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On the Ulam stability of the Cauchy-Jensen equation and the additive-quadratic equation

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Abstract

In this paper, we investigate the Ulam stability of the functional equations

$$2f\left(x+y, \frac{z+w}{2}\right) = f(x,z) + f(x,w) + f(y,z) + f(y,w)$$

and

$$f(x+y, z+w) + f(x+y, z-w) = 2f(x,z) + 2f(x,w) + 2f(y,z) + 2f(y,w)$$

in paranormed spaces.

Keywords: Cauchy-Jensen mapping, additive-quadratic mapping, paranormed space.
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1. Introduction

In 1940, S. M. Ulam proposed the stability problem (see [10]):

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In 1941, this problem was solved by D. H. Hyers [3] in the case of Banach space. Thereafter, we call that type the Hyers-Ulam stability. In 1978, Th. M. Rassias [9] extended the Hyers-Ulam stability by considering variables. It also has been generalized to the function case by P. Găvruta [2]. For more details on this topic, we also refer to [1, 4, 6] and references therein.

We recall some basic facts concerning Fréchet spaces (see [11]).

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Definition 1.1. Let X be a vector space. A *paranorm* on X is a function $P : X \to \mathbb{R}$ such that for all $x, y \in X$
(i) $P(0) = 0$;
(ii) $P(-x) = P(x)$;
(iii) $P(x + y) \leq P(x) + P(y)$ (triangle inequality);
(iv) If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_n - x) \to 0$, then $P(t_n x_n - tx) \to 0$ (continuity of scalar multiplication).

The pair (X, P) is called a *paranormed space* if P is a paranorm on X. Note that

$$P(nx) \leq nP(x)$$

for all $n \in \mathbb{N}$ and all $x \in (X, P)$. The paranorm P on X is called *total* if, in addition, P satisfies (v) $P(x) = 0$ implies $x = 0$. A *Fréchet space* is a total and complete paranormed space. Note that each seminorm P on X is a paranorm, but the converse need not be true. In recent, C. Park [5] obtained some stability results in paranormed spaces.

Let X and Y be vector spaces. A mapping $f : X \times X \to Y$ is called a *Cauchy-Jensen mapping* (respectively, *additive-quadratic mapping*) if it satisfies the system of equations

$$f(x+y,z) = f(x,z) + f(y,z), \;\; 2f\Big(x, \frac{y+z}{2}\Big) = f(x,y) + f(x,z)$$

$$\text{(respectively, } f(x+y,z) = f(x,z) + f(y,z), \;\; f(x,y+z) + f(x,y-z) = 2f(x,y) + 2f(x,z)).$$

The authors [7, 8] considered the following functional equations:

$$2f\left(x+y, \frac{z+w}{2}\right) = f(x,z) + f(x,w) + f(y,z) + f(y,w) \tag{1.1}$$

and

$$f(x+y,z+w) + f(x+y,z-w) = 2f(x,z) + 2f(x,w) + 2f(y,z) + 2f(y,w). \tag{1.2}$$

It is easy to show that the functions $f(x,y) = ax^2 + bx$ and $f(x,y) = axy^2$ satisfy the functional equations (1.1) and (1.2), respectively. Also, they solved the solutions of (1.1) and (1.2).

From now on, assume that (X, P) is a Fréchet space and $(Y, \|\cdot\|)$ is a Banach space.

In this paper, we investigate the Ulam stability of the functional equations (1.1) and (1.2) in paranormed spaces.

2. Ulam stability of the Cauchy-Jensen functional equation (1.1)

Theorem 2.1. *Let r, θ be positive real numbers with $r > \log_2 6$, and let $f : Y \times Y \to X$ be a mapping satisfying $f(x, 0) = 0$ for all $x \in Y$ such that*

$$P\left(2f\Big(x+y, \frac{z+w}{2}\Big) - f(x,z) - f(x,w) - f(y,z) - f(y,w)\right) \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r) \tag{2.1}$$

for all x, y, z, $w \in Y$. Then there exists a unique mapping $F : Y \times Y \to X$ satisfying (1.1) such that

$$P\big(2f(x,y) - F(x,y)\big) \leq 2\theta\Big(\frac{15}{2^r - 6}\|x\|^r + \frac{13 + 2\cdot 3^r}{3^r - 6}\|y\|^r\Big) \tag{2.2}$$

for all $x, y \in Y$.

Proof. Letting $y = x$ in (2.1), we gain

$$P\Big(2f\Big(2x, \frac{z+w}{2}\Big) - 2f(x,z) - 2f(x,w)\Big) \le \theta(2\|x\|^r + \|z\|^r + \|w\|^r) \tag{2.3}$$

for all $x, z, w \in Y$. Letting $w = -z$ in (2.3)), we get

$$P(2f(x,z) + 2f(x,-z)) \le 2\theta(\|x\|^r + \|z\|^r) \tag{2.4}$$

for all $x, z \in Y$. Replacing z by $-z$ and w by $-z$ in (2.3)), we have

$$P(2f(2x,-z) - 4f(x,-z)) \le 2\theta(\|x\|^r + \|z\|^r) \tag{2.5}$$

for all $x, z \in Y$. By (2.4) and (2.5), we obtain

$$\begin{aligned} P(4f(x,z) + 2f(2x,-z)) &\le 2P(2f(x,z) + 2f(x,-z)) + P(2f(2x,-z) - 4f(x,-z)) \\ &\le 6\,\theta(\|x\|^r + \|z\|^r) \end{aligned}$$

for all $x, z \in Y$. Putting $w = -3z$ in (2.3)), we gain

$$P(2f(2x,-z) - 2f(x,z) - 2f(x,-3z)) \le \theta\,[\,2\|x\|^r + (1+3^r)\|z\|^r\,]$$

for all $x, z \in Y$. By the above two inequalities, we see that

$$P(6f(x,z) + 2f(x,-3z)) \le \theta\,[\,8\|x\|^r + (7+3^r)\|z\|^r\,] \tag{2.6}$$

for all $x, z \in Y$. Replacing z by $3z$ in (2.5), we gain

$$P(2f(2x,-3z) - 4f(x,-3z)) \le 2\theta(\|x\|^r + 3^r\|z\|^r)$$

for all $x, z \in Y$. By (2.6) and the above inequality, we get

$$\begin{aligned} P(12f(x,z) + 2f(2x,-3z)) &\le 2P(6f(x,z) + 2f(x,-3z)) + P(2f(2x,-3z) - 4f(x,-3z)) \\ &\le 2\theta\,[\,9\|x\|^r + (7+2\cdot 3^r)\|z\|^r\,] \end{aligned}$$

for all $x, z \in Y$. Replacing z by $-z$ in the above inequality, we have

$$\begin{aligned} P(12f(x,-z) + 2f(2x,3z)) &\le 2P(6f(x,-z) + 2f(x,3z)) + P(2f(2x,3z) - 4f(x,3z)) \\ &\le 2\theta\,[\,9\|x\|^r + (7+2\cdot 3^r)\|z\|^r\,] \end{aligned}$$

for all $x, z \in Y$. By (2.4) and the above inequality, we obtain

$$\begin{aligned} P(12f(x,z) - 2f(2x,3z)) &\le 6P(2f(x,z) + 2f(x,-z)) + P(-12f(x,-z) - 2f(2x,3z)) \\ &\le 2\theta\,[\,15\|x\|^r + (13+2\cdot 3^r)\|z\|^r\,] \end{aligned}$$

for all $x, z \in Y$. Replacing x by $\frac{x}{2^{j+1}}$ and z by $\frac{z}{3^{j+1}}$ in the above inequality, we see that

$$P\Big(12f\Big(\frac{x}{2^{j+1}}, \frac{z}{3^{j+1}}\Big) - 2f\Big(\frac{x}{2^j}, \frac{z}{3^j}\Big)\Big) \le 2\theta\left[\frac{15}{2^{(j+1)r}}\|x\|^r + \frac{13+2\cdot 3^r}{3^{(j+1)r}}\|z\|^r\right]$$

for all nonnegative integers j and all $x, z \in Y$. For given integers $l, m (0 \le l < m)$, we obtain that

$$\begin{aligned} P\Big(2\cdot 6^m f\Big(\frac{x}{2^m}, \frac{z}{3^m}\Big) - 2\cdot 6^l f\Big(\frac{x}{2^l}, \frac{z}{3^l}\Big)\Big) &\le \sum_{j=l}^{m-1} P\Big(2\cdot 6^{j+1} f\Big(\frac{x}{2^{j+1}}, \frac{z}{3^{j+1}}\Big) - 2\cdot 6^j f\Big(\frac{x}{2^j}, \frac{z}{3^j}\Big)\Big) \\ &\le 2\theta \sum_{j=l}^{m-1} 6^j \left[\frac{15}{2^{(j+1)r}}\|x\|^r + \frac{13+2\cdot 3^r}{3^{(j+1)r}}\|z\|^r\right] \end{aligned} \tag{2.7}$$

for all $x, z \in Y$. By (2.7), the sequence $\{2 \cdot 6^j f(\frac{x}{2^j}, \frac{z}{3^j})\}$ is a Cauchy sequence in X for all $x, z \in Y$. Since X is complete, the sequence $\{2 \cdot 6^j f(\frac{x}{2^j}, \frac{z}{3^j})\}$ converges for all $x, z \in Y$. Define $F : Y \times Y \to X$ by $F(x, z) := \lim_{j\to\infty} 2 \cdot 6^j f\big(\frac{x}{2^j}, \frac{z}{3^j}\big)$ for all $x, z \in Y$. By (2.1), we see that

$$\begin{aligned}
&P\bigg(2F\Big(x+y, \frac{z+w}{2}\Big) - F(x,z) - F(x,w) - F(y,z) - F(y,w)\bigg) \\
&= \lim_{j\to\infty} P\bigg(6^j\Big[4f\Big(\frac{x+y}{2^j}, \frac{z+w}{3^j}\Big) - 2f\Big(\frac{x}{2^j}, \frac{z}{3^j}\Big) - 2f\Big(\frac{x}{2^j}, \frac{w}{3^j}\Big) - 2f\Big(\frac{y}{2^j}, \frac{z}{3^j}\Big) - 2f\Big(\frac{y}{2^j}, \frac{w}{3^j}\Big)\Big]\bigg) \\
&\le \lim_{j\to\infty} 2\cdot 6^j P\bigg(2f\Big(\frac{x+y}{2^j}, \frac{z+w}{3^j}\Big) - f\Big(\frac{x}{2^j}, \frac{z}{3^j}\Big) - f\Big(\frac{x}{2^j}, \frac{w}{3^j}\Big) - f\Big(\frac{y}{2^j}, \frac{z}{3^j}\Big) - f\Big(\frac{y}{2^j}, \frac{w}{3^j}\Big)\bigg) \\
&\le 2\theta \lim_{j\to\infty} 6^j\Big(\frac{\|x\|^r + \|y\|^r}{2^{jr}} + \frac{\|z\|^r + \|w\|^r}{3^{jr}}\Big) = 0
\end{aligned}$$

for all $x, y, z, w \in Y$. Since X is total, F satisfies (1.1). Setting $l = 0$ and taking $m \to \infty$ in (2.7), one can obtain the inequality (2.2).

Let $F' : Y \times Y \to X$ be another mapping satisfying (1.1) and (2.2). By [7], there exist bi-additive mappings $B, B' : Y \times Y \to X$ and additive mappings $A, A' : Y \to X$ such that $F(x, y) = B(x, y) + A(x)$ and $F'(x, y) = B'(x, y) + A'(x)$ for all $x, y \in Y$. Since $r > \log_2 6$, we obtain that

$$\begin{aligned}
P(F(x,y) - F'(x,y)) &= P\bigg(6^n\Big[B\Big(\frac{x}{2^n}, \frac{y}{3^n}\Big) + A\Big(\frac{x}{2^n}\Big) - B'\Big(\frac{x}{2^n}, \frac{y}{3^n}\Big) - A'\Big(\frac{x}{2^n}\Big)\Big]\bigg) \\
&\le 6^n\bigg[P\bigg(F\Big(\frac{x}{2^n}, \frac{y}{3^n}\Big) - 2f\Big(\frac{x}{2^n}, \frac{y}{3^n}\Big)\bigg) + P\bigg(2f\Big(\frac{x}{2^n}, \frac{y}{3^n}\Big) - F'\Big(\frac{x}{2^n}, \frac{y}{3^n}\Big)\bigg)\bigg] \\
&\le 4\cdot 6^n\theta\Big(\frac{15}{(2^r-6)2^{nr}}\|x\|^r + \frac{13+2\cdot 3^r}{(3^r-6)3^{nr}}\|y\|^r\Big) \to 0 \quad \text{as} \quad n \to \infty
\end{aligned}$$

for all $x, y \in Y$. Hence F is a unique mapping satisfying (1.1) and (2.2), as desired. □

Theorem 2.2. *Let r be a positive real number with $r < \log_3 6$, and let $f : X \times X \to Y$ be a mapping satisfying $f(x, 0) = 0$ for all $x \in X$ such that*

$$\Big\|2f\Big(x+y, \frac{z+w}{2}\Big) - f(x,z) - f(x,w) - f(y,z) - f(y,w)\Big\| \le P(x)^r + P(y)^r + P(z)^r + P(w)^r \tag{2.8}$$

for all x, y, z, $w \in X$. Then there exists a unique mapping $F : X \times X \to Y$ satisfying (1.1) *such that*

$$\big\|f(x,y) - F(x,y)\big\| \le \frac{18}{6-2^r}P(x)^r + \frac{15+3^{r+1}}{6-3^r}P(y)^r \tag{2.9}$$

for all $x, y \in X$.

Proof. Letting $y = x$ in (2.8), we gain

$$\Big\|2f\Big(2x, \frac{z+w}{2}\Big) - 2f(x,z) - 2f(x,w)\Big\| \le 2P(x)^r + P(z)^r + P(w)^r \tag{2.10}$$

for all $x, z, w \in X$. Putting $w = -z$ in (2.10), we get

$$\|2f(x,z) + 2f(x,-z)\| \le 2\big[P(x)^r + P(z)^r\big] \tag{2.11}$$

for all $x, z \in X$. Replacing z by $-z$ and w by $-z$ in (2.10), we have

$$\|f(2x,-z) - 2f(x,-z)\| \le 2\big[P(x)^r + P(z)^r\big] \tag{2.12}$$

for all $x, z \in X$. By (2.11) and (2.12), we obtain

$$\|f(2x, -z) + 2f(x, z)\| \le 4\big[P(x)^r + P(z)^r\big] \tag{2.13}$$

for all $x, z \in X$. Setting $w = -3z$ in (2.10), we gain

$$\|2f(2x, -z) - 2f(x, z) - 2f(x, -3z)\| \le 2P(x)^r + (1 + 3^r)P(z)^r$$

for all $x, z \in X$. By (2.13) and the above inequality, we get

$$\|6f(x, z) + 2f(x, -3z)\| \le 10P(x)^r + (9 + 3^r)P(z)^r \tag{2.14}$$

for all $x, z \in X$. Replacing z by $3z$ in (2.12), we have

$$\|f(2x, -3z) - 2f(x, -3z)\| \le 2\big[P(x)^r + 3^r P(z)^r\big]$$

for all $x, z \in X$. By (2.14) and the above inequality, we gain

$$\|6f(x, z) + f(2x, -3z)\| \le 12P(x)^r + (9 + 3^{r+1})P(z)^r$$

for all $x, z \in X$. Replacing z by $-z$ in the above inequality, we get

$$\|6f(x, -z) + f(2x, 3z)\| \le 12P(x)^r + (9 + 3^{r+1})P(z)^r$$

for all $x, z \in X$. By (2.11) and the above inequality, we have

$$\|6f(x, z) - f(2x, 3z)\| \le 18P(x)^r + (15 + 3^{r+1})P(z)^r$$

for all $x, z \in X$. Replacing x by $2^j x$ and z by $3^j z$ in the above inequality and dividing 6^{j+1}, we see that

$$\left\|\frac{1}{6^j} f(2^j x, 3^j z) - \frac{1}{6^{j+1}} f(2^{j+1}x, 3^{j+1}z)\right\| \le \frac{1}{6^{j+1}}[18 \cdot 2^{jr} P(x)^r + (15 + 3^{r+1})3^{jr} P(z)^r]$$

for all nonnegative integers j and all $x, z \in X$. For given integers $l, m (0 \le l < m)$, we obtain that

$$\left\|\frac{1}{6^l} f(2^l x, 3^l z) - \frac{1}{6^m} f(2^m x, 3^m z)\right\| \le \sum_{j=l}^{m-1} \frac{1}{6^{j+1}}[18 \cdot 2^{jr} P(x)^r + (15 + 3^{r+1})3^{jr} P(z)^r] \tag{2.15}$$

for all $x, z \in X$. By (2.15), the sequence $\{\frac{1}{6^j} f(2^j x, 3^j y)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{\frac{1}{6^j} f(2^j x, 3^j y)\}$ converges for all $x, y \in X$. Define $F : X \times X \to Y$ by $F(x, y) := \lim_{j\to\infty} \frac{1}{6^j} f(2^j x, 3^j y)$ for all $x, y \in X$.

By (2.8), we see that

$$\begin{aligned}
&\frac{1}{6^j}\left\|2f\left(2^j(x + y), \frac{3^j(z + w)}{2}\right) - f(2^j x, 3^j z) - f(2^j x, 3^j w) - f(2^j y, 3^j z) - f(2^j y, 3^j w)\right\| \\
&\quad\le \frac{1}{6^j}[P(2^j x)^r + P(2^j y)^r + P(3^j z)^r + P(3^j w)^r] \\
&\quad\le \frac{1}{6^j}\big(2^{rj}[P(x)^r + P(y)^r] + 3^{rj}[P(z)^r + P(w)^r]\big)
\end{aligned}$$

for all $x, y, z, w \in X$. Letting $j \to \infty$, F satisfies (1.1). By Theorem 4 in [7], F is a Cauchy-Jensen mapping. Setting $l = 0$ and taking $m \to \infty$ in (2.15), one can obtain the inequality (2.9). Let $G : X \times X \to Y$ be another Cauchy-Jensen mapping satisfying (2.9). Since $0 < r < \log_3 6$, we obtain that

$$\begin{aligned}
\|F(x,y)-G(x,y)\| &= \frac{1}{2^n}\|F(2^nx,y)-F(2^nx,0)+G(2^nx,0)-G(2^nx,y)\| \\
&= \frac{1}{6^n}\|F(2^nx,3^ny)-F(2^nx,0)+G(2^nx,0)-G(2^nx,3^ny)\| \\
&\le \frac{1}{6^n}\|F(2^nx,3^ny)-F(2^nx,0)-f(2^nx,3^ny)+f(2^nx,0)\| \\
&\quad +\frac{1}{6^n}\|-f(2^nx,0)+f(2^nx,3^ny)+G(2^nx,0)-G(2^nx,3^ny)\| \\
&\le \frac{1}{6^n}(\|F(2^nx,3^ny)-f(2^nx,3^ny)\|+\|-F(2^nx,0)+f(2^nx,0)\|) \\
&\quad +\frac{1}{6^n}(\|-f(2^nx,0)+G(2^nx,0)\|+\|f(2^nx,3^ny)-G(2^nx,3^ny)\|) \\
&\le \frac{2}{6^n}\left[\frac{36\cdot 2^{nr}}{6-2^r}P(x)^r+\frac{3^{nr}(15+3^{r+1})}{6-3^r}P(y)^r\right]\to 0 \quad\text{as}\quad n\to\infty
\end{aligned}$$

for all $x, y \in X$. Hence F is a unique Cauchy-Jensen mapping, as desired. □

3. Ulam stability of the additive-quadratic functional equation (1.2)

Theorem 3.1. *Let r, θ be positive real numbers with $r > \log_2 8 = 3$, and let $f : Y \times Y \to X$ be a mapping satisfying $f(x, 0) = 0$ for all $x \in Y$ such that*

$$\begin{aligned}
P(f(x+y,&z+w)+f(x+y,z-w)-2f(x,z)-2f(x,w)-2f(y,z)-2f(y,w)) \\
&\le \theta(\|x\|^r+\|y\|^r+\|z\|^r+\|w\|^r) \qquad (3.1)
\end{aligned}$$

for all x, y, z, $w \in Y$. Then there exists a unique mapping $F : Y \times Y \to X$ satisfying (1.2) *such that*

$$P\big(f(x,y)-F(x,y)\big) \le \frac{2\theta}{2^r-8}(\|x\|^r+\|y\|^r) \qquad (3.2)$$

for all $x, y \in Y$.

Proof. Letting $y = x$ and $w = z$ in (3.1), we gain

$$P(f(2x,2z)-8f(x,z)) \le 2\theta(\|x\|^r+\|z\|^r)$$

for all $x, z \in Y$. Replacing x by $\frac{x}{2^{j+1}}$ and z by $\frac{z}{2^{j+1}}$ in the above inequality, we see that

$$P\left(f\left(\frac{x}{2^j},\frac{z}{2^j}\right)-8f\left(\frac{x}{2^{j+1}},\frac{z}{2^{j+1}}\right)\right) \le \frac{2\theta}{2^{(j+1)r}}(\|x\|^r+\|z\|^r)$$

for all nonnegative integers j and all $x, z \in Y$. Thus we obtain that

$$\begin{aligned}
&P\left(8^jf\left(\frac{x}{2^j},\frac{z}{2^j}\right)-8^{j+1}f\left(\frac{x}{2^{j+1}},\frac{z}{2^{j+1}}\right)\right) \\
&\le 8^jP\left(f\left(\frac{x}{2^j},\frac{z}{2^j}\right)-8f\left(\frac{x}{2^{j+1}},\frac{z}{2^{j+1}}\right)\right) \le \frac{2}{2^r}\left(\frac{8}{2^r}\right)^j\theta(\|x\|^r+\|z\|^r)
\end{aligned}$$

for all nonnegative integers j and all $x, z \in Y$. For given integers $l, m (0 \le l < m)$, we have

$$P\left(8^l f\left(\frac{x}{2^l}, \frac{z}{2^l}\right) - 8^m f\left(\frac{x}{2^m}, \frac{z}{2^m}\right)\right) \le \sum_{j=l}^{m-1} \frac{2}{2^r}\left(\frac{8}{2^r}\right)^j \theta(\|x\|^r + \|z\|^r) \tag{3.3}$$

for all $x, z \in Y$. By (3.3), the sequence $\{8^j f(\frac{x}{2^j}, \frac{z}{2^j})\}$ is a Cauchy sequence in X for all $x, z \in Y$. Since X is complete, the sequence $\{8^j f(\frac{x}{2^j}, \frac{z}{2^j})\}$ converges for all $x, z \in Y$. Define $F : Y \times Y \to X$ by $F(x, z) := \lim_{j\to\infty} 8^j f\big(\frac{x}{2^j}, \frac{z}{2^j}\big)$ for all $x, z \in Y$. By (3.1), we see that

$$\begin{aligned}
&P\big(F(x+y, z+w) + F(x+y, z-w) - 2F(x,z) - 2F(x,w) - 2F(y,z) - 2F(y,w)\big)\\
&\quad= \lim_{j\to\infty} P\bigg(8^j\Big[f\Big(\frac{x+y}{2^j}, \frac{z+w}{2^j}\Big) + f\Big(\frac{x+y}{2^j}, \frac{z-w}{2^j}\Big)\\
&\qquad\quad - 2f\Big(\frac{x}{2^j}, \frac{z}{2^j}\Big) - 2f\Big(\frac{x}{2^j}, \frac{w}{2^j}\Big) - 2f\Big(\frac{y}{2^j}, \frac{z}{2^j}\Big) - 2f\Big(\frac{y}{2^j}, \frac{w}{2^j}\Big)\Big]\bigg)\\
&\quad\le \lim_{j\to\infty} 8^j P\bigg(f\Big(\frac{x+y}{2^j}, \frac{z+w}{2^j}\Big) + f\Big(\frac{x+y}{2^j}, \frac{z-w}{2^j}\Big)\\
&\qquad\quad - 2f\Big(\frac{x}{2^j}, \frac{z}{2^j}\Big) - 2f\Big(\frac{x}{2^j}, \frac{w}{2^j}\Big) - 2f\Big(\frac{y}{2^j}, \frac{z}{2^j}\Big) - 2f\Big(\frac{y}{2^j}, \frac{w}{2^j}\Big)\bigg)\\
&\quad\le \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r) \lim_{j\to\infty}\Big(\frac{8}{2^r}\Big)^j = 0
\end{aligned}$$

for all $x, y, z, w \in Y$. Since X is total, F satisfies (1.2). Setting $l = 0$ and taking $m \to \infty$ in (3.3), one can obtain the inequality (3.2).

Let $F' : Y \times Y \to X$ be another mapping satisfying (1.2) and (3.2). By [8], there exist multi-additive mappings $M, M' : Y \times Y \times Y \to X$ such that $F(x, y) = M(x, y, y)$, $F'(x, y) = M'(x, y, y)$, $M(x, y, z) = M(x, z, y)$ and $M'(x, y, z) = M'(x, z, y)$ for all $x, y, z \in Y$. Since $r > 3$, we obtain that

$$\begin{aligned}
P(F(x,y) - F'(x,y)) &= P\bigg(8^n\Big[M\Big(\frac{x}{2^n}, \frac{y}{2^n}, \frac{y}{2^n}\Big) - M'\Big(\frac{x}{2^n}, \frac{y}{2^n}, \frac{y}{2^n}\Big)\Big]\bigg)\\
&\le 8^n P\bigg(M\Big(\frac{x}{2^n}, \frac{y}{2^n}, \frac{y}{2^n}\Big) - M'\Big(\frac{x}{2^n}, \frac{y}{2^n}, \frac{y}{2^n}\Big)\bigg)\\
&\le 8^n\bigg[P\bigg(F\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) - f\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big)\bigg) + P\bigg(f\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) - F'\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big)\bigg)\bigg]\\
&\le \Big(\frac{8}{2^r}\Big)^n \frac{4\theta}{2^r - 8}(\|x\|^r + \|y\|^r) \to 0 \quad \text{as} \quad n \to \infty
\end{aligned}$$

for all $x, y \in Y$. Hence F is a unique mapping satisfying (1.2) and (3.2), as desired. □

Theorem 3.2. *Let r be a positive real number with $r < \log_2 8 = 3$, and let $f : X \times X \to Y$ be a mapping satisfying $f(x, 0) = 0$ for all $x \in X$ such that*

$$\begin{aligned}
&\|f(x+y, z+w) + f(x+y, z-w) - 2f(x,z) - 2f(x,w) - 2f(y,z) - 2f(y,w)\|\\
&\qquad \le P(x)^r + P(y)^r + P(z)^r + P(w)^r
\end{aligned} \tag{3.4}$$

for all x, y, z, $w \in X$. Then there exists a unique mapping $F : X \times X \to Y$ satisfying (1.2) *such that*

$$\big\|f(x,y) - F(x,y)\big\| \le \frac{2}{8 - 2^r}[P(x)^r + P(y)^r] \tag{3.5}$$

for all $x, y \in X$.

Proof. Letting $y = x$ and $w = z$ in (3.4), we gain

$$\|f(2x, 2z) - 8f(x, z)\| \leq 2[P(x)^r + P(z)^r]$$

for all $x, z \in X$. Replacing x by $2^j x$ and z by $2^j z$ in the above inequality, we see that

$$\left\|\frac{1}{8}f(2^{j+1}x, 2^{j+1}z) - f(2^j x, 2^j z)\right\| \leq \frac{2^{jr}}{4}[P(x)^r + P(z)^r]$$

for all nonnegative integers j and all $x, z \in X$. Thus we obtain that

$$\left\|\frac{1}{8^{j+1}}f(2^{j+1}x, 2^{j+1}z) - \frac{1}{8^j}f(2^j x, 2^j z)\right\| \leq \frac{1}{4}\left(\frac{2^r}{8}\right)^j [P(x)^r + P(z)^r]$$

for all nonnegative integers j and all $x, z \in X$. For given integers $l, m (0 \leq l < m)$, we have

$$\begin{aligned} \left\|\frac{1}{8^l}f(2^l x, 2^l z) - \frac{1}{8^m}f(2^m x, 2^m z)\right\| &\leq \sum_{j=l}^{m-1}\left\|\frac{1}{8^j}f(2^j x, 2^j z) - \frac{1}{8^{j+1}}f(2^{j+1}x, 2^{j+1}z)\right\| \\ &\leq \sum_{j=l}^{m-1}\frac{1}{4}\left(\frac{2^r}{8}\right)^j [P(x)^r + P(z)^r] \end{aligned} \tag{3.6}$$

for all $x, z \in X$. By (3.6), the sequence $\{\frac{1}{8^j}f(2^j x, 2^j z)\}$ is a Cauchy sequence in Y for all $x, z \in X$. Since Y is complete, the sequence $\{\frac{1}{8^j}f(2^j x, 2^j z)\}$ converges for all $x, z \in X$. Define $F : X \times X \to Y$ by $F(x, z) := \lim_{j\to\infty} \frac{1}{8^j}f(2^j x, 2^j z)$ for all $x, z \in X$. By (3.4), we see that

$$\begin{aligned} &\big\|F(x+y, z+w) + F(x+y, z-w) - 2F(x, z) - 2F(x, w) - 2F(y, z) - 2F(y, w)\big\| \\ &= \lim_{j\to\infty}\bigg\|\frac{1}{8^j}\big[f(2^j(x+y), 2^j(z+w)) + f(2^j(x+y), 2^j(z-w)) \\ &\qquad - 2f(2^j x, 2^j z) - 2f(2^j x, 2^j w) - 2f(2^j y, 2^j z) - 2f(2^j y, 2^j w)\big]\bigg\| \\ &= \lim_{j\to\infty}\frac{1}{8^j}\big\|f(2^j(x+y), 2^j(z+w)) + f(2^j(x+y), 2^j(z-w)) \\ &\qquad - 2f(2^j x, 2^j z) - 2f(2^j x, 2^j w) - 2f(2^j y, 2^j z) - 2f(2^j y, 2^j w)\big\| \\ &\leq [P(x)^r + P(y)^r + P(z)^r + P(w)^r]\lim_{j\to\infty}\left(\frac{2^r}{8}\right)^j = 0 \end{aligned}$$

for all $x, y, z, w \in X$. Thus F is a mapping satisfying (1.2). Setting $l = 0$ and taking $m \to \infty$ in (3.6), one can obtain the inequality (3.5).

Let $G : X \times X \to Y$ be another additive-quadratic mapping satisfying (3.5). Since $0 < r < 3$, we have

$$\begin{aligned} \|F(x, y) - G(x, y)\| &= \frac{1}{8^n}\|F(2^n x, 2^n y) - G(2^n x, 2^n y)\| \\ &\leq \frac{1}{8^n}\|F(2^n x, 2^n y) - f(2^n x, 2^n y)\| + \frac{1}{8^n}\|f(2^n x, 2^n y) - G(2^n x, 2^n y)\| \\ &\leq \left(\frac{2^r}{8}\right)^n \frac{4}{8 - 2^r}[P(x)^r + P(y)^r] \to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

for all $x, y \in X$. Hence F is a unique additive-quadratic mapping, as desired. □

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