasoning as in Theorem 3.2, using condition (iv), we obtain that $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in (X, ρ) .

Using condition (i) it follows that it is Cauchy in (X, d) as well.

By (ii) and (iii) it is easy to prove that $\{x_n\}_{n \in \mathbb{N}}$ converges in (X, d) to the unique fixed point of f . □

Remark 3.7. It is an open problem to find conditions under which the operator $f : Y \rightarrow Y$ defined as in Theorem 3.2 has the limit shadowing property, that is, assuming that there exist $z_n \in Y, n \in \mathbb{N}$, such that $d(z_{n+1}, f(z_n)) \rightarrow 0$ as $n \rightarrow \infty$, it follows that there exists $x \in Y$ such that

$$d(z_n, f^n(x)) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

References

- [1] V. Berinde, *Contractii generalizate și aplicații*, Editura Cub Press, Baia Mare (1997).3
- [2] W. A. Kirk, P. S. Srinivasan, P. Veeramani, *Fixed points for mappings satisfying cyclical contractive conditions*, *Fixed Point Theory*, **4** (2003), 79–89.1, 1.1, 2
- [3] M. Păcurar, I. A. Rus, *Fixed point theory for cyclic φ -contractions*, *Nonlinear Anal.*, **72** (2010), 1181–1187.1, 2.3
- [4] M. A. Petric, *Some results concerning cyclical contractive mappings*, *Gen. Math.*, **18** (2010), 213–226.1
- [5] A. Petruşel, *Ćirić type fixed point theorems*, *Stud. Univ. Babeş-Bolyai math.*, **59** (2014), 233–245.1
- [6] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, *Trans. Amer. Math. Soc.*, **226** (1977), 257–290.1
- [7] I. A. Rus, *Generalized contractions and applications*, Cluj University Press, Cluj-Napoca, (2001).2
- [8] I. A. Rus, *Cyclic representations and fixed points*, *Ann. T. Popoviciu Seminar Funct. Eq. Approx. Convexity*, **3** (2005), 171–178.2
- [9] I. A. Rus, A. Petruşel, G. Petruşel, *Fixed Point Theory*, Cluj University Press, (2008).2.2, 2
- [10] I. A. Rus, M. A. Şerban, *Some generalizations of a Cauchy lemma and applications*, *Topics in Mathematics, Computer Science and Philosophy*, Presa Univ. Clujean, Cluj-Napoca, (2008), 173-181.2.1, 2.3