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Pseudo almost automorphic and weighted pseudo almost automorphic mild solutions to a partial functional differential equation in Banach spaces

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This paper is dedicated to Professor Ljubomir Ćirić

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Abstract

In this paper we prove some existence and uniqueness results for pseudo almost automorphic and weighted pseudo almost automorphic mild solutions to a class of partial functional differential equation in Banach spaces. The main technique is based upon some appropriate composition theorems combined with the Banach contraction mapping principle and fractional powers of operators.

Keywords: Pseudo almost automorphic, Weighted pseudo almost automorphic, Partial functional differential equations, Fractional powers of operators.
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1. Introduction

In this paper, we are mainly concerned with the existence and uniqueness of pseudo almost automorphic and weighted pseudo almost automorphic mild solutions for a class of partial neutral functional differential

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equations in the abstract form

$$\frac{d}{dt}[u(t) + f(t, u(h_1(t)))] = Au(t) + g(t, u(h_2(t))), t \in \mathbb{R}, \tag{1.1}$$

where $A : D(A) \subset X \to X$ is the infinitesimal generator of an analytic semigroup of linear operators $\{T(t)\}_{t\geq 0}$ on a Banach space $(X, |\cdot|)$ and there exist positive numbers M, δ such that $\|T(t)\| \leq Me^{-\delta t}$ for $t \geq 0$, and $f(\cdot)$, $g(\cdot)$, $h_i(\cdot)$, $i = 1, 2$, are appropriate functions specified later.

The concept of pseudo almost automorphic function, which was initiated by Xiao et al. in [1], is more general than that of almost automorphic function created by Bochner in [2, 3]. Since then, those functions has been studied and developed extensively. For more details on those functions we refer the reader to [4, 5, 6, 7, 8, 9] and the references therein. Very recently, J. Blot et al [10] have introduced the concept of weighted pseudo almost automorphic, which generalizes the concept of weighted pseudo almost periodic [11, 12, 13, 14]. In [10], the authors have proved some useful properties of the space of weighted pseudo almost automorphic functions and established a general existence and uniqueness theorem for weighted pseudo almost automorphic mild solutions to some semi-linear differential equations. The existence of almost automorphic, pseudo almost automorphic and weighted pseudo almost automorphic solutions are among the most attractive topics in qualitative theory of differential equations due to their significance and applications in physics, mechanics and mathematical biology [15]. In recent years, the existence of almost automorphic and pseudo almost automorphic solutions on different kinds of differential equations have been considered in many publications such as [6, 16, 17, 18, 19, 20, 21, 22, 23, 24] and references therein. Especially, authors in [23] have proved the existence of almost automorphic and weighted pseudo almost automorphic solutions to a semi-linear evolution equation in a Banach space X such as

$$x'(t) = A(t)x(t) + f(t, x(t)), t \in \mathbb{R},$$

where the family $\{A(t), t \in \mathbb{R}\}$ of operators in X generates an exponentially stable evolution family $\{U(t, s), t \geq s\}$ and $f : \mathbb{R} \times X \to X$ an almost automorphic function(resp. a weighted pseudo almost automorphic function).

And in [24], authors have investigated the existence of almost automorphic and pseudo-almost automorphic mild solutions to the following equation

$$\frac{du(t)}{dt} = Au(t) + \frac{d}{dt}F_1(t, u(h_1(t))) + F_2(t, u(h_2(t))),$$

where $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ on a Banach space X.

Motivated by the above mentioned works [10, 23, 24], the main purpose of this paper is to deal with the existence and uniqueness of pseudo almost automorphic and weighted pseudo almost automorphic mild solutions to the problem (1.1). We obtain the new results by using fractional powers of linear operators and the Banach contraction mapping principle.

The rest of this paper is organized as follows: In section 2 we recall some basic definitions, lemmas and preliminary facts which will be need in the sequel. Our main results and their proofs are arranged in Section 3.

2. Preliminaries

This section is concerned with some notations, definitions, lemmas and preliminary facts which are used in what follows.

Throughout the paper, let $(X, |\cdot|)$ be a Banach space and $C(\mathbb{R}, X)$ stand for the collection of continuous functions from $\mathbb{R}$ into X. We denote by $BC(\mathbb{R}, X)$ the Banach space of all bounded continuous functions from $\mathbb{R}$ into X endowed with the supremum norm defined by $\|x\|_{BC(\mathbb{R},X)} := \sup_{t\in\mathbb{R}} \{|x(t)|\}$. Furthermore, $BC(\mathbb{R} \times X, X)$ is the space of all bounded continuous functions $\mathcal{F} : \mathbb{R} \times X \to X$.

Let $\mathbb{U}$ denote the set of all functions (weights) $\rho : \mathbb{R} \to (0, \infty)$, which are locally integrable over $\mathbb{R}$ such that $\rho > 0$ almost everywhere. For a given $r > 0$ and for each $\rho \in \mathbb{U}$, we set

$$m(r, \rho) = \int_{-r}^{r} \rho(x)dx.$$

We denote by $\mathbb{U}_\infty$ the set of all $\rho \in \mathbb{U}$ with $\lim_{r\to\infty} m(r, \rho) = \infty$ and $\mathbb{U}_b$ the set of all $\rho \in \mathbb{U}_\infty$ such that ρ is bounded and $\inf_{x\in\mathbb{R}} \rho(x) > 0$.

It is clear that $\mathbb{U}_b \subset \mathbb{U}_\infty \subset \mathbb{U}$, with strict inclusions.

Let $0 \in \rho(A)$, then it is possible to define the fractional power $(-A)^\alpha$, for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D((-A)^\alpha)$. Furthermore, the subspace $D((-A)^\alpha)$ is dense in X and the expression

$$|x|_\alpha = |(-A)^\alpha x|, \ x \in D((-A)^\alpha),$$

defines a norm on $D((-A)^\alpha)$. Hereafter we denote by X_α the Banach space $D((-A)^\alpha)$ with norm $|x|_\alpha$.

The following properties hold by [25, Lemma 2.1] and [26].

Lemma 2.1. *Let $0 < \gamma \leq \mu \leq 1$. Then the following properties hold:*

(i) X_μ is a Banach space and $X_\mu \hookrightarrow X_\gamma$ is continuous.

(ii) The function $s \to (-A)^\mu T(s)$ is continuous in the uniform operator topology on $(0, \infty)$ and there exists $M_\mu > 0$ such that $\|(-A)^\mu T(t)\| \leq M_\mu e^{-\delta t} t^{-\mu}$ for each $t > 0$.

(iii) For each $x \in D((-A)^\mu)$ and $t \geq 0$, $(-A)^\mu T(t)x = T(t)(-A)^\mu x$.

(iv) $(-A)^{-\mu}$ is a bounded linear operator in X with $D((-A)^\mu) = Im((-A)^{-\mu})$.

Definition 2.2. A continuous function $\mathcal{F} : \mathbb{R} \to X$ is said to be almost automorphic if for every sequence of real numbers $(s_n{}')_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$\mathcal{G}(t) := \lim_{n\to\infty} \mathcal{F}(t + s_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} \mathcal{G}(t - s_n) = \mathcal{F}(t)$$

for each $t \in \mathbb{R}$. The collection of such functions will be denote by $AA(X)$.

We recall that every almost periodic function is almost automorphic, but the class of almost automorphic functions is larger than the class of almost periodic solutions. For example

$$f(t) = \cos\left(\frac{1}{2 + \sin\sqrt{2}t + \sin t}\right), \ t \in \mathbb{R}$$

is almost automorphic but not almost periodic.

Definition 2.3. A continuous function $\mathcal{F} : \mathbb{R} \times X \to X$ is said to be almost automorphic if $\mathcal{F}(t, x)$ is almost automorphic for each $t \in \mathbb{R}$ uniformly for all $x \in \mathscr{B}$, where $\mathscr{B}$ is any bounded subset of X. The collection of such functions will be denote by $AA(\mathbb{R} \times X, X)$.

Lemma 2.4. *([7]) $\left(AA(X), \|\cdot\|_{AA(X)}\right)$ is a Banach space endowed with the supremum norm given by*

$$\|\mathcal{F}\|_{AA(X)} = \sup_{t\in\mathbb{R}} |\mathcal{F}(t)|.$$

The notation $PAA_0(X)$ stands for for the spaces of functions

$$PAA_0(X) = \left\{\mathcal{F} \in BC(\mathbb{R}, X) : \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} |\mathcal{F}(t)|dt = 0\right\}.$$

Similarly, the notation $PAA_0(\mathbb{R} \times X, X)$ stands for for the spaces of functions

$$PAA_0(\mathbb{R} \times X, X) = \left\{ \begin{array}{c} \mathcal{F} \in BC(\mathbb{R} \times X, X) : \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} |\mathcal{F}(t, x)|dt = 0, \\ \text{uniformly in } x \text{ in any bounded subset of } X \end{array} \right\}.$$

Definition 2.5. A continuous function $\mathcal{F}:\mathbb{R}\to X$(resp. $\mathbb{R}\times X\to X$) is called pseudo almost automorphic if it can be decomposed as $\mathcal{F}=\mathscr{G}+\phi$, where $\mathscr{G}\in AA(X)$(resp. $AA(\mathbb{R}\times X,X)$) and $\phi\in PAA_0(X)$(resp. $PAA_0(\mathbb{R}\times X,X)$). The class of all such functions will be denote by $PAA(X)$(resp. $PAA(\mathbb{R}\times X,X)$).

Lemma 2.6. *([27, Theorem 2.2]) $\big(PAA(X),\|\cdot\|_{PAA(X)}\big)$ is a Banach space endowed with the supremum norm given by*

$$\|\mathcal{F}\|_{PAA(X)}=\sup_{t\in\mathbb{R}}|\mathcal{F}(t)|.$$

Lemma 2.7. *([27, Lemma 2.4]) Assume $\mathcal{F}=\mathscr{G}+\phi\in PAA(\mathbb{R}\times X,X)$, where $\mathscr{G}(t,x)\in AA(\mathbb{R}\times X,X)$ and $\phi(t,x)\in PAA_0(\mathbb{R}\times X,X)$, and suppose that $\mathcal{F}(t,x)$ is uniformly continuous in any bounded subset $\mathscr{K}\subset X$ uniformly for $t\in\mathbb{R}$. If $x(t)\in PAA(\mathbb{R},X)$, then $\mathcal{F}(\cdot,x(\cdot))\in PAA(\mathbb{R},X)$.*

Now for $\rho\in\mathbb{U}_\infty$, we define

$$PAA_0(\mathbb{R},\rho)=\left\{\mathcal{F}\in BC(\mathbb{R},X):\lim_{r\to\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}|\mathcal{F}(t)|\rho(t)dt=0\right\};$$

$$PAA_0(\mathbb{R}\times X,\rho)=\left\{\begin{array}{c}\mathcal{F}\in BC(\mathbb{R}\times X,X):\lim_{r\to\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}|\mathcal{F}(t,x)|\rho(t)dt=0\\ \text{uniformly in } x\in X\end{array}\right\}.$$

Definition 2.8. ([10]) A bounded continuous function $\mathcal{F}:\mathbb{R}\to X$(resp. $\mathbb{R}\times X\to X$) is called weighted pseudo almost automorphic if it can be decomposed as $\mathcal{F}=\mathscr{G}+\phi$, where $\mathscr{G}\in AA(X)$(resp. $AA(\mathbb{R}\times X,X)$) and $\phi\in PAA_0(\mathbb{R},\rho)$(resp. $PAA_0(\mathbb{R}\times X,\rho)$). The class of all such functions will be denote by $WPAA(\mathbb{R},\rho)$(resp. $WPAA(\mathbb{R}\times X,\rho)$).

Remark 2.9. ([10, Remark 2.2.]) When $\rho=1$, we obtain the standard spaces $PAA(\mathbb{R},X)$ and $PAA(\mathbb{R}\times X,X)$.

Lemma 2.10. *[5, Theorem 3.4] Let $\rho\in\mathbb{U}_\infty$. Suppose that $PAA_0(\mathbb{R},\rho)$ is translation invariant. Then the decomposition of weighted pseudo almost automorphic functions is unique.*

Lemma 2.11. *[28, Theorem 2.15] Let $\rho\in\mathbb{U}_\infty$. If $PAA_0(\mathbb{R},\rho)$ is translation invariant, then $\big(WPAA(\mathbb{R},\rho),\|\cdot\|_{WPAA(\mathbb{R},\rho)}\big)$ is a Banach space endowed with the supremum norm given by*

$$\|\mathcal{F}\|_{WPAA(\mathbb{R},\rho)}=\sup_{t\in\mathbb{R}}|\mathcal{F}(t)|.$$

Lemma 2.12. *([10, Corollary 2.11.]) Let $\mathcal{F}=\mathscr{G}+\phi\in WPAA(\mathbb{R}\times X,\rho)$ where $\rho\in\mathbb{U}_\infty$, $\mathscr{G}\in AA(\mathbb{R}\times X,X)$ and $\phi\in PAA_0(\mathbb{R}\times X,\rho)$. Assume both $\mathcal{F}$ and $\mathscr{G}$ are Lipschitzian in $x\in X$ uniformly in $t\in\mathbb{R}$. If $x(t)\in WPAA(\mathbb{R},\rho)$ then the function $\mathcal{F}(\cdot,x(\cdot))\in WPAA(\mathbb{R},\rho)$.*

Lemma 2.13. *([10, Lemma 3.1.]) Let $\mathcal{F}=\mathscr{G}+\phi\in WPAA(\mathbb{R},\rho)$ where $\rho\in\mathbb{U}_\infty$ and $\{T(t)\}_{t\geq0}$ is an exponentially stable semigroup. Then $F(t):=\int_{-\infty}^{t}T(t-s)\mathcal{F}(s)ds\in WPAA(\mathbb{R},\rho)$.*

The next result is a straightforward consequence of Lemma 2.13 when $\rho=1$.

Lemma 2.14. *Let $\mathcal{F}=\mathscr{G}+\phi\in PAA(X)$ and $\{T(t)\}_{t\geq0}$ is an exponentially stable semigroup. If $F:\mathbb{R}\to X$ be the function defined by*

$$F(t)=\int_{-\infty}^{t}T(t-s)\mathcal{F}(s)ds,\ t\geq s\in\mathbb{R},$$

then $F(\cdot)\in PAA(X)$.

Definition 2.15. A function $u \in BC(\mathbb{R}, X)$ is called a pseudo almost automorphic mild solution of Eq. (1.1) on $\mathbb{R}$ if $u \in PAA(X)$ and the function $s \to AT(t-s)f(s, u(h_1(s)))$ is integrable on $(-\infty, t)$ for each $t \in \mathbb{R}$, and $u(t)$ satisfies

$$\begin{aligned} u(t) &= T(t-a)[u(a) + f(a, u(h_1(a)))] - f(t, u(h_1(t))) - \int_a^t AT(t-s)f(s, u(h_1(s)))\,ds \\ &\quad + \int_a^t T(t-s)g(s, u(h_2(s)))\,ds \end{aligned}$$

for all $t \geq a$ and all $a \in \mathbb{R}$.

Definition 2.16. A function $u \in BC(\mathbb{R}, X)$ is called a weighted pseudo almost automorphic mild solution of Eq. (1.1) on $\mathbb{R}$ if $u \in WPAA(\mathbb{R}, \rho)$ and the function $s \to AT(t-s)f(s, u(h_1(s)))$ is integrable on $(-\infty, t)$ for each $t \in \mathbb{R}$, and $u(t)$ satisfies

$$\begin{aligned} u(t) &= T(t-a)[u(a) + f(a, u(h_1(a)))] - f(t, u(h_1(t))) - \int_a^t AT(t-s)f(s, u(h_1(s)))\,ds \\ &\quad + \int_a^t T(t-s)g(s, u(h_2(s)))\,ds \end{aligned}$$

for all $t \geq a$ and all $a \in \mathbb{R}$.

Now we list the following basic assumptions of this paper:
(H1) (I) There exists a positive number $\alpha \in (0,1)$ such that $f : \mathbb{R} \times X \to X_\alpha$ is continuous and $(-A)^\alpha f \in PAA(\mathbb{R} \times X, X)$. Let $L_f^{(1)} > 0$ be such that for each $(t,x), (t,y) \in \mathbb{R} \times X$

$$|(-A)^\alpha f(t,x) - (-A)^\alpha f(t,y)| \leq L_f^{(1)}|x-y|.$$

(II) There exists a positive number $\alpha \in (0,1)$ such that $f : \mathbb{R} \times X \to X_\alpha$ is continuous and $(-A)^\alpha f = \varphi_1 + \psi_1 \in WPAA(\mathbb{R} \times X, \rho)$, and there exist positive numbers $L_f^{(2)}$, L_{φ_1} such that for each $(t,x), (t,y) \in \mathbb{R} \times X$

$$|(-A)^\alpha f(t,x) - (-A)^\alpha f(t,y)| \leq L_f^{(2)}|x-y|, \ |\varphi_1(t,x) - \varphi_2(t,y)| \leq L_{\varphi_1}|x-y|.$$

(H2) (I) $g \in PAA(\mathbb{R} \times X, X)$ and there exists a positive number $L_g^{(1)}$ such that for each $(t,x), (t,y) \in \mathbb{R} \times X$

$$|g(t,x) - g(t,y)| \leq L_g^{(1)}|x-y|.$$

(II) $g = \varphi_2 + \psi_2 \in WPAA(\mathbb{R} \times X, \rho)$, and there exist positive numbers $L_g^{(2)}$, L_{φ_2} such that for each $(t,x), (t,y) \in \mathbb{R} \times X$

$$|g(t,x) - g(t,y)| \leq L_g^{(2)}|x-y|, \ |\varphi_2(t,x) - \varphi_2(t,y)| \leq L_{\varphi_2}|x-y|.$$

(H3) ([24]) The functions $h_i : \mathbb{R} \to \mathbb{R}$, $h_i(\mathbb{R}) = \mathbb{R}$ are continuously differentiable on $\mathbb{R}$, and for $u(\cdot) \in AA(\mathbb{X})$, $u(h_i(\cdot)) \in AA(\mathbb{X})$, $h_i'(t) > 0$, $i = 1, 2$, are nondecreasing with

$$\limsup_{r \to \infty} \left(\frac{|h_i(-r)| + |h_i(r)|}{r h_i'(-r)} \right) < \infty.$$

(H4) The functions $h_i : \mathbb{R} \to \mathbb{R}$, $h_i(\mathbb{R}) = \mathbb{R}$ are continuously differentiable on $\mathbb{R}$, and for $u(\cdot) \in AA(\mathbb{X})$, $u(h_i(\cdot)) \in AA(\mathbb{X})$, $h_i'(t) > 0$, $i = 1, 2$, are nondecreasing with

$$\limsup_{r \to \infty} \left(\frac{m(r_i^*, \rho)}{m(r, \rho) h_i'(-r)} \right) < \infty, \text{ and } 0 < \sup_{t \in \mathbb{R}} \frac{\rho(t)}{\rho(h_i(t))} < \infty,$$

where $r_i^* = |h_i(-r)| + |h_i(r)|$ for $i = 1, 2$.

3. Main results

In this section, we present and prove our main results. In order to establish our main results, we need the following auxiliary results.

Lemma 3.1. *Let $\alpha \in (0,1]$ and $(-A)^\alpha v \in WPAA(\mathbb{R},\rho)$. If $u(\cdot) : \mathbb{R} \to X$ be the function defined by*

$$u(t) = \int_{-\infty}^{t} AT(t-s)v(s)ds, \ t \geq s,$$

then $u(\cdot) \in WPAA(\mathbb{R},\rho)$.

Proof. . First we observe that $u(\cdot)$ is well defined. Since $(-A)^\alpha v \in WPAA(\mathbb{R},\rho)$, then $(-A)^\alpha v$ is bounded, we assume that there exists $M_1 > 0$, such that $\|(-A)^\alpha v\|_{WPPA(\mathbb{R},\rho)} \leq M_1$. So

$$\begin{aligned} |u(t)| &\leq \int_{-\infty}^{t} |AT(t-s)v(s)|ds \\ &\leq \int_{-\infty}^{t} \|(-A)^{1-\alpha}T(t-s)\||(-A)^\alpha v(s)|ds \\ &\leq M_1 \int_{-\infty}^{t} M_{1-\alpha}e^{-\delta(t-s)}(t-s)^{\alpha-1}ds \\ &\leq M_1 M_{1-\alpha} \int_{0}^{+\infty} \sigma^{\alpha-1}e^{-\delta\sigma}d\sigma \\ &\leq M_1 M_{1-\alpha}\delta^{-\alpha}\Gamma(\alpha), \end{aligned}$$

where $\Gamma(\cdot)$ is the gamma function. Thus $s \to AT(t-s)v(s)$ is integrable on $(-\infty, t)$ for each $t \in \mathbb{R}$ and so that $u(t)$ is a bounded continuous functions. Now we prove that $u(\cdot) \in WPAA(\mathbb{R},\rho)$. We let $(-A)^\alpha v(t) = m(t) + n(t)$, where $m(\cdot) \in AA(X)$ and $n(\cdot) \in PAA_0(\mathbb{R},\rho)$.

Then

$$\begin{aligned} u(t) &= -\int_{-\infty}^{t} (-A)^{1-\alpha}T(t-s)(-A)^\alpha v(s)ds \\ &= -\int_{-\infty}^{t} (-A)^{1-\alpha}T(t-s)m(s)ds - \int_{-\infty}^{t} (-A)^{1-\alpha}T(t-s)n(s)ds. \end{aligned}$$

Let $\mathfrak{G}(t) = -\int_{-\infty}^{t}(-A)^{1-\alpha}T(t-s)m(s)ds$, $\mathfrak{H}(t) = -\int_{-\infty}^{t}(-A)^{1-\alpha}T(t-s)n(s)ds$. Clearly, $u(t) = \mathfrak{G}(t)+\mathfrak{H}(t)$.

Now we show that $\mathfrak{G}(t) \in AA(X)$. Let $({s_n}')_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers. Since $m(t) \in AA(X)$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ of $({s_n}')_{n\in\mathbb{N}}$ such that $\varphi(t) := \lim_{n\to\infty} m(t+s_n)$ is well defined for each $t \in \mathbb{R}$, and $m(t) = \lim_{n\to\infty} \varphi(t-s_n)$ for each $t \in \mathbb{R}$.

Now, we consider

$$\begin{aligned} \mathfrak{G}(t+s_n) &= -\int_{-\infty}^{t+s_n} (-A)^{1-\alpha}T(t+s_n-s)m(s)ds \\ &= -\int_{-\infty}^{t} (-A)^{1-\alpha}T(t-s)m(s+s_n)ds \\ &= -\int_{-\infty}^{t} (-A)^{1-\alpha}T(t-s)m_n(s)ds, \end{aligned}$$

where $m_n(s) = m(s + s_n)$, $n = 1, 2, \cdots$. Also, we have

$$\begin{aligned}
|\mathfrak{G}(t+s_n)| &\leq \int_{-\infty}^{t} |(-A)^{1-\alpha}T(t-s)m_n(s)|ds \\
&\leq \int_{-\infty}^{t} M_{1-\alpha}e^{-\alpha(t-s)}(t-s)^{\alpha-1}|m_n(s)|ds \\
&\leq M_{1-\alpha}\delta^{-\alpha}\Gamma(\alpha)\|m\|_{AA(X)},
\end{aligned}$$

for $n = 1, 2, \cdots$. By the property (ii) of Lemma 2.1 it follows that

$$(-A)^{1-\alpha}T(t-s)m_n(s) \to (-A)^{1-\alpha}T(t-s)\varphi(s), \ \textit{as } n \to \infty,$$

for each $s \in \mathbb{R}$ fixed and any $t \geq s$, and we get

$$\lim_{n\to\infty} \mathfrak{G}(t+s_n) = -\int_{-\infty}^{t} (-A)^{1-\alpha}T(t-s)\varphi(s)ds,$$

by the Lebesgue's dominated convergence theorem. Analogously to the above proof, it can be shown that

$$\lim_{n\to\infty} \left\{ -\int_{-\infty}^{t-s_n} (-A)^{1-\alpha}T(t-s_n-s)\varphi(s)ds \right\} = \mathfrak{G}(t).$$

This shows that $\mathfrak{G}(t) \in AA(X)$.

Next we prove that $\mathfrak{H}(t) \in PAA_0(\mathbb{R}, \rho)$, that is we need to prove that

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} |\mathfrak{H}(t)|\rho(t)dt = 0.$$

We have

$$\begin{aligned}
\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} |\mathfrak{H}(t)|\rho(t)dt &\leq \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r}\int_{-\infty}^{t} |(-A)^{1-\alpha}T(t-s)n(s)|\rho(s)dsdt \\
&\leq \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} dt \int_{-\infty}^{-r} |(-A)^{1-\alpha}T(t-s)n(s)|\rho(s)ds \\
&\quad + \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} dt \int_{-r}^{t} |(-A)^{1-\alpha}T(t-s)n(s)|\rho(s)ds \\
&= I_1 + I_2,
\end{aligned}$$

where

$$I_1 := \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} dt \int_{-\infty}^{-r} |(-A)^{1-\alpha}T(t-s)n(s)|\rho(s)ds$$

and

$$I_2 := \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} dt \int_{-r}^{t} |(-A)^{1-\alpha}T(t-s)n(s)|\rho(s)ds.$$

We get

$$
\begin{aligned}
I_1 &= \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} dt \int_{-\infty}^{-r} |(-A)^{1-\alpha} T(t-s) n(s)| \rho(s) ds \\
&\le \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} dt \int_{-\infty}^{-r} M_{1-\alpha} e^{-\delta(t-s)} (t-s)^{\alpha-1} |n(s)| \rho(s) ds \\
&\le \lim_{r\to\infty} \frac{\|n\|_{BC(\mathbb{R},X)}}{m(r,\rho)} \int_{-r}^{r} \rho(t) dt \int_{-\infty}^{-r} M_{1-\alpha} e^{-\delta(t-s)} (t-s)^{\alpha-1} ds \\
&\le \lim_{r\to\infty} \frac{M_{1-\alpha}\|n\|_{BC(\mathbb{R},X)}}{m(r,\rho)} \int_{-r}^{r} \rho(t) dt \int_{t+r}^{+\infty} \sigma^{\alpha-1} e^{-\delta\sigma} d\sigma \\
&\le \lim_{r\to\infty} \frac{M_{1-\alpha}\|n\|_{BC(\mathbb{R},X)}}{m(r,\rho)} \int_{-r}^{r} \rho(t) dt \int_{2r}^{+\infty} \sigma^{\alpha-1} e^{-\delta\sigma} d\sigma \\
&\le \lim_{r\to\infty} M_{1-\alpha}\|n\|_{BC(\mathbb{R},X)} \int_{2r}^{+\infty} (2r)^{\alpha-1} e^{-\delta\sigma} d\sigma \\
&\le \frac{M_{1-\alpha}\|n\|_{BC(\mathbb{R},X)}}{(2r)^{1-\alpha} e^{2\delta r} \delta},
\end{aligned}
$$

where converges to zero as $r \to \infty$.

$$
\begin{aligned}
I_2 &= \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} dt \int_{-r}^{t} |(-A)^{1-\alpha} T(t-s) n(s)| \rho(s) ds \\
&\le \lim_{r\to\infty} \frac{M_{1-\alpha}}{m(r,\rho)} \int_{-r}^{r} dt \int_{-r}^{t} e^{-\delta(t-s)} (t-s)^{\alpha-1} |n(s)| \rho(s) ds \\
&\le \lim_{r\to\infty} \frac{M_{1-\alpha}}{m(r,\rho)} \int_{-r}^{r} |n(t)| \rho(t) dt \int_{0}^{t+r} \sigma^{\alpha-1} e^{-\delta\sigma} d\sigma \\
&\le \lim_{r\to\infty} \frac{M_{1-\alpha}}{m(r,\rho)} \int_{-r}^{r} |n(t)| \rho(t) dt \int_{0}^{+\infty} \sigma^{\alpha-1} e^{-\delta\sigma} d\sigma \\
&\le \lim_{r\to\infty} M_{1-\alpha} \delta^{-\alpha} \Gamma(\alpha) \frac{1}{m(r,\rho)} \int_{-r}^{r} |n(t)| \rho(t) dt.
\end{aligned}
$$

Since $n(\cdot) \in PAA_0(\mathbb{R},\rho)$, then $\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} |n(t)|\rho(t)dt = 0$. Therefore $\lim_{r\to\infty} I_2 = 0$. In view of the above it is clear that $u \in WPAA(\mathbb{R},\rho)$. The proof is achieved. □

The next result is a straightforward consequence of Lemma 3.1 when $\rho = 1$, and so we omit its proof.

Lemma 3.2. *Let $\alpha \in (0,1]$ and $(-A)^\alpha v \in PAA(X)$. If $u(\cdot) : \mathbb{R} \to X$ be the function defined by*

$$u(t) = \int_{-\infty}^{t} AT(t-s) v(s) ds, \ t \ge s,$$

then $u(\cdot) \in PAA(X)$.

Lemma 3.3. *([24, Lemma 4.1]) Assume that both $h_1(\cdot)$ and $h_2(\cdot)$ satisfy (H3). If $u \in PAA(X)$, then $u(h_i(\cdot)) \in PAA(X)$ for $i = 1, 2$.*

The proof of the following lemma is similar to that of Lemma 3.2 in [29]. For the completeness, we give the detailed proof here.

Lemma 3.4. *Assume that both $h_1(\cdot)$ and $h_2(\cdot)$ satisfy (H4). If $u \in WPAA(\mathbb{R},\rho)$, then $u(h_i(\cdot)) \in WPAA(\mathbb{R},\rho)$ for $i = 1, 2$.*

Proof. . Let $u(\cdot) = v(\cdot) + w(\cdot)$, where $v(\cdot) \in AA(X)$ and $w(\cdot) \in PAA_0(\mathbb{R}, \rho)$. By condition (H4) and the definition of almost automorphic functions, we can easily conclude that $v(h_i(t)) \in AA(X)$.

On the other hand, we need to prove that $w(h_i(t)) \in PAA_0(\mathbb{R}, \rho)$. We have

$$\begin{aligned}
0 &\leq \frac{1}{m(r,\rho)} \int_{-r}^{r} |w(h_i(t))|\rho(t)\,dt \\
&= \frac{1}{m(r,\rho)} \int_{-r}^{r} |w(h_i(t))|\rho(h_i(t)) \frac{\rho(t)}{\rho(h_i(t))} \frac{h_i'(t)}{h_i'(t)} dt \\
&\leq \frac{1}{m(r,\rho)h_i'(-r)} \sup_{t\in\mathbb{R}} \frac{\rho(t)}{\rho(h_i(t))} \int_{-r}^{r} |w(h_i(t))|\rho(h_i(t))\, h_i'(t)dt \\
&\leq \frac{1}{m(r,\rho)h_i'(-r)} \sup_{t\in\mathbb{R}} \frac{\rho(t)}{\rho(h_i(t))} \int_{-|h_i(-r)|}^{|h_i(r)|} |w(t)|\rho(t)dt \\
&\leq \frac{1}{m(r,\rho)h_i'(-r)} \sup_{t\in\mathbb{R}} \frac{\rho(t)}{\rho(h_i(t))} \int_{-|h_i(-r)|-|h_i(r)|}^{|h_i(-r)|+|h_i(r)|} |w(t)|\rho(t)dt \\
&= \frac{m(r_i^*,\rho)}{m(r,\rho)h_i'(-r)} \sup_{t\in\mathbb{R}} \frac{\rho(t)}{\rho(h_i(t))} \frac{1}{m(r_i^*,\rho)} \int_{-r_i^*}^{r_i^*} |w(t)|\rho(t)dt,
\end{aligned}$$

where $r_i^* = |h_i(-r)| + |h_i(r)|, i = 1, 2$. Since (H4) and $w(\cdot) \in PAA_0(\mathbb{R}, \rho)$, then the last inequality converges to zero as $r \to \infty$. Thus $w(h_i(t)) \in PAA_0(\mathbb{R}, \rho)$. As a consequence of the above proof, we can see that $u(h_i(t)) \in WPAA(\mathbb{R}, \rho)$ for $i = 1, 2$. The proof is finished. □

Remark 3.5. Let $\rho = 1$, then the condition (H4) is reduced to the condition (H3). It is consistent with the fact that the space $WPAA(\mathbb{R} \times X, \rho)$ is turned into the space $PAA(\mathbb{R} \times X, X)$ when $\rho = 1$.

Now, we state and prove our main results.

Theorem 3.6. *Assume the conditions (H1)(I), (H2)(I) and (H3) hold, then the problem (1.1) has a unique pseudo almost automorphic mild solution on $\mathbb{R}$ provide that*

$$\|(-A)^{-\alpha}\| L_f^{(1)} + M_{1-\alpha}\delta^{-\alpha}\Gamma(\alpha)L_f^{(1)} + \frac{M}{\delta} L_g^{(1)} < 1, \tag{3.1}$$

where $\Gamma(\cdot)$ is the gamma function.

Proof. Let $\Lambda : PAA(X) \to C(\mathbb{R}, X)$ be the operator defined by

$$\begin{aligned}
\Lambda u(t) &= -(-A)^{-\alpha}(-A)^{\alpha} f(t, u(h_1(t))) + \int_{-\infty}^{t} (-A)^{1-\alpha} T(t-s)(-A)^{\alpha} f(s, u(h_1(s)))\,ds \\
&\quad + \int_{-\infty}^{t} T(t-s) g(s, u(h_2(s)))\,ds \\
&= -f(t, u(h_1(t))) - \int_{-\infty}^{t} AT(t-s) f(s, u(h_1(s)))\,ds \\
&\quad + \int_{-\infty}^{t} T(t-s) g(s, u(h_2(s)))\,ds, t \in \mathbb{R}.
\end{aligned}$$

First we prove that $\Lambda u(t)$ is well defined. From the continuity of $s \to AT(t-s)$ and $s \to T(t-s)$ in the uniform operator topology on $(-\infty, t)$ for each $t \in \mathbb{R}$ and the estimate

$$\begin{aligned}
|AT(t-s) f(s, u(h_1(s)))| &= |(-A)^{1-\alpha} T(t-s)(-A)^{\alpha} f(s, u(h_1(s)))| \\
&\leq M_{1-\alpha} e^{-\delta(t-s)} (t-s)^{\alpha-1} \|(-A)^{\alpha} f(s, u(h_1(s)))\|_{BC(\mathbb{R}\times X, X)},
\end{aligned}$$

it follows that $s \to AT(t-s) f(s, u(h_1(s)))$ and $s \to T(t-s) g(s, u(h_2(s)))$ are integrable on $(-\infty, t)$ for every $t \in \mathbb{R}$ and so that Λu is well defined and continuous. Moreover, from Lemmas 3.3, 2.7, 3.2 and 2.14 we infer that $\Lambda u(t) \in PAA(X)$, that is, Λ maps $PAA(X)$ into itself.

Next, we show that Λ is a contraction on $PAA(X)$. Indeed, for each $t \in \mathbb{R}$, $u, v \in PAA(X)$, we have

$$\begin{aligned}
& |\Lambda u(t) - \Lambda v(t)| \\
\leq\ & |f(t, u(h_1(t))) - f(t, v(h_1(t)))| \\
& + \int_{-\infty}^{t} |AT(t-s) f(s, u(h_1(s))) - AT(t-s) f(s, v(h_1(s)))| ds \\
& + \int_{-\infty}^{t} |T(t-s) g(s, u(h_2(s))) - T(t-s) g(s, v(h_2(s)))| ds \\
\leq\ & \|(-A)^{-\alpha}\| |(-A)^{\alpha} f(t, u(h_1(t))) - (-A)^{\alpha} f(t, v(h_1(t)))| \\
& + \int_{-\infty}^{t} \|(-A)^{1-\alpha} T(t-s)\| |(-A)^{\alpha} f(s, u(h_1(s))) - (-A)^{\alpha} f(s, v(h_1(s)))| ds \\
& + \int_{-\infty}^{t} M e^{-\delta(t-s)} |g(s, u(h_2(s))) - g(s, v(h_2(s)))| ds \\
\leq\ & \|(-A)^{-\alpha}\| L_f^{(1)} |u(h_1(t)) - v(h_1(t))| \\
& + L_f^{(1)} \int_{-\infty}^{t} M_{1-\alpha} e^{-\delta(t-s)} (t-s)^{\alpha-1} |u(h_1(s)) - v(h_1(s))| ds \\
& + L_g^{(1)} \int_{-\infty}^{t} M e^{-\delta(t-s)} |u(h_2(s)) - v(h_2(s))| ds \\
\leq\ & \|(-A)^{-\alpha}\| L_f^{(1)} \|u - v\|_{PAA(X)} + L_f^{(1)} M_{1-\alpha} \delta^{-\alpha} \Gamma(\alpha) \|u - v\|_{PAA(X)} \\
& + L_g^{(1)} \frac{M}{\delta} \|u - v\|_{PAA(X)} \\
=\ & [\|(-A)^{-\alpha}\| L_f^{(1)} + M_{1-\alpha} \delta^{-\alpha} \Gamma(\alpha) L_f^{(1)} + \frac{M}{\delta} L_g^{(1)}] \|u - v\|_{PAA(X)}.
\end{aligned}$$

Thus

$$\|\Lambda u - \Lambda v\|_{PAA(X)} \leq [\|(-A)^{-\alpha}\| L_f^{(1)} + M_{1-\alpha} \delta^{-\alpha} \Gamma(\alpha) L_f^{(1)} + \frac{M}{\delta} L_g^{(1)}] \|u - v\|_{PAA(X)},$$

which implies that Λ is a contraction by (3.1). By the contraction principle, we conclude that there exists a unique fixed point $u(\cdot)$ for Λ in $PAA(X)$, such that $\Lambda u = u$, that is

$$u(t) = -f(t, u(h_1(t))) - \int_{-\infty}^{t} AT(t-s) f(s, u(h_1(s))) \, ds + \int_{-\infty}^{t} T(t-s) g(s, u(h_2(s))) \, ds,$$

for all $t \in \mathbb{R}$. If we let $u(a) = -f(a, u(h_1(a))) - \int_{-\infty}^{a} AT(a-s) f(s, u(h_1(s))) \, ds + \int_{-\infty}^{a} T(a-s) g(s, u(h_2(s))) \, ds$, then

$$\begin{aligned}
T(t-a) u(a) \ =\ & -T(t-a) f(a, u(h_1(a))) - \int_{-\infty}^{a} AT(t-s) f(s, u(h_1(s))) \, ds \\
& + \int_{-\infty}^{a} T(t-s) g(s, u(h_2(s))) \, ds.
\end{aligned}$$

But for $t \geq a$,

$$\begin{aligned}
& \int_a^t T(t-s)g\left(s, u(h_2(s))\right) ds \\
= \ & \int_{-\infty}^t T(t-s)g\left(s, u(h_2(s))\right) ds - \int_{-\infty}^a T(t-s)g\left(s, u(h_2(s))\right) ds \\
= \ & u(t) + f\left(t, u(h_1(t))\right) + \int_{-\infty}^t AT(t-s)f\left(s, u(h_1(s))\right) ds \\
& -T(t-a)[u(a) + f\left(a, u(h_1(a))\right)] \\
& - \int_{-\infty}^a AT(t-s)f\left(s, u(h_1(s))\right) ds \\
= \ & u(t) + f\left(t, u(h_1(t))\right) + \int_a^t AT(t-s)f\left(s, u(h_1(s))\right) ds \\
& -T(t-a)[u(a) + f\left(a, u(h_1(a))\right)].
\end{aligned}$$

In conclusion,

$$\begin{aligned}
u\left(t\right) &= T(t-a)\left[u(a) + f\left(a, u(h_1(a))\right)\right] - f\left(t, u(h_1(t))\right) \\
& \quad - \int_a^t AT(t-s)f\left(s, u(h_1(s))\right) ds + \int_a^t T(t-s)g\left(s, u(h_2(s))\right) ds
\end{aligned}$$

is a mild solution of equation (1.1) and $u \in PAA(X)$. This finishes the proof. □

Theorem 3.7. *Assume the conditions (H1)(II), (H2)(II) and (H4) are satisfied, then the problem (1.1) has a unique weighted pseudo almost automorphic mild solution on* $\mathbb{R}$ *provide that*

$$\|(-A)^{-\alpha}\| L_f^{(2)} + M_{1-\alpha}\delta^{-\alpha}\Gamma(\alpha)L_f^{(2)} + \frac{M}{\delta}L_g^{(2)} < 1, \tag{3.2}$$

where $\Gamma(\cdot)$ *is the gamma function.*

Proof. We define the operator $\Lambda : WPAA(\mathbb{R}, \rho) \to C(\mathbb{R}, X)$ as

$$\begin{aligned}
\Lambda u(t) &= -(-A)^{-\alpha}(-A)^{\alpha} f\left(t, u(h_1(t))\right) + \int_{-\infty}^t (-A)^{1-\alpha}T(t-s)(-A)^{\alpha} f\left(s, u(h_1(s))\right) ds \\
& \quad + \int_{-\infty}^t T(t-s)g\left(s, u(h_2(s))\right) ds \\
&= -f\left(t, u(h_1(t))\right) - \int_{-\infty}^t AT(t-s)f\left(s, u(h_1(s))\right) ds \\
& \quad + \int_{-\infty}^t T(t-s)g\left(s, u(h_2(s))\right) ds, t \in \mathbb{R}.
\end{aligned}$$

The same arguments used in the proof of Theorem 3.6, we can prove that $\Lambda u(t)$ is well defined and continuous. Moreover, from Lemmas 3.4, 2.12, 3.1 and 2.13, we infer that $\Lambda u(t) \in WPAA(\mathbb{R}, \rho)$, that is, Λ maps $WPAA(\mathbb{R}, \rho)$ into itself.

Now we prove that Λ is a contraction on $WPAA(\mathbb{R}, \rho)$. Indeed, for each $t \in \mathbb{R}$, $u, v \in WPAA(\mathbb{R}, \rho)$, we have

$$
\begin{aligned}
& |\Lambda u(t)-\Lambda v(t)| \\
\leq\ & |f(t,u(h_1(t)))-f(t,v(h_1(t)))| \\
& +\int_{-\infty}^{t}|AT(t-s)f(s,u(h_1(s)))-AT(t-s)f(s,v(h_1(s)))|ds \\
& +\int_{-\infty}^{t}|T(t-s)g(s,u(h_2(s)))-T(t-s)g(s,v(h_2(s)))|ds \\
\leq\ & \|(-A)^{-\alpha}\||(-A)^{\alpha}f(t,u(h_1(t)))-(-A)^{\alpha}f(t,v(h_1(t)))| \\
& +\int_{-\infty}^{t}\|(-A)^{1-\alpha}T(t-s)\||(-A)^{\alpha}f(s,u(h_1(s)))-(-A)^{\alpha}f(s,v(h_1(s)))|ds \\
& +\int_{-\infty}^{t}Me^{-\delta(t-s)}|g(s,u(h_2(s)))-g(s,v(h_2(s)))|ds \\
\leq\ & \|(-A)^{-\alpha}\|L_f^{(2)}|u(h_1(t))-v(h_1(t))| \\
& +L_f^{(2)}\int_{-\infty}^{t}M_{1-\alpha}e^{-\delta(t-s)}(t-s)^{\alpha-1}|u(h_1(s))-v(h_1(s))|ds \\
& +L_g^{(2)}\int_{-\infty}^{t}Me^{-\delta(t-s)}|u(h_2(s))-v(h_2(s))|ds \\
\leq\ & \|(-A)^{-\alpha}\|L_f^{(2)}\|u-v\|_{WPAA(\mathbb{R},\rho)}+L_f^{(2)}M_{1-\alpha}\delta^{-\alpha}\Gamma(\alpha)\|u-v\|_{WPAA(\mathbb{R},\rho)} \\
& +L_g^{(2)}\frac{M}{\delta}\|u-v\|_{WPAA(\mathbb{R},\rho)} \\
=\ & [\|(-A)^{-\alpha}\|L_f^{(2)}+M_{1-\alpha}\delta^{-\alpha}\Gamma(\alpha)L_f^{(2)}+\frac{M}{\delta}L_g^{(2)}]\|u-v\|_{WPAA(\mathbb{R},\rho)}.
\end{aligned}
$$

Thus

$$\|\Lambda u-\Lambda v\|_{WPAA(\mathbb{R},\rho)}\leq[\|(-A)^{-\alpha}\|L_f^{(2)}+M_{1-\alpha}\delta^{-\alpha}\Gamma(\alpha)L_f^{(2)}+\frac{M}{\delta}L_g^{(2)}]\|u-v\|_{WPAA(\mathbb{R},\rho)}.$$

It follows that Λ is a contraction from (3.2). By the contraction principle, we draw a conclusion that there exists a unique fixed point $u(\cdot)$ for Λ in $WPAA(\mathbb{R},\rho)$, such that $\Lambda u=u$. Moreover, using the same proof as in Theorem 3.6, we can see that $u(t)=T(t-a)[u(a)+f(a,u(h_1(a)))]-f(t,u(h_1(t)))-\int_a^t AT(t-s)f(s,u(h_1(s)))\,ds+\int_a^t T(t-s)g(s,u(h_2(s)))\,ds$ is a mild solution of equation (1.1) and $u\in WPAA(\mathbb{R},\rho)$. This completes the proof. □

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