



# Two different distributions of limit cycles in a quintic system

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## Abstract

In this paper, the conditions for bifurcations of limit cycles from a third-order nilpotent critical point in a class of quintic systems are investigated. Treat the system coefficients as parameters, we obtain explicit expressions for the first fourteen quasi-Lyapunov constants. As a result, fourteen or fifteen small amplitude limit cycles with different distributions could be created from the third-order nilpotent critical point by two different perturbations. ©2015 All rights reserved.

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## 1. Introduction

Recently years, many works have been devoted to study the center–focus problem which is also related to the so-called *cyclicity* of the point, see [1, 3, 4]. As far as the maximum number of small-amplitude limit cycles are concerned, there have been many results. For an elementary center or an elementary focus, one of the best-known results is  $M(2) = 3$ , which was solved by Bautin [2]. For  $n = 3$ , Yu and Tian have proved that there could be twelve limit cycles around a center point in a planar cubic-degree polynomial system [12]. For  $n = 4$ , an example of a quartic system with eight limit cycles bifurcated from a fine focus [5] was given. As far as bifurcation of limit cycles from degenerate critical points were concerned, they also have been investigated intensively. Especially, for nilpotent critical point, there were also many results about limit cycles, see [7, 9]. So far, regarding the family of polynomial differential systems, a complete

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classification of centers and isochronous centers has only been solved for quadratic polynomial systems, or simply quadratic systems. Recently, the conditions of center and isochronous center at the origin for a class of non-analytic quintic systems were studied in [8]. A class of nilpotent-Poincaré system was discussed in [10]. Two kinds of bifurcation phenomena in a quartic system were investigated in [11].

In this paper, we consider a quintic systems

$$\begin{aligned} \frac{dx}{dt} &= y + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{14}xy^4 + a_{05}y^5 + a_{31}x^3y \\ &\quad - \frac{3}{2}b_{13}x^2y^2 - 4b_{04}xy^3 + a_{04}y^4, \\ \frac{dy}{dt} &= -2x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 + b_{14}xy^4 + b_{05}y^5 \\ &\quad - \frac{3}{2}a_{31}x^2y^2 + b_{13}xy^3 + b_{04}y^4. \end{aligned} \tag{1.1}$$

We will show that two different distributions of fourteen or fifteen cycles can be given by different perturbations.

The rest of this paper will be organized as follows. In Section 2, some preliminary results in [6] will be given. In Section 3, the linear recursive formulae in [6] are used to compute the first fourteen quasi-Lyapunov constants and then obtain the sufficient and necessary conditions for a center. In Section 4, one kind of different bifurcation are discussed to confirm that fourteen limit cycles can bifurcate from quintic systems. In Section 5, another kind of interesting bifurcation phenomenon was discussed to confirm that fifteen limit cycles can bifurcate from quintic systems.

To perform the computations in this paper, we have used the computer algebra system–MATHEMATICA 7.

## 2. Preliminary results

In this section, some important results taken from [6] for center-focus problem of third-order nilpotent critical points in the planar dynamical systems are presented for convenience in future, for more detail, see [6].

It is well known that the origin of a system with a third-order monodromic critical point can be written in the following form of real autonomous planar system:

$$\begin{aligned} \frac{dx}{dt} &= y + \mu x^2 + \sum_{i+2j=3}^{\infty} a_{ij}x^i y^j = X(x, y), \\ \frac{dy}{dt} &= -2x^3 + 2\mu xy + \sum_{i+2j=4}^{\infty} b_{ij}x^i y^j = Y(x, y). \end{aligned} \tag{2.1}$$

**Theorem 2.1.** *For any positive integer  $s$  and a given real number sequence,*

$$\{c_{0\beta}\}, \beta \geq 3, \tag{2.2}$$

*one can construct successively the terms with the coefficients  $c_{\alpha\beta}$  satisfying  $\alpha \neq 0$  of the formal series,*

$$M(x, y) = y^2 + \sum_{\alpha+\beta=3}^{\infty} c_{\alpha\beta}x^\alpha y^\beta = \sum_{k=2}^{\infty} M_k(x, y), \tag{2.3}$$

*such that*

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) M - (s + 1) \left(\frac{\partial M}{\partial x} X + \frac{\partial M}{\partial y} Y\right) = \sum_{m=3}^{\infty} \omega_m(s, \mu)x^m, \tag{2.4}$$

*where  $M_k(x, y)$  is a  $k$ th-degree homogeneous polynomial of  $x, y$  for all  $k$  and  $s\mu = 0$ .*

**Theorem 2.2.** For  $\alpha \geq 1, \alpha + \beta \geq 3$  in (2.3) and (2.4),  $c_{\alpha\beta}$  can be uniquely determined by the recursive formula,

$$c_{\alpha\beta} = \frac{1}{(s+1)\alpha}(A_{\alpha-1,\beta+1} + B_{\alpha-1,\beta+1}); \tag{2.5}$$

and for  $m \geq 1, \omega_m(s, \mu)$  can be uniquely determined by the recursive formula,

$$\omega_m(s, \mu) = A_{m,0} + B_{m,0}, \tag{2.6}$$

where

$$\begin{aligned} A_{\alpha\beta} &= \sum_{k+j=2}^{\alpha+\beta-1} [k - (s+1)(\alpha - k + 1)]a_{kj}c_{\alpha-k+1,\beta-j}, \\ B_{\alpha\beta} &= \sum_{k+j=2}^{\alpha+\beta-1} [j - (s+1)(\beta - j + 1)]b_{kj}c_{\alpha-k,\beta-j+1}. \end{aligned} \tag{2.7}$$

have been set. The  $m$ th-order quasi-Lyapunov constant is defined as

$$\lambda_m = \frac{\omega_{2m+4}(s, \mu)}{2m - 4s - 1}. \tag{2.8}$$

Clearly, the recursive formulae in Theorem 2.2 are linear with respect to all  $c_{\alpha\beta}$ . Therefore, it is convenient to develop programs for computing quasi-Lyapunov constants by using computer algebraic system such as MATHEMATICA.

### 3. Quasi-Lyapunov constants and center conditions

According to Theorem 2.1, for system (1.1), we can find a positive integer  $s$  and a formal series  $M(x, y) = x^4 + y^2 + o(r^4)$ , such that (2.4) holds. Applying the recursive formulae in Theorem 2.2 to carry out calculations, we have

$$\begin{aligned} \omega_3 &= \omega_4 = \omega_5 = 0, \\ \omega_6 &= -\frac{1}{3}b_{21}(-1 + 4s), \\ \omega_7 &\sim 3(s+1)c_{03}, \\ \omega_8 &\sim -\frac{2}{5}(a_{12} + 3b_{03})(-3 + 4s), \\ \omega_9 &\sim 0, \\ \omega_{10} &\sim -\frac{4}{7}b_{03}(a_{21} + b_{12})(-5 + 4s), \\ \omega_{11} &\sim -\frac{3}{8}(4a_{04} - 3a_{21}b_{13} - 3b_{12}b_{13} + 4a_{04}s + 2a_{21}b_{13}s + 2b_{12}b_{13}s - 10c_{05} - 10sc_{05}), \\ \omega_{12} &\sim -\frac{4}{15}(a_{14} + 5b_{05})(-7 + 4s), \\ \omega_{13} &\sim -\frac{1}{5}(a_{21} + b_{12})(4a_{04} + 2a_{21}b_{13} - 3b_{12}b_{13})(-2 + s), \end{aligned} \tag{3.1}$$

(2.8) and (3.1) yield that

$$\begin{aligned} c_{03} &= 0, \\ c_{05} &= \frac{4a_{04} - 3a_{21}b_{13} - 3b_{12}b_{13} + 4a_{04}s + 2a_{21}b_{13}s + 2b_{12}b_{13}s}{10(1 + s)}. \end{aligned}$$

Furthermore, the quasi-Lyapunov constants can be computed in two cases and we obtain the following results.

**Proposition 3.1.** For system (1.1), one can determine successively the terms of the formal series  $M(x, y) = x^4 + y^2 + o(r^4)$ , such that

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)M - 2\left(\frac{\partial M}{\partial x}X + \frac{\partial M}{\partial y}Y\right) = \sum_{m=1}^{14} \mu_m[(2m - 5)x^{2m+4} + o(r^{30})], \tag{3.2}$$

where  $\mu_m$  is the  $m$ th-order quasi-Lyapunov constant at the origin of system (1.1),  $m = 1, 2, \dots, 14$ .

**Theorem 3.2.** For system (1.1), the first 14 quasi-Lyapunov constants at the origin are given by

$$\begin{aligned} \mu_1 &= -\frac{1}{3}b_{21}, \\ \mu_2 &= \frac{2}{5}(a_{12} + 3b_{03}), \\ \mu_3 &= -\frac{4}{7}b_{03}(a_{21} + b_{12}), \\ \mu_4 &= -\frac{4}{15}(a_{14} + 5b_{05}), \end{aligned} \tag{3.3}$$

**Case 1**  $s = 2$

$$\mu_5 = \frac{5}{77}(a_{21} + b_{12})(8b_{05} - 3a_{31}b_{13}),$$

**Subcase 1.1** If  $a_{31} \neq 0$

$$\begin{aligned} \mu_6 &= \frac{7}{195}(a_{21} + b_{12})(4a_{04}a_{31} + 2a_{21}a_{31}b_{13} + 20b_{04}b_{13} - 3a_{31}b_{12}b_{13}), \\ \mu_7 &= \frac{4}{3456}b_{13}(a_{21} + b_{12})b_{04}(338a_{21}a_{31} + 3780b_{04} - 607a_{31}b_{12}), \\ \mu_8 &= -\frac{2}{136769457825}a_{31}b_{13}(a_{21} + b_{12})(80445352a_{21}^3 - 156314652a_{21}^2b_{12} \\ &\quad - 71017440a_{21}b_{12}^2 + 165742564b_{12}^3 + 5569796925b_{14}), \\ \mu_9 &= \frac{a_{31}b_{13}(a_{21} + b_{12})}{9009513854294703750}(-34278223494715200a_{03}a_{21}^2 + 1220801069532512a_{21}^4 \\ &\quad + 254403983521437750a_{21}a_{31}^2 + 27280597988397600a_{03}a_{21}b_{12} + 1577422505111120a_{21}^3b_{12} \\ &\quad - 456873426028144125a_{31}^2b_{12} + 61558821483112800a_{03}b_{12}^2 - 9680067969729072a_{21}^2b_{12}^2 \\ &\quad - 233037428820256a_{21}b_{12}^3 + 9803651976487424b_{12}^4 + 711277409549581875b_{13}^2), \end{aligned}$$

If  $188200a_{21} - 109097b_{12} \neq 0$

$$\begin{aligned} \mu_{10} &= -\frac{a_{31}b_{13}(a_{21} + b_{12})}{6238962685090061920743750}(338a_{21} - 607b_{12})(-10135591702675084800a_{03}a_{21}^2 \\ &\quad + 728855954800765888a_{21}^4 + 450715180058017875000a_{21}a_{31}^2 - 15079645202623293600a_{03}a_{21}b_{12} \\ &\quad + 4622881761161239120a_{21}^3b_{12} - 261273506901113581875a_{31}^2b_{12} - 4944053499948208800a_{03}b_{12}^2 \\ &\quad - 2612963902077875568a_{21}^2b_{12}^2 - 5712510416295551744a_{21}b_{12}^3 + 794479292142797056b_{12}^4), \\ \mu_{11} &= -\frac{8a_{31}b_{13}(a_{21} + b_{12})^2(338a_{21} - 607b_{12})}{637677123248933811722235822890625(188200a_{21} - 109097b_{12})} \\ &\quad \times (-155642028484585897765178167500000a_{03}^2a_{21} + 336743948141046396161556345000000a_{05}a_{21} \\ &\quad - 40355648151917838258801873729600a_{03}a_{21}^3 + 1148400493414922637218817901376a_{21}^5 \\ &\quad + 90223583324032240640210640487500a_{03}^2b_{12} - 195205921946566092890740236825000a_{05}b_{12} \\ &\quad + 100322882780190497680921124272200a_{03}a_{21}^2b_{12} - 1326293171407512954279596130584a_{21}^4b_{12} \end{aligned}$$

$$-76510898452838076358607208186300a_{03}a_{21}b_{12}^2 - 6170624408595394755838000106236a_{21}^3b_{12}^2 + 13727075208396377958116148254400a_{03}b_{12}^3 + 12301152539987200173257718967391a_{21}^2b_{12}^3 - 5863030103422599488045010426776a_{21}b_{12}^4 - 132125804983772824131265140568b_{12}^5),$$

$$\mu_{12} = \frac{4a_{31}b_{13}(a_{21} + b_{12})^2(338a_{21} - 607b_{12})f_1}{10679729857381660502598890741386436337890625(188200a_{21} - 109097b_{12})^2}$$

$$\mu_{13} = \frac{4a_{31}b_{13}(a_{21} + b_{12})^2(338a_{21} - 607b_{12})f_2}{372280356448415349124270174605458040022939775390625(188200a_{21} - 109097b_{12})^2}$$

$$\mu_{14} = \frac{a_{31}b_{13}(a_{21} + b_{12})^2(338a_{21} - 607b_{12})f_3}{8252562029605892093247171522583951648012519764949218750000(188200 a_{21} - 109097b_{12})^2}.$$

If  $188200a_{21} - 109097b_{12} = 0$

$$\mu_{10} = -\frac{a_{31}b_{13}(a_{21} + b_{12})b_{12}^3}{5084013939209205328125000000000} (1426643860106455452000000a_{03} + 201350610870420881440183b_{12}^2),$$

$$\begin{aligned} \mu_{11} = & -\frac{52169a_{31}b_{13}(a_{21} + b_{12})^2b_{12}^2}{414338609096268815413804791492619015350172500000000000000000} \\ & \times (-150500519428614395475074557359365049484512000000000000a_{05} \\ & + 56268386998393642923015480077596363920739272000000000a_{31}^2b_{12} \\ & + 562763734141083071207110551380816318881287424943257b_{12}^4), \end{aligned}$$

$$\mu_{12} = \frac{4013a_{31}b_{13}(a_{21} + b_{12})b_{12}f_4}{2658937609067166421506187256600432193638074773150000000000000000000}$$

$$\mu_{13} = \frac{4013a_{31}b_{13}(a_{21} + b_{12})^2b_{12}^2f_5}{118016191431192450780373323438877262672731845160211853958507144782000000000000000000000000000000000000}.$$

**Subcase 1.2** If  $a_{31} = 0$

$$\begin{aligned} \mu_6 &= \frac{28}{13}(a_{21} + b_{12})b_{04}b_{13}, \\ \mu_7 &= -\frac{48}{11}b_{04}(a_{21} + b_{12})a_{04}. \end{aligned}$$

**Case 2**  $a_{04} = -\frac{1}{4}(2a_{21} - 3b_{12})b_{13}$ .

$$\begin{aligned} \mu_6 &= \frac{28}{39}(a_{21} + b_{12})b_{04}b_{13}, \\ \mu_7 &= 0, \\ \mu_8 &= -\frac{3}{3094}b_{04}(a_{21} + b_{12})a_{31}b_{13}b_{14}. \end{aligned}$$

Where  $f_i, i = 1, \dots, 6$  are given in Appendix. In the above expressions of  $\mu_k$ , for each  $k = 2, \dots, 14$ ,  $\mu_1 = \mu_2 = \dots = \mu_{k-1} = 0$  have been set.

Theorem 3.2 directly gives the following assertion.

**Proposition 3.3.** *The first fourteen quasi-Lyapunov constants at the origin of system (1.1) are zero if and only if one of the following conditions is satisfied:*

$$b_{21} = 0, a_{12} = -3b_{03}, a_{21} = -b_{12}, a_{14} = -5b_{05}; \tag{3.4}$$

$$b_{21} = a_{12} = b_{03} = a_{14} = b_{05} = a_{04} = b_{13} = 0; \tag{3.5}$$

$$b_{21} = a_{12} = b_{03} = a_{14} = b_{05} = a_{31} = b_{04} = 0. \tag{3.6}$$

From Propositions 3.3 we have the following theorem.

**Theorem 3.4.** *The origin of system (1.1) is a center if and only if the first fourteen quasi-Lyapunov constants are zero, that is, one of the condition in Proposition 3.3 is satisfied.*

*Proof.* When condition (3.4) is satisfied, system (1.1) can be brought to

$$\begin{aligned} \frac{dx}{dt} &= y - b_{12}x^2y - 3b_{03}xy^2 + a_{03}y^3 - 5b_{05}xy^4 + a_{05}y^5 + a_{31}x^3y \\ &\quad - \frac{3}{2}b_{13}x^2y^2 - 4b_{04}xy^3 + a_{04}y^4, \\ \frac{dy}{dt} &= -2x^3 + b_{12}xy^2 + b_{03}y^3 + b_{14}xy^4 + b_{05}y^5 - \frac{3}{2}a_{31}x^2y^2 + b_{13}xy^3 + b_{04}y^4, \end{aligned} \tag{3.7}$$

which has an analytic first integral

$$\begin{aligned} H(x, y) &= \frac{1}{2}y^2 + \frac{1}{2}x^4 - \frac{1}{2}b_{12}x^2y^2 - b_{03}xy^3 + \frac{1}{4}a_{03}y^4 - b_{05}xy^5 + \frac{1}{6}a_{05}y^6 \\ &\quad + \frac{1}{2}a_{31}x^3y - \frac{1}{2}b_{13}x^2y^3 - b_{04}xy^3 + \frac{1}{5}a_{04}y^5. \end{aligned} \tag{3.8}$$

When condition (3.5) holds, system (1.1) can be rewritten as

$$\begin{aligned} \frac{dx}{dt} &= y + a_{03}y^3 + a_{05}y^5 + a_{31}x^3y - 4b_{04}xy^3, \\ \frac{dy}{dt} &= -2x^3 + b_{12}xy^2 + b_{14}xy^4 - \frac{3}{2}a_{31}x^2y^2 + b_{04}y^4, \end{aligned} \tag{3.9}$$

whose vector field is symmetric with respect to the  $x$ -axis.

When condition (3.6) holds, system (1.1) becomes

$$\begin{aligned} \frac{dx}{dt} &= y + a_{21}x^2y + a_{03}y^3 + a_{05}y^5 + a_{04}y^4, \\ \frac{dy}{dt} &= -2x^3 + b_{12}xy^2 + b_{14}xy^4 + b_{13}xy^3, \end{aligned} \tag{3.10}$$

whose vector field is symmetric with respect to the  $y$ -axis. □

#### 4. Existence of fourteen limit cycles

Now, we will prove that fourteen limit cycles enclosing an elementary node at the origin of unperturbed system (1.1) can be bifurcated from the perturbed system of (1.1) the third-order nilpotent critical point  $O(0, 0)$  is a 14th-order weak focus.

$\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = \mu_7 = \mu_8 = \mu_9 = \mu_{10} = \mu_{11} = \mu_{12} = \mu_{13} = 0, \mu_{14} \neq 0$ , means that

**Theorem 4.1.** *The origin of system (1.1) is a 14th-order weak focus if and only if  $188200a_{21} - 109097b_{12} \neq 0$  and*

$$\begin{aligned} b_{21} &= b_{03} = a_{12} = 0, \\ a_{14} &= -\frac{15}{8}a_{31}b_{13}, b_{05} = \frac{3}{8}a_{31}b_{13}, \\ a_{04} &= -\frac{10}{189}(a_{21} + b_{12})b_{13}, \\ b_{04} &= -\frac{1}{3780}a_{31}(338a_{21} - 607b_{12}), \\ b_{14} &= -\frac{52}{5569796925}(4577a_{21} - 5251b_{12})(338a_{21} - 607b_{12})(a_{21} + b_{12}), \end{aligned} \tag{4.1}$$

$$\begin{aligned}
 a_{31}^2 &= -\frac{16(a_{21} + b_{12})}{2394873432826875(188200a_{21} - 109097b_{12})}(-633474481417192800a_{03}a_{21} \\
 &\quad + 45553497175047868a_{21}^3 - 309003343746763050a_{03}b_{12} + 243376612897529577a_{21}^2b_{12} \\
 &\quad - 406686856777396800a_{21}b_{12}^2 + 49654955758924816b_{12}^3), \\
 b_{13}^2 &= \frac{338a_{21} - 607b_{12}}{711277409549581875}(101414862410400a_{03}a_{21} - 3611837483824a_{21}^3 \\
 &\quad - 752674507459875a_{31}^2 + 101414862410400a_{03}b_{12} - 11153277685776a_{21}^2b_{12} \\
 &\quad + 8609551522080a_{21}b_{12}^2 + 16150991724032b_{12}^3).
 \end{aligned}$$

*Proof.* Solving  $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = \mu_7 = \mu_8 = \mu_9 = \mu_{10} = \mu_{11} = 0$ , we obtain above relations. Furthermore, we denote

$$\begin{aligned}
 F_1 &= \text{Factor}[\text{Resultant}[f_1, f_2, b_{12}]], \\
 F_2 &= \text{Factor}[\text{Resultant}[f_1, f_3, b_{12}]],
 \end{aligned}$$

then

$$G = \text{Resultant}[F_1, F_2, a_{03}] = -9.21710139189961 \times 10^{18640} a_{21}^{966} \neq 0$$

so the origin of system (1.1) is a 14th-order weak focus. □

Now, we study the perturbed system of (1.1), given by:

$$\begin{aligned}
 \frac{dx}{dt} &= \delta x + y + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{14}xy^4 + a_{05}y^5 + a_{31}x^3y - \frac{3}{2}b_{13}x^2y^2 - 4b_{04}xy^3 + a_{04}y^4, \\
 \frac{dy}{dt} &= \delta y - 2x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 + b_{14}xy^4 + b_{05}y^5 - \frac{3}{2}a_{31}x^2y^2 + b_{13}xy^3 + b_{04}y^4.
 \end{aligned} \tag{4.2}$$

When conditions in (4.1) hold,

$$\frac{\partial(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8, \mu_9, \mu_{10}, \mu_{11}, \mu_{12}, \mu_{13})}{\partial(b_{21}, a_{12}, b_{03}, a_{14}, b_{05}, a_{04}, b_{04}, b_{14}, b_{13}, a_{31}, b_{12}, a_{03})} \neq 0. \tag{4.3}$$

Further, it follows from Theorem 2.1 in [6] that

**Theorem 4.2.** *If the origin of system (1.1) is a 14th-order weak focus, for  $0 < \delta \ll 1$ , with a small perturbation to the coefficients of system (1.1), then, for system (4.2), in a small neighborhood of the origin, there exist exactly fourteen small amplitude limit cycles enclosing the origin  $O(0, 0)$ , which is an elementary node, see figure 1.*

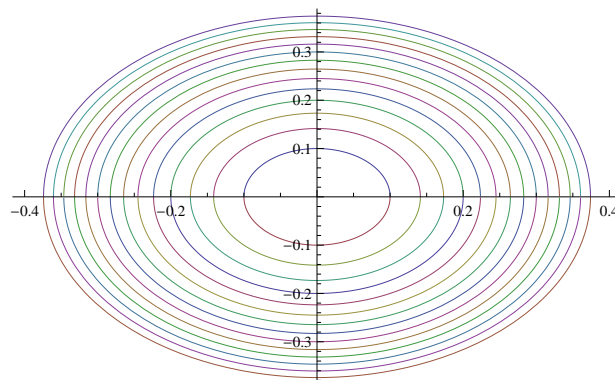


Fig1:Phase portrait of system (4.2)

### 5. Existence of fifteen limit cycles

An interesting bifurcation of limit cycles which is different from the first kind of bifurcation will be considered in this section. It is first time to consider this kind of bifurcation phenomena in a quintic system. The following perturbed system of (1.1)

$$\begin{aligned} \frac{dx}{dt} &= y + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{14}xy^4 + a_{05}y^5 + a_{31}x^3y \\ &\quad - \frac{3}{2}b_{13}x^2y^2 - 4b_{04}xy^3 + a_{04}y^4, \\ \frac{dy}{dt} &= 4\delta\epsilon y - (x^2 - \epsilon^2)(2x - b_{21}y) + b_{12}xy^2 \\ &\quad + b_{03}y^3 + b_{14}xy^4 + b_{05}y^5 - \frac{3}{2}a_{31}x^2y^2 + b_{13}xy^3 + b_{04}y^4. \end{aligned} \tag{5.1}$$

which is called double perturbed system of system (1.1) will be considered in this section. Obviously, when  $0 < |\epsilon| \ll 1$ , system (5.1) has three real singular points in the neighborhood of the origin, namely  $O(0, 0)$  and  $P_{1,2}(\pm\epsilon, 0)$ .

The following transformation

$$\begin{aligned} x &= \epsilon(u \pm 1), \\ y &= 2\epsilon^2 \frac{\delta u - \rho v}{1 \pm \epsilon(\pm a_{21}\epsilon + a_{31}\epsilon^2)}, t = \frac{\tau}{2\rho\epsilon}, \\ \rho &= \sqrt{(1 \pm \epsilon(\pm a_{21}\epsilon + a_{31}\epsilon^2)) - \delta^2}, \end{aligned}$$

can shift  $P_{1,2}(\pm\epsilon, 0)$  of system (5.1) to origin, and obtain a new system in the form of

$$\begin{aligned} \frac{d\xi}{d\tau} &= \Phi(\xi, \eta, \epsilon, \delta) = \frac{\delta\xi}{\rho} - \eta + \sum_{k+j=2}^{\infty} A_{kj}(\epsilon, \delta)\xi^k\eta^j, \\ \frac{d\eta}{d\tau} &= \Psi(\xi, \eta, \epsilon, \delta) = \xi + \frac{\delta\eta}{\rho} + \sum_{k+j=2}^{\infty} B_{kj}(\epsilon, \delta)\xi^k\eta^j, \end{aligned} \tag{5.2}$$

where  $\Phi(\xi, \eta, \epsilon, \delta)$  and  $\Psi(\xi, \eta, \epsilon, \delta)$  are power series in  $(u, v, \epsilon, \delta)$  with nonzero convergence radius. So  $P_{1,2}(\pm\epsilon, 0)$  of (5.1) are fine foci when  $\delta \neq 0$ , and weak foci or centers when  $\delta = 0$ . Especially for  $\delta = 0$ , let  $A = 1 + a_{21}\epsilon^2 - a_{31}\epsilon^3$ , corresponding to  $P_{1,2}(\pm\epsilon, 0)$ , system (5.1) are changed into the same systems

$$\begin{aligned} \frac{du}{dt} &= -v + \frac{1}{8\epsilon^2}(3b_{13}u^2v^2 - 2(a_{21} - 3a_{31}\epsilon)u^2v) - \frac{\sqrt{A}}{8\epsilon^3}a_{31}u^3v \\ &\quad - \frac{1}{2A}(2a_{05}v^5 + 2v^4(a_{04} - a_{14}\epsilon) + 2v^3(a_{03} + 4b_{04}\epsilon) + v^2\epsilon(2a_{12} + 3b_{13}\epsilon)) \\ &\quad + \frac{1}{2\epsilon\sqrt{A}}(4b_{04}uv^3 - a_{14}uv^4 - uv\epsilon(-2a_{21} + 3a_{31}\epsilon) - uv^2(a_{12} + 3b_{13}\epsilon)), \\ \frac{dv}{dt} &= u + \frac{1}{2\epsilon}b_{21}uv - \frac{1}{4\epsilon^2}(b_{13}uv^3 + b_{14}uv^4 - uv^2(b_{12} + 3a_{31}\epsilon)) - \frac{A}{8\epsilon^4}u^3 \\ &\quad + \frac{1}{4\epsilon\sqrt{A}}(v^2\epsilon(2b_{12} + 3a_{31}\epsilon) - 2b_{05}v^5 + 2v^3(-b_{03} + b_{13}\epsilon) + v^4(-b_{04} + b_{14}\epsilon)) \\ &\quad + \frac{\sqrt{A}}{16\epsilon^3}(3a_{31}u^2v^2 - 2b_{21}u^2v - 12\epsilon u^2). \end{aligned} \tag{5.3}$$

The first Lyapunov constant at origin for system (5.3) is given by

$$\begin{aligned} V_1 &= -i(-4(a_{12} + 3b_{03})\epsilon^2 + b_{21}(2 + \epsilon^2(2a_{21} - 2b_{12} - 5a_{31}\epsilon))(1 + \epsilon^2(a_{21} - a_{31}\epsilon)) \\ &\quad + 4\epsilon^4(-3a_{21}b_{03} + 3(a_{31}b_{03} + (a_{21} + b_{12})b_{13})\epsilon + a_{12}(a_{21} + 2b_{12} + a_{31}\epsilon))) \end{aligned}$$



When the the origin of system (1.1) is a 14th-order weak focus, the first Lyapunov constant of system (5.3) at origin is

$$V_1 = 3i(a_{21} + b_{12})b_{13}\varepsilon\sqrt{\frac{\varepsilon^2}{1 + \varepsilon^2(a_{21} - a_{31}\varepsilon)}} \neq 0.$$

Summarizing the above results yields the following theorem.

**Theorem 5.1.** *If the origin of system (1.1) is a 14-order weak focus, choosing proper coefficients in system (1.1), when  $0 < |\varepsilon| \ll 1$ , there exist fifteen limit cycles with the distribution of one limit cycle enclosing each of  $P_{1,2}(\pm\varepsilon, 0)$ , and thirteen limit cycles enclosing both  $(\varepsilon, 0)$  and  $(-\varepsilon, 0)$  in the neighborhood of origin, see figure 2.*

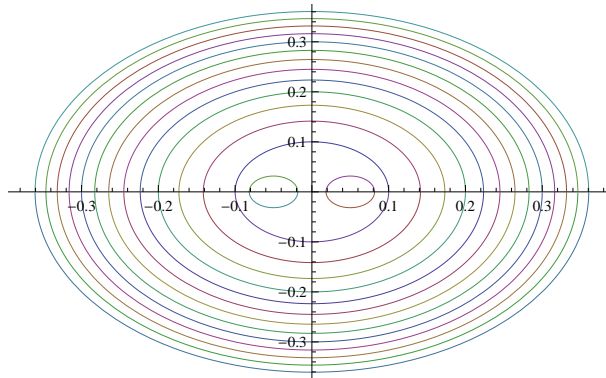


Fig2: Phase portrait of system (5.1)

We have studied an interesting bifurcation which, different from the first kind of bifurcation, can generate 15 limit cycles by perturb the quintic system with a nilpotent critical point. We set a new record of limit cycles bifurcated from an isolated critical point in a quintic systems.

### 6. Appendix

$$\begin{aligned} f_1 = & 398966693980473495044364802094971820834902560000a_{03}^2a_{21}^3 \\ & - 48489636594232851849374317431018923523189427200a_{03}a_{21}^5 \\ & + 795369119945534258251508745369820216529276416a_{21}^7 \\ & - 718400014293604148961831635987602940403519720000a_{03}^2a_{21}^2b_{12} \\ & + 152921183082778076676566272028978847544528804800a_{03}a_{21}^4b_{12} \\ & - 2510833550357441407998313646699136388042145696a_{21}^6b_{12} \\ & + 356317132340031901544189053015226441369740192500a_{03}^2a_{21}b_{12}^2 \\ & - 142549870063093476910061839908358922606626741800a_{03}a_{21}^3b_{12}^2 \\ & - 3455428951965305229795989093550467441472614016a_{21}^5b_{12}^2 \\ & - 34019195527703640579808183650038388221353777500a_{03}^2b_{12}^3 \\ & + 42046193252714160892045029852498992695979871300a_{03}a_{21}^2b_{12}^3 \\ & + 21051975770330342368859581962672863442883594940a_{21}^4b_{12}^3 \\ & - 12498424762400134909638954262681702039342989300a_{03}a_{21}b_{12}^4 \\ & - 28761099927352612229977897096247888639860250035a_{21}^3b_{12}^4 \\ & + 9286053737618789385106074298291885023438915600a_{03}b_{12}^5 \\ & + 17232637296910981974952758407427232827427356021a_{21}^2b_{12}^5 \end{aligned}$$

$$\begin{aligned}
& - 5393891532223612123819453540678290723571468976a_{21}b_{12}^6 \\
& + 1041396652581455625890548077863148594041425624b_{12}^7. \\
f_2 = & - 24714563406696142329573684187411776522882888937500000000a_{03}^3a_{21}^2 \\
& + 1017727874198815914002109649602054965255954430301285920000a_{03}^2a_{21}^4 \\
& - 148422427930752803914126116795233952782387191533318950400a_{03}a_{21}^6 \\
& + 2682109232981897661133268285053254093403584075746227712a_{21}^8 \\
& + 286533977043605636528108419106701656037933531818750000000a_{03}^3a_{21}b_{12} \\
& - 1299674420031486596597836223967266965076173923629962920000a_{03}^2a_{21}^3b_{12} \\
& + 460070887103259573203935871277792464270858686632567667200a_{03}a_{21}^5b_{12} \\
& - 8276801652742140668107662909968091300645710060209596320a_{21}^7b_{12} \\
& - 83049939674618076855225941018288604061558008822609375000a_{03}^3b_{12}^2 \\
& - 69440448669777049376409085900289684617949062201082467500a_{03}^2a_{21}^2b_{12}^2 \\
& - 443621265781240896460800763193165192038203974173872669000a_{03}a_{21}^4b_{12}^2 \\
& - 10520825400343865025006684044789304316223697816709103584a_{21}^6b_{12}^2 \\
& + 415688757403312721826752538448018350674540966641244130000a_{03}^2a_{21}b_{12}^3 \\
& + 217764784583944053962045612525475786665634995519422126500a_{03}a_{21}^3b_{12}^3 \\
& + 63018255228791867093127279940419729229402897749466265308a_{21}^5b_{12}^3 \\
& - 55511671671230969621513993548815913178060524469785680000a_{03}^2b_{12}^4 \\
& - 150405163297367214664803827521331260798692606577521099750a_{03}a_{21}^2b_{12}^4 \\
& - 81212786659593733006465951076881491007503048628520436715a_{21}^4b_{12}^4 \\
& + 62638746628191947529935527186175452863981811173582531600a_{03}a_{21}b_{12}^5 \\
& + 44237642447032098451551492019908714431134104741251139502a_{21}^3b_{12}^5 \\
& + 4160463031238415844056885463309129897149201232764013200a_{03}b_{12}^6 \\
& - 11935302354245668551207290055170662557283042763600481520a_{21}^2b_{12}^6 \\
& + 1287096054394367010552995175059704614495457345335390736a_{21}b_{12}^7 \\
& + 715421429145973100067384633366635978935926698260090208b_{12}^8. \\
f_3 = & - 242425111786067976611212182225062143222606705035240605770200000000a_{03}^3a_{21}^3 \\
& + 286176795709517811613300792495519634395876066229883531326988480000a_{03}^2a_{21}^5 \\
& - 44125762841748835770765043941258701655034818510078013953397017600a_{03}a_{21}^7 \\
& + 837471878111822708660446314728994722724368314674489772604105728a_{21}^9 \\
& + 312138302472308913359697042864978793128990046500043773448830000000a_{03}^3a_{21}^2b_{12} \\
& - 305263154519473514379906570282196551888125765520185141662879200000a_{03}^2a_{21}^4b_{12} \\
& + 131354595996150828785722536371551778663609512818927726919469563200a_{03}a_{21}^6b_{12} \\
& - 2447136222999108280771835384233498744422637393908750630069481152a_{21}^8b_{12} \\
& - 93007133216868706311168625103660998300617139060256504669049375000a_{03}^3a_{21}b_{12}^2 \\
& - 67491646847285835801489252463937674675646180975607063536040075000a_{03}^2a_{21}^3b_{12}^2 \\
& - 114147067067937716252939070627255381838999506368623854930267049200a_{03}a_{21}^5b_{12}^2
\end{aligned}$$

$$\begin{aligned}
& - 3436789214613228110060304335826676630363142487122204734533059616a_{21}^7b_{12}^2 \\
& - 6679407945723866273209731197111782267936176447253382010391875000a_{03}^3b_{12}^3 \\
& + 49675063664528688899519027071566553193432156094515703994477462500a_{03}^2a_{21}^2b_{12}^3 \\
& + 40244690648760616897412434507383958963317715594253349285594897000a_{03}a_{21}^4b_{12}^3 \\
& + 18155731747727760359007038150618851901069482854402072485014314696a_{21}^6b_{12}^3 \\
& + 51667687988519077921098737236038449592418283405559297256423762500a_{03}^2a_{21}b_{12}^4 \\
& - 13725916410357744991150569902830560754549755488285599769406776250a_{03}a_{21}^3b_{12}^4 \\
& - 20664374965517969399829070811074229972170966626262965504290147118a_{21}^5b_{12}^4 \\
& - 12619458923680791320787500237912021139695476494806687407032345000a_{03}^2b_{12}^5 \\
& - 11928164637411067449940797361536394448675173021913953233870482850a_{03}a_{21}^2b_{12}^5 \\
& + 8085098291409163429839177180184469247412418128808767492480312503a_{21}^4b_{12}^5 \\
& + 13650448892781387770450511811599955198758287787657241068621418200a_{03}a_{21}b_{12}^6 \\
& + 119818520509067685276902575048298554460514128879384659016185383a_{21}^3b_{12}^6 \\
& - 691512581416122560482950680254949027460046163262203156122777200a_{03}b_{12}^7 \\
& - 1277626368239126953717300563535651351675419666165946771422284586a_{21}^2b_{12}^7 \\
& + 590394321779533618268138537203223052339144766435359212612924376a_{21}b_{12}^8 \\
& + 33950330025292611485792925477013496666630065842418847365459792b_{12}^9, \\
f_4 = & 4196091763511947009497011770511840390597811942750000000000000000a_{31}^4 \\
& + 100702447347231847733691331480916247238597348008919323000000000a_{31}^2b_{12}^3 \\
& - 1202931658308219887576021440917870023330474424551091103542581b_{12}^6, \\
f_5 = & 961791304412896182650862376639025129593197438322536576517122889429170000000000000000a_{31}^4 \\
& + 240490664152978819978234052280101093876830358741540951676460504724830703604000000000a_{31}^2b_{12}^3 \\
& - 2269154758844849936490830061600992479517647152406514419340067910036554090451307671b_{12}^6.
\end{aligned}$$

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