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Positive solutions for a class of q-fractional boundary value problems with p-Laplacian

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Abstract

By meaning of the upper and lower solutions method, we study the existence of positive solutions for a class of q-fractional boundary value problems with p-Laplacian.

Keywords: q-fractional boundary value problem, p-Laplacian, positive solution, upper and lower solutions method.
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1. Introduction

In this paper we investigate the existence of positive solutions for the q-fractional boundary value problems with p-Laplacian

$$\begin{cases} D_q^{\beta}(\varphi_p(D_q^{\alpha}u(t))) = f(t, u(t)),\ t \in (0, 1), \\ u(0) = 0,\ u(1) = \int_0^1 h(t)u(t)d_qt,\ D_q^{\alpha}u(0) = 0,\ D_q^{\alpha}u(1) = bD_q^{\alpha}u(\eta), \end{cases} \tag{1.1}$$

where D_q^{α}, D_q^{β} are the fractional q-derivative of the Riemann-Liouville type with $1 < \alpha, \beta \leq 2$, $0 \leq b \leq 1$, $0 < \eta < 1$, $\varphi_p(s) = |s|^{p-2}s$, $\varphi_p^{-1} = \varphi_r$, $p^{-1} + r^{-1} = 1$, $p > 1$,$r > 1$, and $f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R}^+), h \in C([0, 1], \mathbb{R}^+)(\mathbb{R}^+ := [0, +\infty))$.

Fractional differential equations can describe many phenomena in various fields of science and engineering such as physics, mechanics, chemistry, control, engineering, etc. In recent years there are a large number of papers dealing with the existence of solutions (or positive solutions) of nonlinear fractional differential equations by virtue of techniques of nonlinear analysis, for example, see [2, 3, 4, 5, 7, 8, 10, 11, 12, 13] and the references therein.

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In [2], R. Almeida and N. Martins discussed the fractional q-difference equation

$$\begin{cases} {}^C D_q^\alpha[x](t) = g(t, x(t)), 0 \le t \le 1, \\ x(0) = \gamma_0, D_q[x](0) = \gamma_1, \\ x(1) = \gamma_2 \int_0^\eta x(s) d_q s, \end{cases} \tag{1.2}$$

and presented some sufficient conditions regarding the existence and uniqueness of solutions for (1.2). Their arguments are based on fixed point theorems: Banach fixed point theorem, Krasnoselskii fixed point theorem and Leray-Schauder alternative.

As known to all, the upper and lower solutions method is an effective tool to deal with the existence of solutions for nonlinear differential equations, see [5, 7, 10, 11, 12]. However, to the best of our knowledge, few results exist in the literatures devoted to investigate integral boundary conditions by applying the method. Motivated by the above works, in this paper we apply the upper and lower solutions method as well as the Schauder fixed point theorem to establish a new existence result of at least one positive solution for (1.1).

2. Preliminaries

Let $q \in (0, 1)$ and define

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad a \in \mathbb{R}.$$

The q-analogue of the power function $(a - b)^n$ with $\mathbb{N}_0$ is

$$(a - b)^0 = 1, \quad (a - b)^n = \prod_{k=0}^{n-1} (a - bq^k), \quad n \in \mathbb{N},\ a, b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a - b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a - bq^n}{a - bq^{\alpha+n}}.$$

Note that, if $b = 0$ then $a^{(\alpha)} = a^\alpha$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1 - q)^{(x-1)}}{(1 - q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\},$$

and satisfies $\Gamma_q(x + 1) = [x]\Gamma_q(x)$. The q-derivative of a function f is here defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \quad (D_q f)(0) = \lim_{x \to 0} (D_q f)(x),$$

and q-derivatives of higher order by

$$(D_q^0 f)(x) = f(x) \quad \text{and} \quad (D_q^n f)(x) = D_q(D_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The q-integral of a function f defined in the interval $[0, b]$ is given by

$$(I_q f)(x) = \int_0^x f(t) d_q t = x(1 - q) \sum_{n=0}^{\infty} f(xq^n) q^n, \quad x \in [0, b].$$

If $a \in [0, b]$ and f is defined in the interval $[0, b]$, its integral from a to b is defined by

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t.$$

Similarly as done for derivatives, an operator I_q^n can be defined, i.e.,

$$(I_q^0 f)(x) = f(x) \quad \text{and} \quad (I_q^n f)(x) = I_q(I_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, i.e.,

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

Basic properties of the two operators can be found in the book [6]. We now point out three formulas that will be used later (${}_iD_q$ denotes the derivative with respect to variable i)

$$\begin{aligned}
&[a(t-s)]^{(\alpha)} = a^{\alpha}(t-s)^{(\alpha)},\\
&{}_tD_q(t-s)^{(\alpha)} = [\alpha]_q(t-s)^{(\alpha-1)},\\
&\Big({}_xD_q \int_0^x f(x,t)d_qt\Big)(x) = \int_0^x {}_xD_q f(x,t)d_qt + f(qx,x).
\end{aligned} \tag{2.1}$$

We note that if $\alpha > 0$ and $a \le b \le t$, then $(t-a)^{(\alpha)} \ge (t-b)^{(\alpha)}$ (see [3]). The following definition was considered first in [1].

Definition 2.1. Let $\alpha \ge 0$ and f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is $(I_q^0 f)(x) = f(x)$ and

$$(I_q^{\alpha} f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)^{(\alpha-1)} f(t)d_qt, \quad \alpha > 0,\ x \in [0,1].$$

Definition 2.2. (see [9]) The fractional q-derivative of the Riemann-Liouville type of order $\alpha \ge 0$ is defined by $(D_q^0 f)(x) = f(x)$ and

$$(D_q^{\alpha} f)(x) = (D_q^m I_q^{m-\alpha} f)(x), \quad \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Next, we list some properties that are already known in the literature. Its proof can be found in [1, 9].

Lemma 2.3. *Let $\alpha, \beta \ge 0$ and f be a function defined on $[0,1]$. Then the next formulas hold:*
(i) $(I_q^{\beta} I_q^{\alpha} f)(x) = (I_q^{\alpha+\beta} f)(x)$,
(ii) $(D_q^{\alpha} I_q^{\alpha} f)(x) = f(x)$.

Lemma 2.4. *(see [3]) Let $\alpha > 0$ and p be a positive integer. Then the following equality holds:*

$$(I_q^{\alpha} D_q^p f)(x) = (D_q^p I_q^{\alpha} f)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} (D_q^k f)(0).$$

Throughout this paper we always assume that the following condition holds:
(H1) $\kappa := 1 - \int_0^1 h(t)t^{\alpha-1}d_qt > 0$.

Lemma 2.5. *Suppose that (H1) holds. Let $y \in C[0,1]$ and $1 < \alpha \le 2$. Then*

$$\begin{cases} D_q^{\alpha} u(t) + y(t) = 0, t \in (0,1),\\ u(0) = 0, u(1) = \displaystyle\int_0^1 h(t)u(t)d_qt, \end{cases} \tag{2.2}$$

is equivalent to

$$u(t) = \int_0^1 G(t, qs)y(s)d_qs,$$

where

$$G(t, s) = g(t, s) + \frac{t^{\alpha-1}}{\kappa} \int_0^1 h(t)g(t, s)d_qt,$$

$$g(t, s) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} (t(1-s))^{(\alpha-1)} - (t-s)^{(\alpha-1)}, & 0 \le s \le t \le 1, \\ (t(1-s))^{(\alpha-1)}, & 0 \le t \le s \le 1. \end{cases}$$

Proof. By Lemma 2.4 we have

$$u(t) = -I_q^\alpha y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2},\ c_1, c_2 \in \mathbb{R}.$$

From $u(0) = 0$ we obtain $c_2 = 0$. Consequently,

$$u(t) = -I_q^\alpha y(t) + c_1 t^{\alpha-1} = -\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s)d_qs + c_1 t^{\alpha-1}.$$

Hence $u(1) = \int_0^1 h(t)u(t)d_qt$ implies that

$$c_1 = \int_0^1 h(t)u(t)d_qt + \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s)d_qs,$$

and

$$\begin{aligned} u(t) &= -\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s)d_qs + t^{\alpha-1} \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s)d_qs + t^{\alpha-1} \int_0^1 h(t)u(t)d_qt \\ &= \int_0^1 g(t, qs)y(s)d_qs + t^{\alpha-1} \int_0^1 h(t)u(t)d_qt. \end{aligned} \tag{2.3}$$

Multiplying $h(t)$ on both sides of (2.3) and integrating over $[0, 1]$, we find

$$\int_0^1 h(t)u(t)d_qt = \int_0^1 h(t) \int_0^1 g(t, qs)y(s)d_qsd_qt + \int_0^1 h(t)t^{\alpha-1}d_qt \int_0^1 h(t)u(t)d_qt.$$

By (H1) we have

$$\int_0^1 h(t)u(t)d_qt = \frac{1}{\kappa} \int_0^1 h(t) \int_0^1 g(t, qs)y(s)d_qsd_qt.$$

Combining this with (2.3) we obtain

$$\begin{aligned} u(t) &= \int_0^1 g(t, qs)y(s)d_qs + \frac{t^{\alpha-1}}{\kappa} \int_0^1 h(t) \int_0^1 g(t, qs)y(s)d_qsd_qt \\ &= \int_0^1 G(t, qs)y(s)d_qs. \end{aligned}$$

This completes the proof. □

Lemma 2.6. *Suppose that (H1) holds. Let $y \in C[0, 1]$, $1 < \alpha, \beta \le 2$, $0 \le b \le 1$, $0 < \eta < 1$. Then*

$$\begin{cases} D_q^\beta(\varphi_p(D_q^\alpha u(t))) = y(t),\ t \in (0, 1), \\ u(0) = 0,\ u(1) = \int_0^1 h(t)u(t)d_qt,\ D_q^\alpha u(0) = 0,\ D_q^\alpha u(1) = bD_q^\alpha u(\eta), \end{cases} \tag{2.4}$$

is equivalent to

$$u(t)=\int_0^1 G(t,qs)\varphi_r\left(\int_0^1 H(s,q\tau)y(\tau)d_q\tau\right)d_qs,$$

where

$$H(t,s)=m(t,s)+\frac{b^{p-1}t^{\beta-1}}{1-b^{p-1}\eta^{\beta-1}}m(\eta,s),$$

$$m(t,s)=\frac{1}{\Gamma_q(\beta)}\begin{cases}(t(1-s))^{(\beta-1)}-(t-s)^{(\beta-1)}, & 0\le s\le t\le 1,\\ (t(1-s))^{(\beta-1)}, & 0\le t\le s\le 1.\end{cases}$$

Proof. By Lemma 2.4 we have

$$\varphi_p(D_q^\alpha u(t))=I_q^\beta y(t)+c_3t^{\beta-1}+c_4t^{\beta-2},\ c_3,c_4\in\mathbb{R}.$$

From $D_q^\alpha u(0)=0$ we obtain $c_4=0$. Consequently,

$$\varphi_p(D_q^\alpha u(t))=I_q^\beta y(t)+c_3t^{\beta-1}=\int_0^t\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}y(s)d_qs+c_3t^{\beta-1}.$$

By (2.4) we obtain

$$\varphi_p(D_q^\alpha u(1))=\int_0^1\frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)}y(s)d_qs+c_3,$$
$$\varphi_p(D_q^\alpha u(\eta))=\int_0^\eta\frac{(\eta-qs)^{(\beta-1)}}{\Gamma_q(\beta)}y(s)d_qs+c_3\eta^{\beta-1},$$

and

$$\int_0^1\frac{(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)}y(s)d_qs+c_3=b^{p-1}\int_0^\eta\frac{(\eta-qs)^{(\beta-1)}}{\Gamma_q(\beta)}y(s)d_qs+c_3b^{p-1}\eta^{\beta-1}.$$

Hence

$$c_3=b^{p-1}\int_0^\eta\frac{(\eta-qs)^{(\beta-1)}}{(1-b^{p-1}\eta^{\beta-1})\Gamma_q(\beta)}y(s)d_qs-\int_0^1\frac{(1-qs)^{(\beta-1)}}{(1-b^{p-1}\eta^{\beta-1})\Gamma_q(\beta)}y(s)d_qs.$$

As a result,

$$\begin{aligned}\varphi_p(D_q^\alpha u(t))&=\int_0^t\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}y(s)d_qs-t^{\beta-1}\int_0^1\frac{(1-qs)^{(\beta-1)}}{(1-b^{p-1}\eta^{\beta-1})\Gamma_q(\beta)}y(s)d_qs\\&\quad+t^{\beta-1}b^{p-1}\int_0^\eta\frac{(\eta-qs)^{(\beta-1)}}{(1-b^{p-1}\eta^{\beta-1})\Gamma_q(\beta)}y(s)d_qs\\&=\int_0^t\frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}y(s)d_qs-\int_0^1\frac{(t(1-qs))^{(\beta-1)}}{\Gamma_q(\beta)}y(s)d_qs+\int_0^1\frac{(t(1-qs))^{(\beta-1)}}{\Gamma_q(\beta)}y(s)d_qs\\&\quad-\int_0^1\frac{(t(1-qs))^{(\beta-1)}}{(1-b^{p-1}\eta^{\beta-1})\Gamma_q(\beta)}y(s)d_qs+t^{\beta-1}b^{p-1}\int_0^\eta\frac{(\eta-qs)^{(\beta-1)}}{(1-b^{p-1}\eta^{\beta-1})\Gamma_q(\beta)}y(s)d_qs\\&=-\int_0^1 m(t,qs)y(s)d_qs-\frac{b^{p-1}t^{\beta-1}}{1-b^{p-1}\eta^{\beta-1}}\int_0^1 m(\eta,qs)y(s)d_qs\\&=-\int_0^1 H(t,qs)y(s)d_qs.\end{aligned}$$

Consequently,

$$\begin{cases}D_q^\alpha u(t)+\varphi_r\left(\int_0^1 H(t,qs)y(s)d_qs\right)=0,\\ u(0)=0,u(1)=\int_0^1 h(t)u(t)d_qt.\end{cases}$$

Combining this with Lemma 2.5, we have

$$u(t) = \int_0^1 G(t, qs)\varphi_r \left(\int_0^1 H(s, q\tau) y(\tau) d_q\tau \right) d_qs.$$

This completes the proof. $\square$

Lemma 2.7. *$G(t, s)$ and $H(t, s)$ defined above have the following properties:*
(i) *G, H are continuous on $[0, 1] \times [0, 1]$ and $G(t, qs) \geq 0$, $H(t, qs) \geq 0$ for all $t, s \in [0, 1]$,*
(ii) *for any $t, s \in [0, 1]$,*

$$\sigma_1(qs)t^{\alpha-1} \leq G(t, qs) \leq \sigma_2(qs)t^{\alpha-1},$$

where

$$\sigma_1(qs) = \frac{1}{\kappa} \int_0^1 h(t)g(t, qs)d_qt, \ \ \sigma_2(qs) = \frac{1}{\kappa} \int_0^1 h(t)g(t, qs)d_qt + \frac{(1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}.$$

Lemma 2.8. *(see [11, Lemma 2.8]) Let $u \in C[0, 1]$ satisfy $u(0) = 0$, $u(1) = \varphi_p(b)u(\eta)$ and $D_q^\beta u(t) \geq 0$ for all $t \in (0, 1)$. Then $u(t) \leq 0$ for $t \in [0, 1]$.*

Let $E := \{u | u, \varphi_p(D_q^\alpha u) \in C^2[0, 1]\}$. Now we introduce the following definitions about the upper and lower solutions for (1.1).

Definition 2.9. A function ϕ is called a lower solution for (1.1), if $\phi \in E$ satisfies

$$\begin{cases} D_q^\beta(\varphi_p(D_q^\alpha \phi(t))) \leq f(t, \phi(t)), \ t \in (0, 1), \\ \phi(0) \leq 0, \ \phi(1) \leq \displaystyle\int_0^1 h(t)\phi(t)d_qt, \ D_q^\alpha\phi(0) \geq 0, \ D_q^\alpha\phi(1) \geq bD_q^\alpha\phi(\eta). \end{cases}$$

Definition 2.10. A function ψ is called an upper solution for (1.1), if $\psi \in E$ satisfies

$$\begin{cases} D_q^\beta(\varphi_p(D_q^\alpha \psi(t))) \geq f(t, \psi(t)), \ t \in (0, 1), \\ \psi(0) \geq 0, \ \psi(1) \geq \displaystyle\int_0^1 h(t)\psi(t)d_qt, \ D_q^\alpha\psi(0) \leq 0, \ D_q^\alpha\psi(1) \leq bD_q^\alpha\psi(\eta). \end{cases}$$

Define $A : E \to E$

$$(Au)(t) = \int_0^1 G(t, qs)\varphi_r \left(\int_0^1 H(s, q\tau) f(\tau, u(\tau)) d_q\tau \right) d_qs.$$

Then, by Lemma 2.6 we obtain that the existence of solutions for (1.1) is equivalent to the existence of fixed points for the operator A. Furthermore, the continuity G, H and f enables us to prove A is a completely continuous operator.

3. Main results

Theorem 3.1. *Suppose that (H1) and the following conditions hold:*
(H2) *$f \in C([0, 1] \times [0, +\infty), (0, +\infty))$ and $f(t, u)$ is increasing in u,*
(H3) *there exists $c \in (0, 1)$ such that*

$$f(t, \mu u) \geq \mu^{c(p-1)} f(t, u), \ \forall \mu \in [0, 1], \ t \in [0, 1], \ \textit{where } p > 1.$$

Then (1.1) has at least one positive solution.

Proof. We divide four steps.

Step 1. If u is a positive solution for (1.1), then there exist $m_1, m_2 > 0$ such that

$$m_1\rho(t) \le u(t) \le m_2\rho(t), \tag{3.1}$$

where

$$\rho(t) = \int_0^1 G(t,qs)\varphi_r\left(\int_0^1 H(s,q\tau)d_q\tau\right)d_qs.$$

Indeed, $u \in C[0,1]$ implies that there exists $M > 0$ such that

$$|u(t)| \le M,\ \forall t \in [0,1].$$

By (H2) we can choose

$$m_1 := \min_{t\in[0,1],u\in[0,M]} \sqrt[p-1]{f(t,u(t))} > 0, \quad m_2 := \max_{t\in[0,1],u\in[0,M]} \sqrt[p-1]{f(t,u(t))} > 0.$$

Then

$$m_1\rho(t) \le u(t) = (Au)(t) = \int_0^1 G(t,qs)\varphi_r\left(\int_0^1 H(s,q\tau)f(\tau,u(\tau))d_q\tau\right)d_qs \le m_2\rho(t).$$

Step 2. The existence of upper and lower solutions for (1.1).

Let

$$\xi(t) = \int_0^1 G(t,qs)\varphi_r\left(\int_0^1 H(s,q\tau)f(\tau,\rho(\tau))d_q\tau\right)d_qs.$$

Then by Lemma 2.6 we obtain ξ is a positive solution for the problem

$$\begin{cases} D_q^\beta(\varphi_p(D_q^\alpha u(t))) = f(t,\rho(t)),\ t \in (0,1), \\ u(0) = 0,\ u(1) = \int_0^1 h(t)u(t)d_qt,\ D_q^\alpha u(0) = 0,\ D_q^\alpha u(1) = bD_q^\alpha u(\eta). \end{cases} \tag{3.2}$$

Furthermore,

$$\xi(0) = 0,\ \xi(1) = \int_0^1 h(t)\xi(t)d_qt,\ D_q^\alpha\xi(0) = 0,\ D_q^\alpha\xi(1) = bD_q^\alpha\xi(\eta). \tag{3.3}$$

By Step 1 we obtain there exist $\kappa_1 > 0$, $\kappa_2 > 0$ such that

$$\kappa_1\rho(t) \le \xi(t) \le \kappa_2\rho(t).$$

Let $\xi_1(t) = \delta_1\xi(t)$, $\xi_2(t) = \delta_2\xi(t)$, where

$$0 < \delta_1 < \min\left\{\frac{1}{\kappa_2}, \kappa_1^{\frac{c}{1-c}}\right\},\ \delta_2 > \max\left\{\frac{1}{\kappa_1}, \kappa_2^{\frac{c}{1-c}}\right\}.$$

Then

$$\begin{aligned} f(t,\xi_1(t)) = f(t,\delta_1\xi(t)) = f\left(t, \delta_1\frac{\xi(t)}{\rho(t)}\rho(t)\right) &\ge \left(\delta_1\frac{\xi(t)}{\rho(t)}\right)^{c(p-1)} f(t,\rho(t)) \\ &\ge (\delta_1\kappa_1)^{c(p-1)}f(t,\rho(t)) \ge \delta_1^{p-1}f(t,\rho(t)), \end{aligned}$$

and

$$D_q^\beta(\varphi_p(D_q^\alpha\xi_1(t))) = D_q^\beta(\varphi_p(D_q^\alpha\delta_1\xi(t))) = \delta_1^{p-1}D_q^\beta(\varphi_p(D_q^\alpha\xi(t))) = \delta_1^{p-1}f(t,\rho(t)) \le f(t,\xi_1(t)).$$

Moreover, from (3.3) we have

$$\xi_1(0) = 0,\ \xi_1(1) = \int_0^1 h(t)\xi_1(t)d_qt,\ D_q^\alpha\xi_1(0) = 0,\ D_q^\alpha\xi_1(1) = bD_q^\alpha\xi_1(\eta).$$

Therefore, by Definition 2.9 we obtain ξ_1 is a lower solution for (1.1).

On the other hand,

$$\begin{aligned}\delta_2^{p-1} f(t,\rho(t)) &= \delta_2^{p-1} f\left(t, \frac{\rho(t)}{\xi_2(t)}\xi_2(t)\right) = \delta_2^{p-1} f\left(t, \frac{\rho(t)}{\delta_2\xi(t)}\xi_2(t)\right) \geq \delta_2^{p-1}\left(\frac{\rho(t)}{\delta_2\xi(t)}\right)^{c(p-1)} f(t,\xi_2(t)) \\ &\geq \delta_2^{p-1}\left(\frac{1}{\delta_2\kappa_2}\right)^{c(p-1)} f(t,\xi_2(t)) \geq \delta_2^{p-1}\delta_2^{-(p-1)} f(t,\xi_2(t)) = f(t,\xi_2(t)),\end{aligned}$$

and

$$D_q^\beta(\varphi_p(D_q^\alpha \xi_2(t))) = D_q^\beta(\varphi_p(D_q^\alpha \delta_2\xi(t))) = \delta_2^{p-1} D_q^\beta(\varphi_p(D_q^\alpha \xi(t))) = \delta_2^{p-1} f(t,\rho(t)) \geq f(t,\xi_2(t)).$$

Moreover, from (3.3) we have

$$\xi_2(0) = 0,\ \xi_2(1) = \int_0^1 h(t)\xi_2(t)d_qt,\ D_q^\alpha\xi_2(0) = 0,\ D_q^\alpha\xi_2(1) = bD_q^\alpha\xi_2(\eta).$$

Therefore, by Definition 2.10 we obtain ξ_2 is an upper solution for (1.1).

Step 3. We prove that the following problem has at least one positive solution:

$$\begin{cases} D_q^\beta(\varphi_p(D_q^\alpha u(t))) = g(t,u(t)),\ t\in(0,1), \\ u(0)=0,\ u(1) = \int_0^1 h(t)u(t)d_qt,\ D_q^\alpha u(0) = 0,\ D_q^\alpha u(1) = bD_q^\alpha u(\eta), \end{cases} \tag{3.4}$$

where

$$g(t,u(t)) = \begin{cases} f(t,\xi_1(t)), & u(t) < \xi_1(t), \\ f(t,u(t)), & \xi_1(t)\leq u(t)\leq \xi_2(t), \\ f(t,\xi_2(t)), & u(t) > \xi_2(t). \end{cases}$$

To see this, we consider the operator $B : C[0,1] \to C[0,1]$

$$(Bu)(t) = \int_0^1 G(t,qs)\varphi_r\left(\int_0^1 H(s,q\tau)g(\tau,u(\tau))d_q\tau\right)d_qs.$$

By [11, Page 10 and 11], we obtain B is a compact operator, by using the Schauder fixed point theorem, the operator B has at least a fixed point, i.e., (3.4) has at least one positive solution.

Step 4. We prove (1.1) has at least one positive solution. Suppose that u^* is a positive solution for (3.4), according to Step 3 we only need to prove

$$\xi_1(t) \leq u^*(t) \leq \xi_2(t) \text{ for } t\in[0,1].$$

The method is similar for the two inequalities. We only prove $u^*(t)\leq\xi_2(t)$ for $t\in[0,1]$. Suppose by contradiction that $u^*(t) > \xi_2(t)$. From (3.4) we have

$$D_q^\beta(\varphi_p(D_q^\alpha u^*(t))) = g(t,u^*(t)) = f(t,\xi_2(t)).$$

On the other hand, since ξ_2 is an upper solution for (1.1), we have

$$D_q^\beta(\varphi_p(D_q^\alpha \xi_2(t))) \geq f(t,\xi_2(t)).$$

Let $z(t) = \varphi_p(D_q^\alpha\xi_2(t)) - \varphi_p(D_q^\alpha u^*(t))$. Then

$$\begin{aligned}&D_q^\beta z(t) = D_q^\beta(\varphi_p(D_q^\alpha\xi_2(t))) - D_q^\beta(\varphi_p(D_q^\alpha u^*(t))) \geq f(t,\xi_2(t)) - f(t,\xi_2(t)) = 0, \\ &z(0) = 0,\ z(1) = \varphi_p(b)z(\eta).\end{aligned}$$

Thus by Lemma 2.8 we have $z(t) \leq 0$, $t \in [0,1]$, which implies that

$$\varphi_p(D_q^\alpha \xi_2(t)) \leq \varphi_p(D_q^\alpha u^*(t)),\ t \in [0,1].$$

Since φ_p is monotone increasing, we obtain $D_q^\alpha \xi_2(t) \leq D_q^\alpha u^*(t)$, i.e., $D_q^\alpha(\xi_2 - u^*)(t) \leq 0$. Combining Lemma 2.5, we have $(\xi_2 - u^*)(t) \geq 0$. Therefore, $\xi_2(t) \geq u^*(t), t \in [0,1]$, a contradiction to the assumption that $u^*(t) > \xi_2(t)$.

Consequently, $\xi_1(t) \leq u^*(t) \leq \xi_2(t)$ for $t \in [0,1]$, i.e., u^* is a positive solution for (1.1). This completes the proof. □

Remark 3.2. In [7], the authors had the following condition:

(H_f) $f(t,u) \in C([0,1] \times [0,+\infty),(0,+\infty))$ is nondecreasing relative to u and there exists a positive constant $c < 1$ such that

$$\mu^c f(t,u) \leq f(t,\mu u),\ \forall 0 \leq \mu \leq 1.$$

Moreover, their example is $f(t,u) = t + u^c$, $0 < c < 1$. This is a sublinear function. We note that if $p \geq 2$, this example also satisfies our condition (H3). However, if $f(t,u) = e^t + u^\sigma$, where $\sigma > 1, u \in [0,+\infty), t \in [0,1]$, then (H_f) doesn't hold for all $u \in [0,+\infty)$, but (H3) still holds with $p \geq \frac{\sigma}{c} + 1$. In a word, for some appropriate values of p, our nonlinear term f is allowed to grow superlinearly or sublinearly.

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