SOLVABILITY OF A NONLINEAR BOUNDARY VALUE PROBLEM

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Abstract. In this paper we consider three point boundary value problems of second order. We introduce new and sufficient conditions that allow us to obtain the existence of a nontrivial solution by using Leray Schauder nonlinear alternative. As an application, we give some examples to illustrate our results.

1. Introduction and preliminaries

This paper is devoted to the study of the existence of nontrivial solution for the following second order three point boundary value problem (BVP):

$$u'' + f(t, u) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$u(0) = \alpha u'(0), \quad u(1) = \beta u'(\eta), \tag{1.2}$$

where $\eta \in (0, 1)$, α, $\beta \in \mathbb{R}$, $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$.

The parameters α and β are arbitrary in $\mathbb{R}$ such that $1 + \alpha - \beta \neq 0$. We do not assume any monotonicity condition on the nonlinearity f, we assume that $f(t, 0) \neq 0$ and there exist two nonnegative functions $k, h \in L^1[0, 1]$ such that

$$|f(t, x)| \leq k(t) |x|^p + h(t), \forall (t, x) \in [0, 1] \times \mathbb{R},$$

where $0 < p \leq 2$, so our conditions are new and more general than the conditions found in the literature. The case $p > 2$ still an open question. The importance of these conditions is that they appear in the study of integrodifferential equations and integral inequalities [2, pp. 6]. The second order equations (1.1) are used to model various phenomena in physics, chemistry and epidemiology. In general

nonlinearities that refer to source terms represent specific physical laws, in chemistry, for example, if $f(t, u) = ug(u)e^{\frac{u-1}{\varepsilon}}$, then it represents Arrheninus law for chemistry reactions, where the positive parameter ε represents the activation energy for the reaction and the continuous function g represents the concentration of the chemical product, see [1]. The nonlocal conditions (1.2) arise in the study of the equilibrium states of a heated bar [17], in this situation two controllers at $t = 0$ and $t = \eta$ alter the heat according to the temperatures detected by a sensor at $t = 1$.

Many of the results involving nonlocal boundary value problems are studied in [4-22]. Using the Leray-Schauder nonlinear alternative some results on the existence of solutions for the equation (1.1) subject to the conditions $u(0) = 0$, $u(1) = \alpha u(\eta)$ can be found in [12, 20]. Similar boundary value problem with the conditions $u(0) = \alpha u'(0)$, $u(1) = \beta u(\eta)$ is considered in [18]. Motived by a work of Guidotti and Merino [11] and using fixed point index theory or Sperner's Lemma, Infante and Webb in [5] studied the boundary conditions $u'(0) = 0$, $\sigma u'(1) + u(\eta) = 0$ and Webb [20] provides explicit optimal constants for the problem. Infante and Webb [6], Palamides, Infante and Pietramala [17] and Fan and Ma [9] studied $u'(0) + \alpha[u] = 0$, $\sigma u'(1) + u(\eta) = 0$. Infante in [7] considered the more general boundary conditions $u'(0) + \alpha[u] = 0$, $\sigma u'(1) + u(\eta) = \beta[u]$, where $\alpha[u]$ and $\beta[u]$ are bounded linear functionals on $C[0,1]$. Our aim is to give new conditions on the nonlinearity f, then, using Leray Schauder nonlinear alternative, we establish the existence of nontrivial solution of the BVP (1.1)-(1.2). As an application, some examples to illustrate our results are given.

This paper is organized as follows. First, we list some preliminary material to be used later. Then, in Section 3, we present and prove our main results which consist in existence theorems and corollaries. We end our work with some illustrating examples.

2. Preliminary Lemmas

Let $E = C[0,1]$, with the norm $||y|| = \max_{t \in [0,1]} |y(t)|$. Firstly we state two preliminary results.

Lemma 2.1. *Let $y \in E$. If $\beta \neq \alpha + 1$, then the three point BVP*

$$\begin{cases} u''(t) + y(t) = 0, & 0 < t < 1, \\ u(0) = \alpha u'(0), & u(1) = \beta u'(\eta), \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = -\int_0^t (t-s)\, y(s) ds + \frac{t+\alpha}{1+\alpha-\beta} \int_0^1 (1-s)\, y(s) ds - \beta \frac{t+\alpha}{1+\alpha-\beta} \int_0^\eta y(s) ds.$$

Proof. The proof is easy, then we omit it. □

We define the integral operator $T : E \to E$, by

$$Tu(t) = -\int_0^t (t-s) f(s, u(s)) \, ds + \frac{t+\alpha}{1+\alpha-\beta} \int_0^1 (1-s) f(s, u(s)) \, ds \quad (2.1)$$
$$-\beta \frac{t+\alpha}{1+\alpha-\beta} \int_0^\eta f(s, u(s)) \, ds$$

By Lemma 2.1, the BVP (1.1)-(1.2) has a solution if and only if the operator T has a fixed point in E. By Ascoli Arzela Theorem we prove that T is completely continuous operator. Now we cite the Leray–Schauder nonlinear alternative.

Lemma 2.2. *[3] Let F be a Banach space and Ω a bounded open subset of F, $0 \in \Omega$. Let $T : \overline{\Omega} \to F$ be a completely continuous operator. Then, either there exists $x \in \partial\Omega$, $\lambda > 1$ such that $T(x) = \lambda x$, or there exists a fixed point $x^* \in \overline{\Omega}$ of T.*

3. Main Results

In this section, we present and prove our main results.

Theorem 3.1. *We assume that $f(t, 0) \neq 0$, $1 + \alpha \neq \beta$, $0 < p < 1$, and there exist two nonnegative functions $k, h \in L^1[0, 1]$, $k(t) \neq 0$ almost everywhere on $[0, 1]$ and such that*

$$|f(t, x)| \leq k(t) |x|^p + h(t), \forall (t, x) \in [0, 1] \times \mathbb{R}, \quad (3.1)$$

$$\left(1 + \frac{|1+\alpha|}{|1+\alpha-\beta|}\right) \int_0^1 (1-s) k(s) ds + \left|\beta \frac{1+\alpha}{1+\alpha-\beta}\right| \int_0^\eta k(s) ds < \frac{1}{2}, \quad (3.2)$$

$$\left(1 + \frac{|1+\alpha|}{|1+\alpha-\beta|}\right) \int_0^1 (1-s) h(s) ds + \left|\beta \frac{1+\alpha}{1+\alpha-\beta}\right| \int_0^\eta h(s) ds > \frac{1}{2}. \quad (3.3)$$

Then the BVP (1.1)-(1.2) has at least one nontrivial solution $u^* \in C[0, 1]$.

Proof. Setting

$$M = \left(1 + \frac{|1+\alpha|}{|1+\alpha-\beta|}\right) \int_0^1 (1-s) k(s) ds + \left|\beta \frac{1+\alpha}{1+\alpha-\beta}\right| \int_0^\eta k(s) ds$$

and

$$N = \left(1 + \frac{|1+\alpha|}{|1+\alpha-\beta|}\right) \int_0^1 (1-s) h(s) ds + \left|\beta \frac{1+\alpha}{1+\alpha-\beta}\right| \int_0^\eta h(s) ds.$$

Since k is nonnegative and $k(t) \neq 0$, *a.e.* $t \in [0, 1]$, then $M \neq 0$, let $m = \dfrac{N}{M}$. By hypothesis (3.3) we have $m \neq 0$. Set $\Omega = \{u \in C[0, 1] : ||u|| < m\}$ and assume that there exist $u \in \partial\Omega$ and $\lambda > 1$ such $Tu = \lambda u$. Then

$$\lambda m = \lambda ||u|| = ||Tu|| = \max_{0 \leq t \leq 1} |(Tu)(t)|$$

$$\leq \ \|u\|^p \left[\left(1 + \frac{|1 + \alpha|}{|1 + \alpha - \beta|}\right) \int_0^1 (1 - s)\, k(s) ds + \left|\beta \frac{1 + \alpha}{1 + \alpha - \beta}\right| \int_0^\eta k(s) ds\right]$$

$$+ \left[\left(1 + \frac{|1 + \alpha|}{|1 + \alpha - \beta|}\right) \int_0^1 (1 - s)\, h(s) ds + \left|\beta \frac{1 + \alpha}{1 + \alpha - \beta}\right| \int_0^\eta h(s) ds\right]$$

$$= \ M\, \|u\|^p + N.$$

From this we get

$$\lambda \leq M^{2-p} N^{p-1} + M. \tag{3.4}$$

Since $0 < p < 1$, $M < 1/2$ and $N > 1/2$ we have $N^{p-1} < (1/2)^{p-1}$ and $M^{2-p} < (1/2)^{2-p}$. Consequently (3.4) becomes $\lambda < 1$; this contradicts the fact that $\lambda > 1$. By Lemma 2.1 we conclude that operator T has a fixed point $u^* \in \overline{\Omega}$ and then the BVP (1.1)-(1.2) has a nontrivial solution $u^* \in C[0,1]$. $\qquad\square$

Theorem 3.2. *We assume that $f(t,0) \neq 0$, $1 + \alpha \neq \beta$, $p = 1$ and there exist nonnegative functions $k, h \in L^1[0,1]$ such that*

$$|f(t,x)| \leq k(t)\,|x| + h(t)\,, \forall\,(t,x) \in [0,1] \times \mathbb{R},$$

$$\left(1 + \frac{|1 + \alpha|}{|1 + \alpha - \beta|}\right) \int_0^1 (1 - s)\, k(s) ds + \left|\beta \frac{1 + \alpha}{1 + \alpha - \beta}\right| \int_0^\eta k(s) ds < 1.$$

Then the BVP (1.1)-(1.2) has at least one nontrivial solution $u^* \in C[0,1]$.

Proof. Since the function f is continuous and $f(t,0) \neq 0$, there exists an interval $[\sigma, \tau] \subset [0,1]$ such that $\min\limits_{\sigma \leq t \leq \tau} |f(t,0)| > 0$ and as $h(t) \geq |f(t,0)|$, *a.e.* $t \in [0,1]$ then $N > 0$. Let $m = \dfrac{N}{1 - M}$, so $m \neq 0$. Assume that there exist $u \in \partial\Omega$, $\lambda > 1$ such $Tu = \lambda u$. Then

$$\lambda m = \lambda \,\|u\| = \|Tu\| = \max_{0 \leq t \leq 1} |(Tu)(t)| \leq M\,\|u\| + N.$$

Therefore, we have $\lambda \leq M + \frac{N}{m} = 1$. This contradicts the fact that $\lambda > 1$. By Lemma 2.1 we conclude that the operator T has a fixed point $u^* \in \overline{\Omega}$ and then the BVP (1.1)-(1.2) has a nontrivial solution $u^* \in C[0,1]$. $\qquad\square$

Theorem 3.3. *We assume that $f(t,0) \neq 0$, $1 + \alpha \neq \beta$, $1 < p \leq 2$, there exist nonnegative functions $k, h \in L^1[0,1]$, $k(t) \neq 0$, a.e. on $[0,1]$ and such that*

$$|f(t,x)| \leq k(t)\,|x|^p + h(t)\,, \forall\,(t,x) \in [0,1] \times \mathbb{R},$$

$$\left(1 + \frac{|1 + \alpha|}{|1 + \alpha - \beta|}\right) \int_0^1 (1 - s)\, k(s) ds + \left|\beta \frac{1 + \alpha}{1 + \alpha - \beta}\right| \int_0^\eta k(s) ds < \frac{1}{2}, \tag{3.5}$$

$$\left(1 + \frac{|1 + \alpha|}{|1 + \alpha - \beta|}\right) \int_0^1 (1 - s)\, h(s) ds + \left|\beta \frac{1 + \alpha}{1 + \alpha - \beta}\right| \int_0^\eta h(s) ds < \frac{1}{2}. \tag{3.6}$$

Then the BVP (1.1)-(1.2) has at least one nontrivial solution $u^* \in C[0,1]$.

Proof. Since k is nonnegative and $k(t) \neq 0$, *a.e.* $t \in [0,1]$, then $M \neq 0$. From $f(t,0) \neq 0$ we deduce that $m = \dfrac{N}{M} \neq 0$. Assume that there exist $u \in \partial\Omega$ and $\lambda > 1$ such $Tu = \lambda u$. Then $\lambda \leq M^{2-p} N^{p-1} + M$. Since $1 \leq p \leq 2$, $M < 1/2$ and

$N < 1/2$ we have $M^{2-p} < (1/2)^{2-p}$ and $N^{p-1} < (1/2)^{p-1}$. Consequently $\lambda < 1$; this contradicts the fact that $\lambda > 1$. By Lemma 2.1 we conclude that operator T has a fixed point $u^* \in \overline{\Omega}$ and then the BVP (1.1)-(1.2) has a nontrivial solution $u^* \in C[0,1]$. $\qquad\square$

Remark 3.4. For $0 < p \leq 2$, and under the conditions of Theorems 3.1, 3.2 and 3.3, we have proved that the BVP (1.1)-(1.2) has a nontrivial solution $u^* \in C[0,1]$. The case $p > 2$ still an open question.

Theorem 3.5. *Under the conditions of Theorem 3.1 $(0 < p < 1)$ and if one of the following conditions is satisfied:*
  (1) *There exist $n > 1$ and $r > 0$ such that*

$$\left(\int_0^1 k^n(s)ds\right)^{\frac{1}{n}} < \frac{\frac{1}{2}(1+q)^{\frac{1}{q}}}{1 + \frac{|1+\alpha|}{|1+\alpha-\beta|} + \left(\beta\frac{1+\alpha}{1+\alpha-\beta}\right)(\eta(1+q))^{\frac{1}{q}}}(\frac{1}{n} + \frac{1}{q} = 1), \quad (3.7)$$

$$\left(\int_0^\eta h^{-\frac{1}{r}}(s)ds\right)^{-r} > \frac{1}{2\left[1 + \frac{|1+\alpha|}{|1+\alpha-\beta|}(1-\eta) + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right]\eta^{r+1}}. \quad (3.8)$$

  (2) *There exist two constants $\mu > -1$, $\tau > -1$ such that*

$$k(s) < \frac{\frac{1}{2}|1+\alpha-\beta|(\mu+1)(\mu+2)}{|1+\alpha-\beta| + |1+\alpha| + |\beta(1+\alpha)|(\mu+2)\eta^{\mu+1}}s^\mu, \quad (3.9)$$

$$h(s) > \frac{(\tau+1)}{2\eta^{\tau+1}\left[\left(1 + \frac{|1+\alpha|}{|1+\alpha-\beta|}\right)(1-\eta) + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right]}s^\tau, \quad (3.10)$$

$$meas\left\{s \in [0,1], k(s) < \frac{\frac{1}{2}(\mu+1)(\mu+2)}{1 + \left|\frac{(1+\alpha)}{1+\alpha-\beta}\right| + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|(\mu+2)\eta^{\mu+1}}s^\mu\right\} > 0$$

$$and\ meas\left\{s \in [0,1], h(s) > \frac{(\tau+1)}{2\left[\left(1 + \frac{|1+\alpha|}{|1+\alpha-\beta|}\right)(1-\eta) + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right]\eta^{\tau+1}}s^\tau\right\} > 0.$$

  (3) *The functions k and h satisfy*

$$k(s)| \leq \frac{\frac{1}{2}}{1 + \left|\frac{(1+\alpha)}{1+\alpha-\beta}\right| + \left(\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right)}, \quad (3.11)$$

$$h(s) > \frac{1}{2\left[\left(1 + \left|\frac{1+\alpha}{1+\alpha-\beta}\right|\right)(1-\eta) + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right]}, \quad (3.12)$$

$$meas\left\{s \in [0,1], k(s) < \frac{\frac{1}{2}\left|1 + \alpha - \beta\right|}{\left|(1+\alpha)\right| + \left|1 + \alpha - \beta\right| + \left|\beta\left(1+\alpha\right)\right|}\right\} > 0$$

$$and \; meas\left\{s \in [0,1], h(s) > \frac{1}{2\eta\left[\left(1 + \frac{|1+\alpha|}{|1+\alpha-\beta|}\right)(1-\eta) + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right]}\right\} > 0.$$

(4) The function f satisfies

$$\omega_1 = \lim_{|x|\to\infty} \sup \max_{t\in[0,1]} \frac{|f(t,x)|}{|x|^p} < \frac{\frac{1}{4}}{1 + \left|\frac{1+\alpha}{1+\alpha-\beta}\right| + \left|\beta\frac{(1+\alpha)}{1+\alpha-\beta}\right|} \tag{3.13}$$

and

$$\omega_2 = \lim_{|x|\to\infty} \sup \max_{t\in[0,1]} |f(t,x)| > \frac{1}{4\eta\left[\left(1 + \left|\frac{1+\alpha}{1+\alpha-\beta}\right|\right)(1-\eta) + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right]}. \tag{3.14}$$

Then the BVP (1.1)-(1.2) has at least one nontrivial solution $u^* \in C[0,1]$.

Proof. Let M and N be as in the proof of Theorem 3.1. To show Theorem 3.5, we only need to prove that $M < \frac{1}{2}$ and $N > \frac{1}{2}$.

Suppose that condition (1) holds. By using Hölder inequality, we get

$$M \le \left(1 + \left|\frac{1+\alpha}{1+\alpha-\beta}\right|\right)\left(\int_0^1 k^n(s)ds\right)^{\frac{1}{n}}\left(\int_0^1 (1-s)^q ds\right)^{\frac{1}{q}}$$

$$+ \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\left(\int_0^\eta k^n(s)ds\right)^{\frac{1}{n}}\left(\int_0^\eta ds\right)^{\frac{1}{q}}.$$

Therefore we have

$$M \le \left(\int_0^1 k^n(s)ds\right)^{\frac{1}{n}}\left[\left(1 + \frac{|1+\alpha|}{|1+\alpha-\beta|}\right)\left(\int_0^1 (1-s)^q ds\right)^{\frac{1}{q}}\right.$$

$$\left. + |\beta\frac{1+\alpha}{1+\alpha-\beta}|\left(\int_0^\eta ds\right)^{\frac{1}{q}}\right].$$

Using (3.7) we obtain

$$M < \frac{\frac{1}{2}(1+q)^{\frac{1}{q}}}{(1 + \frac{|1+\alpha|}{|1+\alpha+\beta|}) + |\beta\frac{1+\alpha}{1+\alpha-\beta}|\left(\eta(1+q)\right)^{\frac{1}{q}}}$$

$$\times \left[(1 + \frac{|1+\alpha|}{|1+\alpha-\beta|})(\frac{1}{1+q})^{\frac{1}{q}} + |\beta\frac{1+\alpha}{1+\alpha-\beta}|\eta^{\frac{1}{q}}\right] = \frac{1}{2}.$$

On the other hand, we have

$$N = \left(1 + \left|\frac{1+\alpha}{1+\alpha-\beta}\right|\right)\int_0^1 (1-s)h(s)ds + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\int_0^\eta h(s)ds\right]$$

$$\ge \left[\left(1 + \frac{|1+\alpha|}{|1+\alpha-\beta|}\right)(1-\eta) + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right]\int_0^\eta h(s)ds.$$

Using the reverse Hölder inequality and applying (3.8), we get

$$
\begin{aligned}
N \;\geq\; & \left[\left(1+\frac{|1+\alpha|}{|1+\alpha-\beta|}\right)(1-\eta)+\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right] \\
& \times\left[\left(\int_0^\eta 1^{1/r+1}ds\right)^{r+1}\left(\int_0^\eta h^{-1/r}(s)ds\right)^{-r}\right] \\
=\; & \left[\left(1+\frac{|1+\alpha|}{|1+\alpha-\beta|}\right)(1-\eta)+\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right]\eta^{r+1}\left(\int_0^\eta h^{-1/r}(s)ds\right)^{-r}>\frac{1}{2}.
\end{aligned}
$$

Suppose that condition (2) holds. Taking into account (3.9) it yields

$$
\begin{aligned}
M \;<\; & \frac{1}{2}\left[\frac{(\mu+1)(\mu+2)}{1+\left|\frac{1+\alpha}{1+\alpha-\beta}\right|+\left|\frac{\beta(1+\alpha)}{1+\alpha-\beta}\right|(\mu+2)\eta^{\mu+1}}\right] \\
& \times\left[\left(1+\frac{|1+\alpha|}{|1+\alpha-\beta|}\right)\int_0^1(1-s)s^\mu ds+\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\int_0^\eta s^\mu ds\right] \\
=\; & \frac{1}{2}\left[\frac{(\mu+1)(\mu+2)}{1+\left|\frac{(1+\alpha)}{1+\alpha-\beta}\right|+\left(\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right)(\mu+2)\eta^{\mu+1}}\right] \\
& \times\left[\left(1+\frac{|1+\alpha|}{|1+\alpha-\beta|}\right)\frac{1}{(\mu+1)(\mu+2)}+\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\frac{\eta^{\mu+1}}{(\mu+1)}\right]=\frac{1}{2}.
\end{aligned}
$$

On the other hand using (3.10) we obtain

$$
\begin{aligned}
N > & \left[(1-\eta)\left(1+\frac{|1+\alpha|}{|1+\alpha-\beta|}\right)+\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right]\int_0^\eta h(s)ds \\
> & \left[(1-\eta)\left(1+\frac{|1+\alpha|}{|1+\alpha-\beta|}\right)+\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right] \\
& \times\frac{(\tau+1)}{2\left[\left(1+\frac{|1+\alpha|}{|1+\alpha-\beta|}\right)(1-\eta)+\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right]\eta^{\tau+1}}\int_0^\eta s^\tau ds \\
= & \frac{1}{2}.
\end{aligned}
$$

Suppose that condition (3) holds. Using the same reasoning as in the proof of the second statement we obtain

$$
\begin{aligned}
M \;<\; & \left[\left(1+\frac{|1+\alpha|}{|1+\alpha-\beta|}\right)+\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right]\int_0^1 k(s)ds \\
<\; & \left[\left(1+\frac{|1+\alpha|}{|1+\alpha-\beta|}\right)+\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right] \\
& \times\left[\frac{\frac{1}{2}|1+\alpha-\beta|}{|(1+\alpha)|+|1+\alpha-\beta|+|\beta(1+\alpha)|}\right]=\frac{1}{2}.
\end{aligned}
$$

We have

$$N > \left[(1-\eta)\left(1 + \frac{|1+\alpha|}{|1+\alpha-\beta|}\right) + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right]\int_0^\eta h(s)ds$$

$$> \left[(1-\eta)\left(1 + \frac{|1+\alpha|}{|1+\alpha-\beta|}\right) + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right]$$

$$\times \frac{1}{2\eta\left[\left(1 + \frac{|1+\alpha|}{|1+\alpha-\beta|}\right)(1-\eta) + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right]}\eta = \frac{1}{2}.$$

Suppose that condition (1) holds. From $\omega_1 = \lim\sup_{|x|\to\infty}\max_{t\in[0,1]}\frac{|f(t,x)|}{|x|^p}$ we deduce that there exists $c_1 > 0$ such that for $|x| > c_1$, we get $|f(t,x)| \leq (\omega_1 + \varepsilon)|x|^p$, $\forall \varepsilon > 0$. For $\varepsilon = \omega_1$ then $|f(t,x)| \leq 2\omega_1|x|^p$. From $\omega_2 = \lim\sup_{|x|\to\infty}\max_{t\in[0,1]}|f(t,x)|$, we deduce that there exists $c_2 > 0$ such that for $|x| > c_2$ we have $|f(t,x)| \leq \omega_2 + \varepsilon$. Choosing $\varepsilon = \omega_2$, we have $|f(t,x)| \leq 2\omega_2, \forall x \in \mathbb{R}\backslash]-c_2, c_2[$ and consequently

$$|f(t,x)| \leq 2\omega_1|x|^p + 2\omega_2, \forall x \in \mathbb{R}\backslash]-c, c[,$$

where $c = \max(c_1, c_2)$. Setting

$$R = \sup\{|f(t,x)| : (t,x) \in [0,1] \times (-c, c)\},$$

we get, $\forall(t,x) \in [0,1] \times \mathbb{R}$, $|f(t,x)| \leq 2\omega_1|x|^p + 2\omega_2 + R = k(t)|x| + h(t)$ where $k(t) = 2\omega_1$ and $h(t) = 2\omega_2 + R$. Using (3.13) we obtain

$$k(t) = 2\omega_1 < \frac{\frac{1}{2}|1+\alpha-\beta|}{|(1+\alpha)| + |1+\alpha-\beta| + |\beta(1+\alpha)|}$$

then from (3.11) we get $M < 1/2$. Using (3.12), we arriv to

$$h(t) = 2\omega_2 + R \geq 2\omega_2 > \frac{1}{2\eta\left[\left(1 + \frac{|1+\alpha|}{|1+\alpha-\beta|}\right)(1-\eta) + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right]},$$

then $N > 1/2$. Applying the third statement we achieve the proof of Theorem 3.5. $\qquad\square$

Corollary 3.6. *Under the conditions of Theorem 3.1 and if one of the following conditions is satisfied:*

(1) There exist $n > 1$ and $r > 0$ such that

$$\left(\int_0^1 k^n(s)ds\right)^{\frac{1}{n}} < \frac{1}{2\left(1 + \frac{|1+\alpha|}{|1+\alpha-\beta|} + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right)}, \left(\frac{1}{n} + \frac{1}{q} = 1\right), \qquad (3.15)$$

$$\left(\int_0^\eta h^{-\frac{1}{r}}(s)ds\right)^{-r} > \frac{1}{2\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\eta^{r+1}} \qquad (3.16)$$

or

$$\left(\int_0^\eta h^{-\frac{1}{r}}(s)ds\right)^{-r} > \frac{1}{2\left(1 + \frac{|1+\alpha|}{|1+\alpha-\beta|}\right)(1-\eta)\eta^{r+1}}. \qquad (3.17)$$

(2) There exist constant $\mu > -1$ and $\tau > -1$ such that

$$k(s) < \frac{\frac{1}{2}(\mu+1)}{1 + \left|\frac{(1+\alpha)}{1+\alpha-\beta}\right| + \left(\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right)} s^{\mu} \qquad (3.18)$$

and

$$h(s) > \frac{(\tau+1)}{2\eta^{\tau+1}\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|} s^{\tau} \qquad (3.19)$$

$$or, \; h(s) > \frac{(\tau+1)}{2\eta^{\tau+1}(1-\eta)\left(1+\left|\frac{1+\alpha}{1+\alpha-\beta}\right|\right)} s^{\tau}, \qquad (3.20)$$

$$meas\left\{s \in [0,1], k(s) < \frac{\frac{1}{2}(\mu+1)}{1 + \left|\frac{(1+\alpha)}{1+\alpha-\beta}\right| + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|} s^{\mu}\right\} > 0$$

$$and \; meas\left\{s \in [0,1], h(s) > \frac{1}{2\eta^{\tau+1}\left[\left(1+\frac{|1+\alpha|}{|1+\alpha-\beta|}\right)(1-\eta) + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right]} s^{\tau}\right\} > 0.$$

(3) The function k and h satisfy

$$k(s) < \frac{\frac{1}{2}}{1 + \left|\frac{(1+\alpha)}{1+\alpha-\beta}\right| + \left(\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right)}, \qquad (3.21)$$

$$h(s) > \frac{1}{2\left(1+\left|\frac{1+\alpha}{1+\alpha-\beta}\right|\right)(1-\eta)} \; or \; h(s) > \frac{1}{2\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|}, \qquad (3.22)$$

$$meas\left\{s \in [0,1], k(s) < \frac{\frac{1}{2}|1+\alpha-\beta|}{|(1+\alpha)| + |1+\alpha-\beta| + |\beta(1+\alpha)|}\right\} > 0$$

$$and \; meas\left\{s \in [0,1], h(s) > \frac{1}{2\eta\left[\left(1+\frac{|1+\alpha|}{|1+\alpha-\beta|}\right)(1-\eta) + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right]}\right\} > 0.$$

(4) The function f satisfies

$$\omega_1 = \lim_{|x|\to\infty} \sup \max_{t\in[0,1]} \frac{|f(t,x)|}{|x|^p} < \frac{\frac{1}{4}}{1 + \left|\frac{1+\alpha}{1+\alpha-\beta}\right| + \left|\beta\frac{(1+\alpha)}{1+\alpha-\beta}\right|} \qquad (3.23)$$

and

$$\omega_2 = \lim_{|x|\to\infty} \sup \max_{t\in[0,1]} |f(t,x)| > \frac{1}{4\left(1+\left|\frac{1+\alpha}{1+\alpha-\beta}\right|\right)(1-\eta)} \qquad (3.24)$$

or

$$\omega_2 = \lim_{|x|\to\infty} \sup \max_{t\in[0,1]} |f(t,x)| > \frac{1}{4\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|}. \qquad (3.25)$$

Then the BVP (1.1)-(1.2) has at least one nontrivial solution $u^* \in C[0,1]$.

Proof. Suppose that condition (1) holds. We prove that $N > \frac{1}{2}$. We have

$$N = \left(1 + \left|\frac{1+\alpha}{1+\alpha-\beta}\right|\right)\int_0^1 (1-s)\,h(s)ds + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\int_0^\eta h(s)ds$$

$$\geq \left(1 + \frac{|1+\alpha|}{|1+\alpha-\beta|}\right)(1-\eta)\int_0^\eta h(s)ds + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\int_0^\eta h(s)ds$$

$$\geq \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\int_0^\eta h(s)ds.$$

Using the reverse Hölder inequality and applying (3.16), we get

$$N \geq \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\int_0^\eta h(s)ds$$

$$> \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\left[\left(\int_0^\eta 1^{1/r+1}ds\right)^{r+1}\left(\int_0^\eta h^{-1/r}(s)ds\right)^{-r}\right]$$

$$= \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\eta^{r+1}\left(\int_0^\eta h^{-1/r}(s)ds\right)^{-r} > \frac{1}{2}.$$

By the same reasoning we prove that if (3.17) yields then $N > \frac{1}{2}$.
Suppose that condition (2) holds. Taking into account (3.18), we obtain

$$M < \frac{1}{2}\left[\frac{(\mu+1)}{1 + \left|\frac{1+\alpha}{1+\alpha-\beta}\right| + \left|\frac{\beta(1+\alpha)}{1+\alpha-\beta}\right|}\right]$$

$$\times \left(\int_0^1 s^\mu ds\right)\left[\left(1 + \frac{|1+\alpha|}{|1+\alpha-\beta|}\right) + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right]$$

$$= \frac{1}{2}\left[\frac{(\mu+1)}{1 + \left|\frac{(1+\alpha)}{1+\alpha-\beta}\right| + \left(\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right)}\right]$$

$$\times \frac{1}{\mu+1}\left[\left(1 + \frac{|1+\alpha|}{|1+\alpha-\beta|}\right) + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right] = \frac{1}{2}.$$

On the other hand, using (3.19) we obtain

$$N > \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\frac{(\tau+1)}{2\eta^{\tau+1}\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|}\int_0^\eta s^\tau ds = \frac{1}{2}.$$

The other proofs follow similarly as in Theorem 3.5. $\square$

Theorem 3.7. *Under the conditions of Theorem 3.2 ($p = 1$) and if one of the following conditions is satisfied:*

(1) *There exists a constant $n > 1$ such that*

$$\left(\int_0^1 k^n(s)ds\right)^{\frac{1}{n}} < \frac{(1+q)^{\frac{1}{q}}}{1 + \left|\frac{(1+\alpha)}{1+\alpha-\beta}\right| + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|(\eta(1+q))^{\frac{1}{q}}}, \quad \left(\frac{1}{n} + \frac{1}{q} = 1\right).$$

(2) *There exists a constant $\mu > -1$ such that*

$$k(s) < \frac{(\mu+1)(\mu+2)}{1 + \left|\frac{(1+\alpha)}{1+\alpha-\beta}\right| + \left(\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right)(\mu+2)\eta^{\mu+1}}s^{\mu}$$

and

$$meas\left\{s \in [0,1], k(s) < \frac{(\mu+1)(\mu+2)}{1 + \left|\frac{(1+\alpha)}{1+\alpha-\beta}\right| + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|(\mu+2)\eta^{\mu+1}}s^{\mu}\right\} > 0.$$

(3) *The function k satisfies*

$$k(s) < \frac{|1+\alpha-\beta|}{|(1+\alpha)| + |1+\alpha-\beta| + |\beta(1+\alpha)|\eta}$$

and

$$meas\left\{s \in [0,1], k(s) < \frac{|1+\alpha-\beta|}{|(1+\alpha)| + |1+\alpha-\beta| + |\beta(1+\alpha)|\eta}\right\} > 0.$$

(4) *The function f satisfies*

$$\omega = \lim_{|x|\to\infty}\sup\max_{t\in[0,1]}\left|\frac{f(t,x)}{x}\right|$$

$$< 1/2\left(\frac{|1+\alpha-\beta|}{|(1+\alpha)| + |1+\alpha-\beta| + |\beta(1+\alpha)|\eta}\right)$$

Then the BVP (1)-(2) has at least one nontrivial solution $u^* \in C[0,1]$.

Proof. The proof is the same as the one in theorem 3.5, indeed $M < \frac{1}{2} < 1$. $\square$

Theorem 3.8. *Under the conditions of Theorem 3.3 ($1 < p \le 2$) and if one of the following conditions is satisfied:*
   (1) *There exist two constants $n, r > 1$ such that*

$$\left(\int_0^1 k^n(s)ds\right)^{\frac{1}{n}} < \frac{\frac{1}{2}(1+q)^{\frac{1}{q}}}{1 + \frac{|1+\alpha|}{|1+\alpha-\beta|} + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|(\eta(1+q))^{\frac{1}{q}}}, \left(\frac{1}{n} + \frac{1}{q} = 1\right). \quad (3.26)$$

and

$$\left(\int_0^1 h^r(s)ds\right)^{\frac{1}{r}} < \frac{\frac{1}{2}(1+l)^{\frac{1}{l}}}{1 + \left|\frac{(1+\alpha)}{1+\alpha-\beta}\right| + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|(\eta(1+l))^{\frac{1}{l}}}, \left(\frac{1}{r} + \frac{1}{l} = 1\right). \quad (3.27)$$

(2) *There exist two constants* μ, $\tau > -1$ *such that*

$$k(s) < \frac{\frac{1}{2}(\mu+1)(\mu+2)}{1 + \left|\frac{(1+\alpha)}{1+\alpha-\beta}\right| + \left(\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right)(\mu+2)\eta^{\mu+1}}s^{\mu}, \tag{3.28}$$

$$h(s) < \frac{\frac{1}{2}(\tau+1)(\tau+2)}{1 + \left|\frac{(1+\alpha)}{1+\alpha-\beta}\right| + \left(\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right)(\tau+2)\eta^{\tau+1}}s^{\tau}, \tag{3.29}$$

$$meas\left\{s \in [0,1], k(s) < \frac{\frac{1}{2}(\mu+1)(\mu+2)}{1 + \left|\frac{(1+\alpha)}{1+\alpha-\beta}\right| + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|(\mu+2)\eta^{\mu+1}}s^{\mu}\right\} > 0$$

$$and\ meas\left\{s \in [0,1], h(s) < \frac{\frac{1}{2}(\tau+1)(\tau+2)}{1 + \left|\frac{(1+\alpha)}{1+\alpha-\beta}\right| + \left(\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right)(\tau+2)\eta^{\tau+1}}s^{\tau}\right\} > 0.$$

(3) *The functions* k *and* h *satisfy*

$$k(s) < \frac{\frac{1}{2}|1+\alpha-\beta|}{|(1+\alpha)| + |1+\alpha-\beta| + |\beta(1+\alpha)|},$$

$$h(s) < \frac{\frac{1}{2}|1+\alpha-\beta|}{|(1+\alpha)| + |1+\alpha-\beta| + |\beta(1+\alpha)|},$$

$$meas\left\{s \in [0,1], k(s) < \frac{\frac{1}{2}|1+\alpha-\beta|}{|(1+\alpha)| + |1+\alpha-\beta| + |\beta(1+\alpha)|}\right\} > 0$$

$$and\ meas\left\{s \in [0,1], h(s) < \frac{\frac{1}{2}|1+\alpha-\beta|}{|(1+\alpha)| + |1+\alpha-\beta| + |\beta(1+\alpha)|}\right\} > 0.$$

Then the BVP (1)-(2) *has at least one nontrivial solution* $u^* \in C[0,1]$.

Proof. It suffices to prove that $M < 1/2$ and $N < 1/2$. The proof of $M < 1/2$ is the same as in Theorem 3.5. Change the role of k and h in Theorem 3.5, we prove that $N < 1/2$. $\qquad\square$

Remark 3.9. It will be interesting if we can formulate in this case $(1 < p \leq 2)$ similar statement as the last statement of Theorem 3.5.

Corollary 3.10. *Under the conditions of Theorem 3.3 and if one of the following conditions is satisfied:*

(1) *There exist two constants* n, $r > 1$ *such that*

$$\left(\int_0^1 k^n(s)ds\right)^{\frac{1}{n}} < \frac{1}{2\left(1 + \frac{|1+\alpha|}{|1+\alpha-\beta|} + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right)}, (\frac{1}{n} + \frac{1}{q} = 1), \tag{3.30}$$

and

$$\left(\int_0^1 h^r(s)ds\right)^{\frac{1}{r}} < \frac{1}{2\left(1 + \frac{|1+\alpha|}{|1+\alpha-\beta|} + \left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right)}, (\frac{1}{r} + \frac{1}{l} = 1). \tag{3.31}$$

(2) There exist two constants $\mu,\ \tau > -1$ such that

$$k(s) < \frac{\frac{1}{2}(\mu+1)}{1+\left|\frac{(1+\alpha)}{1+\alpha-\beta}\right|+\left(\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right)}s^{\mu}, \tag{3.32}$$

$$h(s) < \frac{\frac{1}{2}(\tau+1)}{1+\left|\frac{(1+\alpha)}{1+\alpha-\beta}\right|+\left(\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right)}s^{\tau}, \tag{3.33}$$

$$meas\left\{s\in[0,1],k(s)<\frac{\frac{1}{2}(\mu+1)}{1+\left|\frac{(1+\alpha)}{1+\alpha-\beta}\right|+\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|}s^{\mu}\right\}>0$$

$$and\ meas\left\{s\in[0,1],h(s)<\frac{\frac{1}{2}(\tau+1)}{1+\left|\frac{(1+\alpha)}{1+\alpha-\beta}\right|+\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|}s^{\tau}\right\}>0$$

Then the BVP (1.1)-(1.2) has at least one nontrivial solution $u^* \in C[0,1]$.

4. Examples

In order to illustrate our results, we give some examples.

Example 4.1. Consider the three point BVP

$$\begin{cases} u''+t^2\sqrt{|u|}\cos u-t^4\sqrt{\frac{u^4+u^2}{u^4+u^2+3}}\sin u+te^t=0, & 0<t<1, \\ u(0)=\frac{1}{2}u'(0), \quad u(1)=\frac{1}{2}u'\left(\frac{1}{2}\right). \end{cases} \tag{4.1}$$

We have $f(t,x)=t^2\sqrt{|x|}\cos x-t^4\sqrt{\frac{x^4+x^2}{x^4+x^2+3}}\sin x+te^t$, so $|f(t,x)|\le k(t)|x|^{\frac{1}{2}}+h(t)$, $p=\frac{1}{2}<1$, where $k(s)=(s^2+s^4)$, $h(s)=se^s$, $k,h\ge 0$. Since $M=\frac{5}{2}\int_0^1(1-s)(s^2+s^4)\,ds+\frac{3}{4}\int_0^{\frac{1}{2}}(s^2+s^4)\,ds=0.327\,6<1/2$ and $N=\frac{5}{2}\int_0^1(1-s)se^s\,ds+\frac{3}{4}\int_0^{\frac{1}{2}}se^s\,ds=0.836\,02>1/2$, then from Theorem 3.1, we have that the BVP (4.1) has at least one nontrivial solution u^* in $C[0,1]$.

Example 4.2. Consider the three point BVP

$$\begin{cases} u''+\frac{u^{\frac{1}{3}}}{3}\left(\frac{\sqrt[2]{t^3}}{2}+\arcsin t\right)+2\cos e^t+4\sin t=0, & 0<t<1, \\ u(0)=-\frac{1}{2}u'(0), \quad u(1)=-\frac{1}{2}u'\left(\frac{1}{3}\right). \end{cases} \tag{4.2}$$

We have $f(t,x)=\left(\frac{x^{\frac{1}{3}}}{3}\right)\left(\frac{\sqrt[2]{t^3}}{2}+\arcsin t\right)+2\cos e^t+4\sin t$. So $|f(t,x)|\le$

$\frac{1}{3}\left(\frac{\sqrt[2]{t^3}}{2}+\arcsin t\right)|x|^{\frac{1}{3}}+2\cos e^t+4\sin t=k(t)|x|^{\frac{1}{3}}+h(t),\ 0<p=\frac{1}{3}<1.$ For

$n=\frac{1}{3}<1, r=\frac{1}{2},$ we get $\left(\int_0^1 k^2(s)ds\right)^{\frac{1}{2}}=0.256\,93<\frac{1}{2}\frac{(1+q)^{\frac{1}{q}}}{1+\frac{|1+\alpha|}{|1+\alpha-\beta|}+\left(\beta\frac{1+\alpha}{1+\alpha-\beta}\right)(\eta(1+q)^{\frac{1}{q}})}=$

$0.494\,87$, and $\left(\int_0^{\frac{1}{3}} h^{-2}(s)ds\right)^{-\frac{1}{2}} = 2.3781 > \frac{1}{2\left(1+\frac{|1+\alpha|}{|1+\alpha-\beta|}(1-\eta)+\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right)\eta^{r+1}} = 1.6409$.
Then condition (1) in Theorem 3.5 is satisfied and the BVP (4.2) has at least one nontrivial solution u^* in $C[0,1]$.

Example 4.3. Consider the three point BVP

$$\begin{cases} u'' + \dfrac{u^{\frac{7}{2}}}{2\sqrt[2]{t^3+1}\,(1+u^2)} + \dfrac{\cos e^t\,(1-\sin t)}{3} = 0, & 0 < t < 1, \\ u\,(0) = -2u'\,(0)\,, \quad u\,(1) = 2u'\left(\frac{1}{3}\right). \end{cases} \tag{4.3}$$

We get $f\,(t,x) = \dfrac{x^{\frac{7}{2}}}{2\sqrt[2]{t^3+1}\,(1+x^2)} + \frac{\cos e^t(1-\sin t)}{3}$. So $|f\,(t,x)| \le k(t)\,|x|^{\frac{3}{2}} + h(t)$,
$1 < p = \frac{3}{2} \le 2$ where $k(t) = \dfrac{1}{4\sqrt[2]{t^3+1}}$, $h(t) = \frac{\cos e^t(1-\sin t)}{3} \in L_1\,[0,1]$. Since
$\int_0^1 k^2(s)ds = 0.05222 < \left(\dfrac{1}{2\left(1+\frac{|1+\alpha|}{|1+\alpha-\beta|}+\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right)}\right)^2 = 0.062\,5$ and $\int_0^1 h^3(s)ds =$
$0.0126 < \left(\dfrac{1}{2\left(1+\frac{|1+\alpha|}{|1+\alpha-\beta|}+\left|\beta\frac{1+\alpha}{1+\alpha-\beta}\right|\right)}\right)^3 = 0.0156$, the condition (1) of corollary 3.10 is
satisfied and the BVP (4.3) has at least one nontrivial solution u^* in $C[0,1]$.

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