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Self-similar sets and fractals generated by Ćirić type operators

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Dedicated to Lj. B. Ćirić in recognition of his great contributions to Fixed Point Theory

Abstract

The purpose of this paper is to present fixed point, strict fixed point and fixed set results for (singlevalued and multivalued) generalized contractions of Ćirić type. The connections between fixed point theory and the theory of self-similar sets is also discussed.

Keywords: Fixed point, strict fixed point, metric space, self-similar set, fractal, generalized contraction.
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1. Preliminaries

Fixed point theory and the theory of self-similar sets (sometimes also called "the mathematics of fractals"), are linked by some common approaches and by related results. As it is well known, if f_i are singlevalued continuous self operators on a complete metric space X, then $f = (f_1, ..., f_m)$ is called an iterated function system (briefly IFS), while the operator $T_f : P_{cp}(X) \to P_{cp}(X)$ given by

$$T_f(Y) = \bigcup_{i=1}^{m} f_i(Y), \text{ for each } Y \in P_{cp}(X)$$

is called the fractal operator generated by the IFS f. In this context, a fixed point of T_f is called a self-similar set for f and, sometimes, it is a fractal (i.e., a set with a non-integer Hausdorff dimension). For example, if $f_1, ..., f_m$ are contractions, then T_f is a contraction too and its unique fixed point $A^* \in P_{cp}(X)$ is a self-similar set. If $f_1, ..., f_m$ are similar contractions (see [12]), then $A^* \in P_{cp}(X)$ is a fractal. Moreover,

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for any nonempty compact subset A of X, the sequence $(T_f^n(A))_{n\in\mathbb{N}}$ converges to A^* as $n \to +\infty$. The elements of the sequence of successive approximations for T_f (i.e, the elements of $(T_f^n(A))_{n\in\mathbb{N}}$) are also called pre-fractals.

Notice that a similar result holds if $F_1, ..., F_m$ are multivalued contractions with compact values.

The purpose of this paper is to study some other connections between fixed point theory and the theory of self-similar sets for the case of generalized contractions of Ćirić type (see [1], [2], [3], [9], [10], etc.). Fixed point, strict fixed point and fixed set results for multivalued operators are presented.

We shall begin by presenting some notions and notations that will be used throughout the paper.

Let (X,d) be a metric space and consider the following families of subsets of X:

$$P(X) := \{Y \in \mathcal{P}(X) \,|Y \neq \emptyset\}\,; P_b(X) := \{Y \in P(X)|\ Y \text{ is bounded }\},$$
$$P_{cl}(X) := \{Y \in P(X)|\ Y \text{ is closed}\}\,; P_{cp}(X) := \{Y \in P(X)|\ Y \text{ is compact}\}$$

Let us define the gap functional between the sets A and B in the metric space (X,d) as:

$$D : P(X) \times P(X) \to \mathbb{R}_+,\ D(A,B) = \inf\{d(a,b) \mid a \in A,\ b \in B\}$$

the excess (generalized) functional between the sets A and B as:

$$\rho : P(X) \times P(X) \to \mathbb{R}_+,\ \rho(A,B) := \sup_{a\in A} D(a,B)$$

and the Pompeiu-Hausdorff (generalized) functional as:

$$H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ H(A,B) = \max\{\rho(A,B), \rho(B,A)\}.$$

The generalized diameter functional is denoted by $\delta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}$, and defined by

$$\delta(A,B) = \sup\{d(a,b) \mid a \in A,\ b \in B\}.$$

Denote by $diam(A) := \delta(A,A)$ the diameter of the set A.

The following lemma is well known.

Lemma 1.1. *Let (X,d) be a metric space, $A \in P_b(X)$ and $q > 1$. Then, for every $x \in X$ there exists $a \in A$ such that $\delta(x,A) \leq qd(x,a)$.*

Let $T : X \to P(X)$ be a multivalued operator and $Graphic(T) := \{(x,y)|y \in T(x)\}$ the graphic of T. An element $x \in X$ is called a *fixed point* for T if and only if $x \in T(x)$ and it is called a *strict fixed point* if and only if $\{x\} = T(x)$.

The set $Fix(T) := \{x \in X|\ x \in T(x)\}$ is called the fixed point set of T, while the symbol $SFix(T) := \{x \in X|\ \{x\} = T(x)\}$ denotes the strict fixed point set of T. Notice that $SFix(T) \subseteq Fix(T)$.

If $f : X \to X$ is a singlevalued operator, we will also denote by

$$O(x_0,n) := \{x_0, f(x_0), f^2(x_0), \cdots, f^n(x_0)\}$$

the orbit of order n of the operator f corresponding to $x_0 \in X$, while

$$O(x_0) := \{x_0, f(x_0), f^2(x_0), \cdots, f^n(x_0), \cdots\}$$

is the orbit of f corresponding to $x_0 \in X$. By $Fix(f)$ we denote the fixed point set of t, i.e.,

$$Fix(f) := \{x \in X|\ x = f(x)\}.$$

2. Fixed point and strict fixed point results

The basic metric fixed point theorems for singlevalued and respectively multivalued operators are Banach's contraction principle (1922), respectively Nadler's contraction principle (1969). These results were later extended in many directions, with respect to the spaces and the operators involved (see [2], [4], [6], [7], [9], etc.). A very active research direction was the case of so called generalized contractions.

Let (X, d) be a metric space and $f : X \to X$ be a singlevalued operator. Then, by definition (see I. A. Rus [11]), f is called a weakly Picard operator if:
(i) $Fix(f) \neq \emptyset$;
(ii) for all $x \in X$, the sequence $(f^n(x))_{n\in\mathbb{N}} \to x^*(x) \in Fix(f)$ as $n \to \infty$.
In particular, if $Fix(f) = \{x^*\}$, then f is called a Picard operator.

Moreover, if f is a weakly Picard operator and there exists $\tilde{c} > 0$ such that

$$d(x, x^*(x)) \leq \tilde{c}d(x, f(x)), \text{ for all } x \in X,$$

then f is called a $\tilde{c}$-weakly Picard operator. Similarly, a Picard operator for which there exists $\tilde{c} > 0$ such that

$$d(x, x^*) \leq \tilde{c}d(x, f(x)), \text{ for all } x \in X,$$

is called a $\tilde{c}$-Picard operator.
A nice extension of Banach's contraction principle was given by Ćirić, Reich and Rus in 1971-1972.

Theorem 2.1. *Let (X, d) be a complete metric space and $f : X \to X$ be an operator for which there exist $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$ such that*

$$d(f(x), f(y)) \leq ad(x, y) + bd(x, f(x)) + cd(y, f(y)), \text{ for all } x, y \in X.$$

Then f is a $\tilde{c}$-Picard operator, with $\tilde{c} := \frac{1}{1-\beta}$, where $\beta := \min\{\frac{a+b}{1-c}, \frac{a+c}{1-b}\} < 1$.

An extension of this result was proved in a paper from 1973 by Hardy and Rogers. The result, in terms of Picard operators, is as follows.

Theorem 2.2. *Let (X, d) be a complete metric space and $f : X \to X$ be an operator for which there exist $a, b, c, e, f \in \mathbb{R}_+$ with $a + b + c + e + f < 1$ such that*

$$d(f(x), f(y)) \leq ad(x, y) + bd(x, f(x)) + cd(y, f(y)) + ed(x, f(y)) + fd(y, f(x)), \text{ for all } x, y \in X.$$

Then f is a $\tilde{c}$-Picard operator, with $\tilde{c} := \frac{1}{1-\beta}$, where $\beta := \min\{\frac{a+b+e}{1-c-e}, \frac{a+c+f}{1-b-f}\} < 1$.

The proofs of the above results are based on the fact that f is a graphic contraction, i.e.,

$$d(f(x), f^2(x)) \leq \beta d(x, f(x)), \text{ for all } x \in X.$$

Another general result was given by Rus in 1979, as follows.

Theorem 2.3. *If (X, d) is a complete metric space and $f : X \to X$ is an operator for which there exists a generalized strict comparison function $\varphi : \mathbb{R}^5_+ \to \mathbb{R}_+$ (which means that φ is increasing in each variable and the function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by*

$$\Phi(t) := \varphi(t, t, t, t, t)$$

satisfy the conditions that $\Phi^n(t) \to 0$ as $n \to +\infty$, for all $t > 0$ and $t - \Phi(t) \to +\infty$ as $t \to +\infty$) such that

$$d(f(x), f(y)) \leq \varphi(d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))), \text{ for all } x, y \in X,$$

then f is a Φ-Picard operator (i.e., f is a Picard operator and $d(x, x^*) \leq \Phi(d(x, f(x)))$, for all $x \in X$).

Remark 2.4. Notice that Theorem 2.1 and Theorem 2.2 are particular cases of Theorem 2.3, for adequate choices of the function φ.

In 1974, Ćirić proved, using an approach involving the orbit of order n and the orbit of the operator f, the following very general result.

Theorem 2.5. *If (X,d) is a complete metric space and if $f : X \to X$ is an operator for which there exists $q \in (0,1)$ such that, for all $x, y \in X$, we have*

$$d(f(x), f(y)) \leq q \max\{d(x,y), d(x,f(x)), d(y,f(y)), d(x,f(y)), d(y,f(x))\}, \tag{2.1}$$

then f is a $\tilde{c}$-Picard operator, with $\tilde{c} := \frac{1}{1-q}$.

For example, the mapping $f : [0,1] \to [0,1]$ given by

$$f(x) = \begin{cases} 2x + \frac{1}{2}, & x \in [0, \frac{1}{4}] \\ \frac{1}{2}, & x \in]\frac{1}{4}, 1] \end{cases} \tag{2.2}$$

Then, f is not a contraction, but it is a Ćirić type operator with $q = \frac{2}{3}$. Notice also that $Fix(f) = \{\frac{1}{2}\}$.

Another example (see [3]) is as follows. Let $X_1 := \{\frac{m}{n} \mid m \in \{0, 1, 3, 9, \cdots\}, n \in \{1, 4, 7, \cdots\}\}$ and $X_2 := \{\frac{m}{n} \mid m \in \{1, 3, 9, 27, \cdots\}, n \in \{2, 5, 8, \cdots\}\}$. Consider $X := X_1 \cup X_2$ and $f : X \to X$ defined by:

$$f(x) = \begin{cases} \frac{3x}{5}, & x \in X_1 \\ \frac{x}{8}, & x \in X_2 \end{cases} \tag{2.3}$$

Then, f is a Ćirić type operator with $q = \frac{3}{5}$, but f fails to satisfy Hardy and Rogers' condition in Theorem 2.3. Moreover $Fix(f) = \{0\}$.

Moreover, we can do the following remark.

Remark 2.6. Any Ćirić-Reich-Rus type operator is a Hardy-Rogers type operator and any Hardy-Rogers type operator is a Ćirić type operator. The reverse implications do not hold, as we can see from several examples given in [5], [10], [11].

Passing to the multivalued case, let (X,d) be a metric space and let $T : X \to P_b(X)$ be a multivalued operator with nonempty and bounded values. We will be interested in the study of strict fixed points of multivalued operators satisfying some contractive type conditions with respect to the functional δ.

Notice that the set of strict fixed points of a multivalued operator T is a fixed set with respect to T, in the sense that $T(SFix(T)) = SFix(T)$.

In 1972, S. Reich proved the following very interesting strict fixed point theorem for multivalued operators.

Theorem 2.7. *If (X,d) is a complete metric space and if $T : X \to P_b(X)$ is a multivalued operator for which there exist $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$ such that*

$$\delta(T(x), T(y)) \leq ad(x,y) + b\delta(x, T(x)) + c\delta(y, T(y)), \text{ for all } x, y \in X,$$

then the following conclusions hold:

(i) $Fix(T) = SFix(T) = \{x^*\}$;

(ii) for each $x_0 \in X$ there exists a sequence of successive approximations for T starting from x_0 (which means that $x_{n+1} \in T(x_n)$, for each $n \in \mathbb{N}$) convergent to x^*;

(iii) $d(x_n, x^*) \leq \frac{\beta^n}{1-\beta} d(x_0, x_1)$, for $n \in \mathbb{N}^*$ (where $\beta := \min\{\frac{a+b}{1-c}, \frac{a+c}{1-b}\} < 1$).

An important extension of the above result is the following theorem of Ćirić, given in 1972.

Theorem 2.8. *Let (X,d) be a complete metric space and let $T : X \to P_b(X)$ be a multivalued operator for which there exists $q \in \mathbb{R}_+$ with $q < 1$ such that, for all $x, y \in X$ the following condition holds*

$$\delta(T(x),T(y)) \le q \max\{d(x,y), \delta(x,T(x)), \delta(y,T(y)), D(x,T(y)), D(y,T(x))\}.$$

Then the following conclusions hold:

(i) $Fix(T) = SFix(T) = \{x^*\}$;

(ii) for each $x_0 \in X$ there exists a sequence of successive approximations for T starting from x_0 convergent to x^*;

(iii) $d(x_n,x^*) \le \frac{q^{(1-a)n}}{1-q^{1-a}} d(x_0,x_1)$, for $n \in \mathbb{N}^*$ (where $a \in (0,1)$ is an arbitrary real number).

A very general result was proved by Rus [11], as follows.

Theorem 2.9. *Let (X,d) be a complete metric space and $T : X \to P_b(X)$ be a multivalued operator for which there exists a function $\varphi : \mathbb{R}^5_+ \to \mathbb{R}_+$ satisfying*
i) φ is increasing in each variable
ii) there exists $p > 1$ such that the function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\Phi(t) := \varphi(t, pt, pt, t, t)$$

satisfy the conditions that $\Phi^n(t) \to 0$ as $n \to +\infty$, for all $t > 0$ and $t - \Phi(t) \to +\infty$ as $t \to +\infty$, such that, for all $x, y \in X$, one have

$$\delta(T(x),T(y)) \le \varphi(d(x,y), \delta(x,T(x)), \delta(y,T(y)), D(x,T(y)), D(y,T(x))).$$

Then

$$Fix(T) = SFix(T) = \{x^*\}.$$

Remark 2.10. Notice again that if, in particular:
1) $\varphi(t_1,t_2,t_3,t_4,t_5) := at_1 + bt_2 + ct_3 + et_4 + ft_5$, $(a,b,c,e,f \in \mathbb{R}_+,\ a+b+c+e+f < 1)$
or
2) $\varphi(t_1,t_2,t_3,t_4,t_5) := q \max\{t_1,t_2,t_3,t_4,t_5\}$, $(q \in [0,1[)$,

then we obtain Theorem 2.7 and Theorem 2.8 from our previous considerations.

Remark 2.11. In the above two results, the approach is based on the construction of a selection $t : X \to X$ of T (i.e., $t(x) \in T(x)$, for each $x \in X$), which satisfies the corresponding fixed point theorem (Ćirić-Reich-Rus (Theorem 2.1), Ćirić (Theorem 2.5) and respectively Rus (Theorem 2.3), for the case of singlevalued operators.

Some extensions of the above theorems can be proved as follows. For example, we will give an extension of Reich's Theorem.

Theorem 2.12. *Let (X,d) be a complete metric space and let $T : X \to P_b(X)$ be a multivalued operator with close graph. Suppose there exist $a,b,c \in \mathbb{R}_+$ with $a+b+c < 1$ such that*

$$\delta(T(x),T(y)) \le ad(x,y) + b\delta(x,T(x)) + c\delta(y,T(y)),\ \text{for all } (x,y) \in Graph(T).$$

then the following conclusions hold:

(i) $Fix(T) = SFix(T) \neq \emptyset$;

(ii) *for each* $x_0 \in X$ *there exists a sequence of successive approximations for* T *starting from* x_0 *convergent to* x^**;*

(iii) $d(x_n, x^*) \le \frac{\beta^n}{1-\beta} d(x_0, x_1)$*, for* $n \in \mathbb{N}^*$ *(where* $\beta := \min\{\frac{a+bq}{1-c}, \frac{a+cq}{1-b}\} < 1$*, for any real* q *with* $1 < q < \min\{\frac{1-a-b}{c}, \frac{1-a-c}{b}\}$*).*

Proof. (i)+(ii) Let $x_0 \in X$. Then, by Lemma 1.1, for $1 < q < \frac{1-a-c}{b}$ there exists $x_1 \in T(x_0)$ such that

$$\delta(x_0, T(x_0)) \le q d(x_0, x_1).$$

Thus, we have that

$$\begin{aligned}
\delta(x_1, T(x_1)) &\le \delta(T(x_0), T(x_1)) \le a d(x_0, x_1) + b\delta(x_0, T(x_0)) + c\delta(x_1, T(x_1)) \\
&\le a d(x_0, x_1) + bq d(x_0, x_1) + c\delta(x_1, T(x_1)).
\end{aligned}$$

Hence

$$\delta(x_1, T(x_1)) \le \frac{a+bq}{1-c} d(x_0, x_1). \tag{2.4}$$

For $x_1 \in X$ we can get again $x_2 \in T(x_1)$ such that $\delta(x_1, T(x_1)) \le q d(x_1, x_2)$. Then

$$d(x_1, x_2) \le \delta(x_1, T(x_1)) \le \frac{a+bq}{1-c} d(x_0, x_1). \tag{2.5}$$

On the other hand, as before we have that

$$\begin{aligned}
\delta(x_2, T(x_2)) &\le \delta(T(x_1), T(x_2)) \le a d(x_1, x_2) + b\delta(x_1, T(x_1)) + c\delta(x_2, T(x_2)) \\
&\le a d(x_1, x_2) + bq d(x_1, x_2) + c\delta(x_2, T(x_2)).
\end{aligned}$$

Using (2.5) we obtain

$$\delta(x_2, T(x_2)) \le \frac{a+bq}{1-c} d(x_1, x_2) \le \left(\frac{a+bq}{1-c}\right)^2 d(x_0, x_1). \tag{2.6}$$

For $x_2 \in X$ we obtain $x_3 \in T(x_2)$ such that $\delta(x_2, T(x_2)) \le q d(x_2, x_3)$.
Then

$$d(x_2, x_3) \le \delta(x_2, T(x_2)) \le \left(\frac{a+bq}{1-c}\right)^2 d(x_0, x_1). \tag{2.7}$$

By this procedures we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ with the following properties:

(1) $(x_n, x_{n+1}) \in Graph(T)$ for each $n \in \mathbb{N}$;

(2) $d(x_n, x_{n+1}) \le \left(\frac{a+bq}{1-c}\right)^n d(x_0, x_1)$ for each $n \in \mathbb{N}$;

(3) $\delta(x_n, T(x_n)) \le \left(\frac{a+bq}{1-c}\right)^n d(x_0, x_1)$ for each $n \in \mathbb{N}$;

From (2) we obtain that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Since the metric space X is complete we get that the sequence is convergent, i.e., $x_n \to x^*$ as $n \to \infty$. By the closedness of the operator T and (1) we get that $x^* \in Fix(T)$. We will show now that $x^* \in SFix(T)$.

Indeed, since $x^* \in Fix(T)$ it is clear that $x^* \in Graph(T)$. Using the contractive condition with $x = y = x^*$ we obtain

$$\delta(T(x^*)) = \delta(T(x^*), T(x^*)) \le a d(x^*, x^*) + b\delta(x^*, T(x^*)) + c\delta(x^*, T(x^*)) = (b+c)\delta(T(x^*)).$$

Hence

$$\delta(T(x^*)) \le (b+c)\delta(T(x^*)). \tag{2.8}$$

This implies that $\delta(T(x^*)) = 0$ and hence $T(x^*)$ is a singleton. Thus $x^* \in SFix(T)$. Moreover, we also obtain that $Fix(T) = SFix(T)$.

(iii) The conclusion easily follows by (2). □

3. A fractal operator theory for generalized contractions

We will present now an existence and uniqueness result for the multivalued fractal operator generated by a multivalued generalized contraction. We will follow the method proposed in [6] and [8].

Let (X,d) be a metric space and $F : X \to P(X)$ be a multivalued operator. The fractal operator generated by F is denoted by $\hat{F} : P_{cp}(X) \to P_{cp}(X)$ and is defined by $Y \mapsto \hat{F}(Y)$

$$\hat{F}(Y) := \bigcup_{x\in Y} F(x)\,,\ \text{for each } Y \in P_{cp}(X)$$

A fixed point for $\hat{F}$ is a fixed set for F, i.e., a nonempty compact set A^* with the property $\hat{F}(A^*) = A^*$.

Concerning the above problem, we have the following result proved in [6]. For the sake of completeness we recall here the proof too.

Theorem 3.1. *Let (X,d) be a complete metric space and let $F : X \to P_{cl}(X)$ be an upper semicontinuous multivalued operator. Suppose that there exists a continuous and increasing (in each variable) function $\varphi : \mathbb{R}_+^3 \to \mathbb{R}_+$ such that the function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by*

$$\Psi(t) := \varphi(t,t,t)$$

satisfies the following properties:

(i) $\Psi^n(t) \to 0$ as $n \to +\infty$, for all $t > 0$;

(ii) $t - \Psi(t) \to +\infty$ as $t \to +\infty$.

Suppose also that

$$H(F(x),F(y)) \le \varphi(d(x,y), D(x,F(y)), D(y,F(x))),\ \text{for all } x,y \in X.$$

Then the fractal operator $\hat{F} : P_{cp}(X) \to P_{cp}(X)$ generated by F has a unique fixed point, i.e., there exists a unique $A^* \in P_{cp}(X)$ such that

$$\hat{F}(A^*) = A^*.$$

Proof. We will prove that $\hat{F}$ satisfies all the assumptions of Theorem 2.3 with a three argument function φ, i.e.,

$$H(\hat{F}(A),\hat{F}(B)) \le \varphi(H(A,B), H(A,\hat{F}(B)), H(B,\hat{F}(A))),\ \text{for all } A,B \in P_{cp}(X)\,.$$

Indeed we have:

$$\begin{aligned}
\rho(\hat{F}(A),\hat{F}(B)) &= \sup_{a\in A} \rho(F(a),\hat{F}(B)) = \sup_{a\in A}(\inf_{b\in B} \rho(F(a),F(b)))\\
&\le \sup_{a\in A}(\inf_{b\in B} H(F(a),F(b)))\\
&\le \sup_{a\in A}(\inf_{b\in B}(\varphi(d(a,b), D(a,F(b)), D(b,F(a)))))\\
&\le \sup_{a\in A} \varphi(\inf_{b\in B} d(a,b), \inf_{b\in B} D(a,F(b)), \inf_{b\in B} D(b,F(a)))\\
&= \sup_{a\in A} \varphi(D(a,B), D(a,F(B)), D(F(a),B))\\
&= \varphi(\sup_{a\in A} D(a,B), \sup_{a\in A} D(a,F(B)), \sup_{a\in A} D(F(a),B))\\
&= \varphi(\rho(A,B), \rho(A,F(B)), \rho(F(A),B))\\
&\le \varphi(H(A,B), H(A,F(B)), H(B,F(A)))\\
&= \varphi(H(A,B), H(A,\hat{F}(B)), H(B,\hat{F}(A))).
\end{aligned}$$

By the above inequality and the similar one for $\rho(F(B),F(A))$, we obtain that

$$H(F(A),F(B)) \leq \varphi(H(A,B),H(A,F(B)),H(B,F(A))).$$

As a consequence, by Theorem 2.3, applied for $\hat{F}$, we get that $\hat{F}$ has a unique fixed point in $P_{cp}(X)$, i.e., there exists a unique $A^* \in P_{cp}(X)$ such that $\hat{F}(A^*) = A^*$. □

Moreover, if (X,d) is a metric space and $F_1,...,F_m : X \to P(X)$ are multivalued operators, then the system $F = (F_1,...,F_m)$ is called an iterated multifunction system (IMS).

If the system $F = (F_1,...,F_m)$ is such that, for each $i \in \{1,2,\cdots m\}$, the multivalued operators $F_i : X \to P_{cp}(X)$ are upper semicontinuos, then the operator T_F defined as

$$T_F(Y) = \bigcup_{i=1}^{m} F_i(Y), \text{ for each } Y \in P_{cp}(X)$$

has the property that $T_F : P_{cp}(X) \to P_{cp}(X)$ and it is called the multivalued fractal operator generated by the IMS $F = (F_1,...,F_m)$.

A nonempty compact subset $A^* \subset X$ is said to be a self-similar set corresponding to the iterated multifunction system $F = (F_1,...,F_m)$ if and only if it is a fixed point for the associated multivalued fractal operator, i.e., $T_F(A^*) = A^*$.

In particular, if F_i are singlevalued continuous operators from X to X, then $f = (f_1,...,f_m)$ is called an iterated function system (briefly IFS) and the operator $T_f : P_{cp}(X) \to P_{cp}(X)$ given by

$$T_f(Y) = \bigcup_{i=1}^{m} f_i(Y), \text{ for each } Y \in P_{cp}(X)$$

is called the fractal operator generated by the IFS f. A fixed point of T_f is called a self-similar set corresponding to the IFS f. In certain conditions, the self-similar sets are fractals (see [12]).

An existence and uniqueness result for the multivalued fractal is the following theorem (see [6]).

Theorem 3.2. *Let (X,d) be a complete metric space and let $F_i : X \to P_{cl}(X)$ $(i \in \{1,2,\cdots,m\})$ be upper semicontinuous multivalued operators. Suppose that there exists continuous and increasing (in each variable) functions $\varphi_i : \mathbb{R}^3_+ \to \mathbb{R}_+$ $(i \in \{1,2,\cdots,m\})$ such that the function $\Psi_i : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\Psi_i(t) := \varphi_i(t,t,t), (i \in \{1,2,\cdots,m\})$ satisify, for each $i \in \{1,2,\cdots,m\}$, the following properties:*

(i) $\Psi_i^n(t) \to 0$ as $n \to +\infty$, for all $t > 0$;

(ii) $t - \Psi_i(t) \to +\infty$ as $t \to +\infty$.

Suppose also that, for each $i \in \{1,2,\cdots,m\}$, we have that

$$H(F_i(x),F_i(y)) \leq \varphi_i(d(x,y),D(x,F(y)),D(y,F(x))), \text{ for all } x,y \in X.$$

Then the multivalued fractal operator $T_F : P_{cp}(X) \to P_{cp}(X)$ generated by IMS $F := (F_1,...,F_m)$ has a unique fixed point, i.e., there exists a unique $A^* \in P_{cp}(X)$ such that $T_F(A^*) = A^*$.

Proof. For $A,B \in P_{cp}(X)$ and using the proof of the previous theorem, we have

$$\begin{aligned} H(T_F(A),T_F(B)) &= H(\bigcup_{i=1}^{m} F_i(A), \bigcup_{i=1}^{m} F_i(B)) \\ &\leq \max_{i\in\{1,2,\cdots,m\}} H(F_i(A),F_i(B)) \\ &\leq \max_{i\in\{1,2,\cdots,m\}} \varphi_i(H(A,B),H(A,F_i(B)),H(B,F_i(A))) \\ &\leq \max_{i\in\{1,2,\cdots,m\}} \varphi_i(H(A,B),H(T_F(B),A),H(T_F(A),B)) \\ &= \bar{\varphi}(H(A,B),H(T_F(B),A),H(T_F(A),B)), \end{aligned}$$

where $\bar{\varphi}(t_1,t_2,t_3) := \max_{i\in\{1,2,\cdots,m\}} \varphi_i(t_1,t_2,t_3)$. The conclusion follows again by Theorem 2.3 applied for T_F. $\square$

Let us consider now the case of multivalued Kannan operators. We will follow the method introduced in [8]. For the beginning, we recall the following concepts, see [8].

Definition 3.3. Let (X,d) be a metric space, $Y \in P_{b,cl}(X)$ and $F : Y \to P_{cp}(Y)$ be a multivalued α-Kannan operator, i.e., $\alpha \in [0, \frac{1}{2}[$ and

$$H(F(x),F(y)) \le \alpha(D(x,F(x)) + D(y,F(y))), \text{ for each } x,y \in X.$$

Suppose that that F is u.s.c. Then, we define:

1. the maximal displacement functional corresponding to F, $E_F : P_b(Y) \to \mathbb{R}_+ \cup \{+\infty\}$ by

$$E_F(A) := \sup\{D(x,F(x)) \mid x \in A\}.$$

2. the maximal displacement functional corresponding to the multivalued fractal operator $\hat{F}$ given by $E_{\hat{F}} : P_b(P_b(Y)) \to \mathbb{R}_+ \cup \{+\infty\}$

$$E_{\hat{F}}(U) := \sup\{E_F(A) \mid A \in U\}, \text{ for } U \subset P_b(Y).$$

We recall first the following auxiliary result, see [8].

Lemma 3.4. *Let (X,d) be a metric space, $Y \in P_{b,cl}(X)$ and $F : Y \to P_{cp}(Y)$ be a multivalued α-Kannan operator, Suppose that that F is u.s.c. Then:*

(a) $H(F(A),F(B)) \le \alpha(E_F(A) + E_F(B))$, for all $A,B \in P_{cp}(Y)$;
(b) $E_F(F(A)) \le \frac{\alpha}{1-\alpha} E_F(A)$, for all $A \in P_{cp}(Y)$;
(c) $E_{\hat{F}}(\hat{F}(U)) \le \frac{\alpha}{1-\alpha} E_{\hat{F}}(U)$, for all $U \subset P_{cp}(Y)$;
(d) $\delta_H(\hat{F}(U)) \le 2\alpha E_{\hat{F}}(U)$, for all $U \subset P_{cp}(Y)$,

We have the following result.

Theorem 3.5. *Let (X,d) be a complete metric space, $Y \subset X$ be a nonempty bounded, closed subset and $F : Y \to P_{cp}(Y)$ be a multivalued u.s.c. and α-Kannan operator. Then, there exists a unique $A^* \in P_{cp}(Y)$ such that $\hat{F}(A^*) = A^*$.*

Proof. Consider the following known construction in nonlinear analysis: $U_1 := \overline{\hat{T}(P_{cp}(Y))}$, $U_2 := \overline{\hat{T}(U_1)}$, ..., $U_{n+1} := \overline{\hat{T}(U_n)}$, for $n \in \mathbb{N}^*$. Then $U_{n+1} \subset U_n$ and $U_n \neq \emptyset$, for each $n \in \mathbb{N}^*$.

Then, by Lemma 3.4, we have:

$$\begin{aligned}\delta_H(U_{n+1}) &= \delta_H(\overline{\hat{T}(U_n)}) \le 2\alpha E_{\hat{T}}(U_n) = 2\alpha E_{\hat{T}}(\overline{\hat{T}(U_{n-1})}) \\ &\le 2\alpha \frac{\alpha}{1-\alpha} E_{\hat{T}}(U_{n-1}) \le \cdots \le 2\alpha \left(\frac{\alpha}{1-\alpha}\right)^n E_{\hat{T}}(P_{cp}(Y)) \to 0 \text{ as } n \to +\infty.\end{aligned}$$

By Cantor's Intersection Theorem

$$U_\infty := \bigcap_{n\in\mathbb{N}^*} U_n \neq \emptyset \text{ and } \delta(U_\infty) = 0.$$

As a consequence, $U_\infty = \{A^*\}$. $\square$

A fractal existence theorem for α-Kannan operators is the following.

Theorem 3.6. *Let (X,d) be a complete metric space, $Y \subset X$ be a nonempty bounded, closed subset and $F_i : Y \to P_{cp}(Y)$ $(i \in \{1,2,\cdots,m\})$ be upper semicontinuous multivalued operators. Suppose that, for each $i \in \{1,2,\cdots,m\}$, there exist $\alpha_i \in [0, \frac{1}{2}[$ such that*

$$H(F_i(x), F_i(y)) \le \alpha_i(D(x,F(x)) + D(y,F(y))), \text{ for all } x,y \in Y.$$

Then the multivalued fractal operator $T_F : P_{cp}(Y) \to P_{cp}(Y)$ generated by IMS $F := (F_1, ..., F_m)$ has a unique fixed point, i.e., there exists a unique $A^* \in P_{cp}(Y)$ such that $T_F(A^*) = A^*$.

Proof. We will consider again the construction from the previous theorem.

We denote $U_1 := \overline{T_F(P_{cp}(Y))}$, $U_2 := \overline{T_F(U_1)}$, ..., $U_{n+1} := \overline{T_F(U_n)}$, for $n \in \mathbb{N}^*$.

Notice that $U_{n+1} \subset U_n$ and $U_n \neq \emptyset$ for each $n \in \mathbb{N}^*$, since $T_F^{n+1}(x_n) \in U_n$, for $n \in \mathbb{N}^*$. Then, by a similar approach as above, we have

$$\begin{aligned}\delta_H(U_{n+1}) &= \delta_H(\overline{T_F(U_n)}) = \delta_H(T_F(U_n)) = \max_{1\le i\le m} \delta_H(T_i(U_n)) \\ &\le 2\alpha \left(\frac{\alpha}{1-\alpha}\right)^n \max_{1\le i\le m} E_{T_i}(P_{cp}(Y)) \to 0 \text{ as } n \to +\infty.\end{aligned}$$

As before, the conclusion follows again follows by Cantor's Intersection Theorem. □

The above result is just an existence theorem which do not gives the possibility to approximate the fractal (i.e., to construct pre-fractals) by successive approximations, for example. The following theorem is an existence, uniqueness and approximation result for a fractal via an iterated multi-function system of Kannan type operators, under a more restrictive assumption.

Theorem 3.7. *Let (X,d) be a complete metric space, $Y \subset X$ be a nonempty closed subset and $F : Y \to P_{cp}(Y)$ be a upper semicontinuous and α-Kannan multivalued operator with $\alpha \in [0, \frac{1}{3}[$. Then, there exists a unique $A^* \in P_{cp}(Y)$ such that $\hat{F}(A^*) = A^*$ and, for any $A \in P_{cp}(Y)$, the sequence $(\hat{F}^n(A))_{n\in\mathbb{N}}$ converges to A^*, as $n \to +\infty$.*

Proof. We will prove that $\hat{F}$ satisfies all the assumptions of Hardy-Rogers' Theorem, see Theorem 2.2. More precisely, we will prove that

$$H(\hat{F}(A), \hat{F}(B)) \le 2\alpha H(A,B) + \alpha(H(A,\hat{F}(B)) + H(B,\hat{F}(A))), \text{ for all } A,B \in P_{cp}(X).$$

Indeed we have:

$$\begin{aligned}\rho(\hat{F}(A), \hat{F}(B)) &= \sup_{a\in A} \rho(F(a), \hat{F}(B)) \\ &= \sup_{a\in A}(\inf_{b\in B} \rho(F(a), F(b))) \\ &\le \sup_{a\in A}(\inf_{b\in B} H(F(a), F(b))) \\ &\le \sup_{a\in A} \inf_{b\in B} \alpha(D(a,F(a)) + D(b,F(b))).\end{aligned}$$

Notice now that

$$D(a,F(a)) \le d(a,b) + D(b,F(A)) + H(F(A),F(a))$$

and

$$D(b,F(b)) \le d(b,a) + D(a,F(B)) + H(F(B),F(b)).$$

Then, we have:

$$\begin{aligned}
&\sup_{a\in A}\inf_{b\in B}\alpha(D(a,F(a))+D(b,F(b)))\\
&\qquad\le \alpha\sup_{a\in A}\inf_{b\in B}[2d(a,b)+D(b,F(A))+D(a,F(B))+H(F(A),F(a))+H(F(B),F(b))]\\
&\qquad\le \alpha\sup_{a\in A}[2\inf_{b\in B}d(a,b)+\sup_{b\in B}(D(b,F(A))+D(a,F(B))+H(F(A),F(a))+H(F(B),F(b)))]\\
&\qquad\le \alpha\sup_{a\in A}[2D(a,B)+\rho(B,F(A))+D(a,F(B))+H(F(A),F(a))+H(F(B),F(B))]\\
&\qquad\le \alpha[2\rho(A,B)+\rho(B,F(A))+\rho(A,F(B))+H(F(A),F(A))]\\
&\qquad\le \alpha[2H(A,B)+H(B,F(A))+H(A,F(B))]=2\alpha H(A,B)+\alpha(H(\hat{F}(A),B)+H(A,\hat{F}(B))).
\end{aligned}$$

By the above inequality and the similar one for $\rho(\hat{F}(B),\hat{F}(A))$, we obtain that

$$H(\hat{F}(A),\hat{F}(B))\le 2\alpha H(A,B)+\alpha(H(B,\hat{F}(A))+H(A,\hat{F}(B))).$$

As a consequence, by Theorem 2.2, applied for $\hat{F}$, we get that $\hat{F}$ has a unique fixed point $A^*\in P_{cp}(X)$ and, for any $A\in P_{cp}(Y)$, the sequence $(\hat{F}^n(A))_{n\in\mathbb{N}}$ converges to A^*, as $n\to+\infty$. □

Moreover, for an iterated multi-function system we have:

Theorem 3.8. *Let (X,d) be a complete metric space, $Y\subset X$ be a nonempty closed subset and $F_i:Y\to P_{cp}(Y)$ $(i\in\{1,2,\cdots,m\})$ be upper semicontinuous multivalued operators. Suppose that, for each $i\in\{1,2,\cdots,m\}$, there exist $\alpha_i\in[0,\frac{1}{3}[$ such that*

$$H(F_i(x),F_i(y))\le\alpha_i(D(x,F(x))+D(y,F(y))),\ \textit{for all } x,y\in Y.$$

Then the multivalued fractal operator $T_F:P_{cp}(Y)\to P_{cp}(Y)$ generated by IMS $F:=(F_1,...,F_m)$ has a unique fixed point, i.e., there exists a unique $A^*\in P_{cp}(Y)$ such that $T_F(A^*)=A^*$ and, for any $A\in P_{cp}(Y)$, the sequence $(T_F^n(A))_{n\in\mathbb{N}}$ converges to A^*, as $n\to+\infty$.

Proof. The proof runs in a similar way to the proof of Theorem 3.2, taking into account the conclusion of Theorem 3.7. □

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References

[1] L. B. Ćirić, *Generalized contraction and fixed point theorems*, Publ. Inst. Math., **12** (1971), 19–26. 1
[2] L. B. Ćirić, *Fixed points for generalized multi-valued contractions*, Mat. Vesnik, **9** (1972), 265–272. 1, 2
[3] L. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., **45** (1974), 267–273. 1, 2
[4] L. B. Ćirić, *Contractive type non-self mappings on metric spaces of hyperbolic type*, J. Math. Anal. Appl., **317** (2006), 28–42. 2
[5] G. E. Hardy, T. D. Rogers, *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull., **16** (1973), 201–206. 2.6
[6] A. Petruşel, *Ćirić type fixed point theorems*, Stud. Univ. Babeş-Bolyai Math., **59** (2014), 233–245. 2, 3, 3
[7] A. Petruşel, G. Petruşel, *Multivalued Picard operator*, J. Nonlinear Convex Anal., **13** (2012), 157–171. 2
[8] A. Petruşel, I. A. Rus, M. A. Şerban, *The multifractal operator and iterated multifunction systems generated by nonself multivalued operators*, Set-Valued Var. Anal., (in press). 3, 3, 3
[9] S. Reich, *Fixed point of contractive functions*, Boll. Un. Mat. Ital., **5** (1972), 26–42. 1, 2
[10] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc., **226** (1977), 257–288. 1, 2.6
[11] I. A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, (2001). 2, 2.6, 2
[12] M. Yamaguti, M. Hata, J. Kigani, *Mathematics of Fractals*, Translations Math. Monograph, Vol. 167, Amer. Math. Soc. Providence, (1997). 1, 3