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Orthogonal stability of a cubic-quartic functional equation

Choonkil Park[a]

[a] *Department of Mathematics, Hanyang University, Seoul 133-791, Korea*

This paper is dedicated to Professor Ljubomir Ćirić

Communicated by Professor V. Berinde

Abstract

Using fixed point method, we prove the Hyers-Ulam stability of the orthogonally cubic-quartic functional equation

$$\begin{aligned} f(2x+y)+f(2x-y) &= 3f(x+y)+f(-x-y)+3f(x-y)+f(y-x) \\ &+ 18f(x)+6f(-x)-3f(y)-3f(-y) \end{aligned} \qquad (1)$$

for all x, y with $x \perp y$.

Keywords: Hyers-Ulam stability, orthogonally cubic-quartic functional equation, fixed point, orthogonality space.
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1. Introduction and preliminaries

Assume that X is a real inner product space and $f : X \to \mathbb{R}$ is a solution of the orthogonal Cauchy functional equation $f(x+y) = f(x)+f(y)$, $\langle x, y\rangle = 0$. By the Pythagorean theorem $f(x) = \|x\|^2$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus orthogonal Cauchy equation is not equivalent to the classic Cauchy equation on the whole inner product space.

Email address: `baak@hanyang.ac.kr` (Choonkil Park)

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G. Pinsker [36] characterized orthogonally additive functionals on an inner product space when the orthogonality is the ordinary one in such spaces. K. Sundaresan [46] generalized this result to arbitrary Banach spaces equipped with the Birkhoff-James orthogonality. The orthogonal Cauchy functional equation

$$f(x+y) = f(x) + f(y), \qquad x \perp y,$$

in which $\perp$ is an abstract orthogonality relation, was first investigated by S. Gudder and D. Strawther [17]. They defined $\perp$ by a system consisting of five axioms and described the general semi-continuous real-valued solution of conditional Cauchy functional equation. In 1985, J. Rätz [43] introduced a new definition of orthogonality by using more restrictive axioms than of S. Gudder and D. Strawther. Moreover, he investigated the structure of orthogonally additive mappings. J. Rätz and Gy. Szabó [44] investigated the problem in a rather more general framework.

Let us recall the orthogonality in the sense of J. Rätz; cf. [43].

Suppose X is a real vector space with $\dim X \geq 2$ and $\perp$ is a binary relation on X with the following properties:
(O_1) totality of $\perp$ for zero: $x \perp 0, 0 \perp x$ for all $x \in X$;
(O_2) independence: if $x, y \in X - \{0\}, x \perp y$, then x, y are linearly independent;
(O_3) homogeneity: if $x, y \in X, x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;
(O_4) the Thalesian property: if P is a 2-dimensional subspace of $X, x \in P$ and $\lambda \in \mathbb{R}_+$, which is the set of nonnegative real numbers, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(X, \perp)$ is called an orthogonality space. By an orthogonality normed space we mean an orthogonality space having a normed structure.

Some interesting examples are
(i) The trivial orthogonality on a vector space X defined by (O_1), and for non-zero elements $x, y \in X$, $x \perp y$ if and only if x, y are linearly independent.
(ii) The ordinary orthogonality on an inner product space $(X, \langle ., . \rangle)$ given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.
(iii) The Birkhoff-James orthogonality on a normed space $(X, \|.\|)$ defined by $x \perp y$ if and only if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. Clearly examples (i) and (ii) are symmetric but example (iii) is not. It is remarkable to note, however, that a real normed space of dimension greater than 2 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric. There are several orthogonality notions on a real normed space such as Birkhoff-James, Boussouis, Singer, Carlsson, unitary-Boussouis, Roberts, Phythagorean, isosceles and Diminnie (see [1]–[3], [7, 13, 22]).

The stability problem of functional equations originated from the following question of Ulam [48]: *Under what condition does there exist an additive mapping near an approximately additive mapping?* In 1941, Hyers [18] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1978, Th.M. Rassias [38] extended the theorem of Hyers by considering the unbounded Cauchy difference $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$, $(\varepsilon > 0, p \in [0, 1))$. During the last decades several stability problems of functional equations have been investigated in the spirit of Hyers-Ulam-Rassias. The reader is referred to [11, 19, 24, 42] and references therein for detailed information on stability of functional equations.

R. Ger and J. Sikorska [16] investigated the orthogonal stability of the Cauchy functional equation $f(x+y) = f(x) + f(y)$, namely, they showed that if f is a mapping from an orthogonality space X into a real Banach space Y and $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in X$ with $x \perp y$ and some $\varepsilon > 0$, then there exists exactly one orthogonally additive mapping $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{16}{3}\varepsilon$ for all $x \in X$.

The first author treating the stability of the quadratic equation was F. Skof [45] by proving that if f is a mapping from a normed space X into a Banach space Y satisfying $\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \varepsilon$ for some $\varepsilon > 0$, then there is a unique quadratic mapping $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{\varepsilon}{2}$. P.W. Cholewa [8] extended the Skof's theorem by replacing X by an abelian group G. The Skof's result was later generalized by S. Czerwik [9] in the spirit of Hyers-Ulam-Rassias. The stability problem of functional equations has been extensively investigated by some mathematicians (see [10, 35], [39]–[41]).

The orthogonally quadratic equation

$$f(x+y)+f(x-y)=2f(x)+2f(y),\ x\perp y$$

was first investigated by F. Vajzović [49] when X is a Hilbert space, Y is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, H. Drljević [14], M. Fochi [15], M.S. Moslehian [29, 30] and Gy. Szabó [47] generalized this result. See also [31, 32].

Let X be a set. A function $d : X\times X\to[0,\infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x,y)=0$ if and only if $x=y$;

(2) $d(x,y)=d(y,x)$ for all $x,y\in X$;

(3) $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.1. *[4, 12] Let (X,d) be a complete generalized metric space and let $J : X\to X$ be a strictly contractive mapping with Lipschitz constant $\alpha<1$. Then for each given element $x\in X$, either*

$$d(J^n x, J^{n+1}x)=\infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x)<\infty, \qquad \forall n\ge n_0$;

(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;

(3) y^* is the unique fixed point of J in the set $Y=\{y\in X \mid d(J^{n_0}x,y)<\infty\}$;

(4) $d(y,y^*)\le\frac{1}{1-\alpha}d(y,Jy)$ for all $y\in Y$.

In 1996, G. Isac and Th.M. Rassias [20] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 25, 28, 33, 34, 37]).

In [23], Jun and Kim considered the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{1.1}$$

It is easy to show that the function $f(x)=x^3$ satisfies the functional equation (1.1), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

In [26], Lee et al. considered the following quartic functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \tag{1.2}$$

It is easy to show that the function $f(x)=x^4$ satisfies the functional equation (1.2), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*.

This paper is organized as follows: In Section 2, we prove the Hyers-Ulam stability of the orthogonally cubic-quartic functional equation (1) in orthogonality spaces for an odd mapping.

In Section 3, we prove the Hyers-Ulam stability of the orthogonally cubic-quartic functional equation (1) in orthogonality spaces for an even mapping.

Throughout this paper, assume that $(X,\perp)$ is an orthogonality space and that $(Y,\|.\|_Y)$ is a real Banach space.

2. Stability of the orthogonally cubic-quartic functional equation: an odd mapping case

In this section, applying some ideas from [16, 19], we deal with the stability problem for the orthogonally cubic-quartic functional equation

$$\begin{aligned} Df(x,y) := &f(2x+y) + f(2x-y) - 3f(x+y) - f(-x-y) \\ &-3f(x-y) - f(y-x) - 18f(x) - 6f(-x) + 3f(y) + 3f(-y) = 0 \end{aligned}$$

for all $x, y \in X$ with $x \perp y$: an odd mapping case.

Definition 2.1. A mapping $f : X \to Y$ is called an *orthogonally cubic mapping* if

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x)$$

for all $x, y \in X$ with $x \perp y$.

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x,y) \le 8\alpha\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{2.1}$$

for all $x, y \in X$ with $x \perp y$. Let $f : X \to Y$ be an odd mapping satisfying

$$\|Df(x,y)\|_Y \le \varphi(x,y) \tag{2.2}$$

for $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally cubic mapping $C : X \to Y$ such that

$$\|f(x) - C(x)\|_Y \le \frac{1}{16-16\alpha}\varphi(x,0) \tag{2.3}$$

for all $x \in X$.

Proof. Putting $y = 0$ in (2.2), we get

$$\|2f(2x) - 16f(x)\|_Y \le \varphi(x,0) \tag{2.4}$$

for all $x \in X$, since $x \perp 0$. So

$$\left\|f(x) - \frac{1}{8}f(2x)\right\|_Y \le \frac{1}{16}\varphi(x,0) \tag{2.5}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\|_Y \le \mu\varphi(x,0), \quad \forall x \in X\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [27, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{8}g(2x)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x) - h(x)\|_Y \le \varphi(x,0)$$

for all $x \in X$. Hence

$$\|Jg(x) - Jh(x)\|_Y = \left\| \frac{1}{8} g(2x) - \frac{1}{8} h(2x) \right\|_Y \le \alpha \varphi(x, 0)$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \le \alpha\varepsilon$. This means that

$$d(Jg, Jh) \le \alpha d(g, h)$$

for all $g, h \in S$.

It follows from (2.5) that $d(f, Jf) \le \frac{1}{16}$.

By Theorem 1.1, there exists a mapping $C : X \to Y$ satisfying the following:

(1) C is a fixed point of J, i.e.,

$$C(2x) = 8C(x) \tag{2.6}$$

for all $x \in X$. The mapping C is a unique fixed point of J in the set

$$M = \{g \in S : d(h, g) < \infty\}.$$

This implies that C is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - C(x)\|_Y \le \mu\varphi(x, 0)$$

for all $x \in X$;

(2) $d(J^n f, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{8^n} f(2^n x) = C(x)$$

for all $x \in X$;

(3) $d(f, C) \le \frac{1}{1-\alpha} d(f, Jf)$, which implies the inequality

$$d(f, C) \le \frac{1}{16 - 16\alpha}.$$

This implies that the inequality (2.3) holds.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|DC(x, y)\|_Y &= \lim_{n\to\infty} \frac{1}{8^n} \|Df(2^n x, 2^n y)\|_Y \\ &\le \lim_{n\to\infty} \frac{1}{8^n} \varphi(2^n x, 2^n y) \le \lim_{n\to\infty} \frac{8^n \alpha^n}{8^n} \varphi(x, y) = 0 \end{aligned}$$

for all $x, y \in X$ with $x \perp y$. So

$$DC(x, y) = 0$$

for all $x, y \in X$ with $x \perp y$. Since f is odd, C is odd. Hence $C : X \to Y$ is an orthogonally cubic mapping, i.e.,

$$C(2x + y) + C(2x - y) = 2C(x + y) + 2C(x - y) + 12C(x)$$

for all $x, y \in X$ with $x \perp y$. Thus $C : X \to Y$ is a unique orthogonally cubic mapping satisfying (2.3), as desired. $\square$

From now on, in corollaries, assume that $(X, \perp)$ is an orthogonality normed space.

Corollary 2.3. *Let θ be a positive real number and p a real number with $0 < p < 3$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\|Df(x,y)\|_Y \le \theta(\|x\|^p + \|y\|^p) \tag{2.7}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally cubic mapping $C : X \to Y$ such that

$$\|f(x) - C(x)\|_Y \le \frac{\theta}{2(8-2^p)}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$. Then we can choose $\alpha = 2^{p-3}$ and we get the desired result. □

Theorem 2.4. *Let $f : X \to Y$ be an odd mapping satisfying (2.2) for which there exists a function $\varphi : X^2 \to [0,\infty)$ such that*

$$\varphi(x,y) \le \frac{\alpha}{8}\varphi(2x,2y)$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally cubic mapping $C : X \to Y$ such that

$$\|f(x) - C(x)\|_Y \le \frac{\alpha}{16-16\alpha}\varphi(x,0) \tag{2.8}$$

for all $x \in X$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 8g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (2.4) that $d(f, Jf) \le \frac{\alpha}{16}$. So

$$d(f,C) \le \frac{\alpha}{16-16\alpha}.$$

Thus we obtain the inequality (2.8).

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let θ be a positive real number and p a real number with $p > 3$. Let $f : X \to Y$ be an odd mapping satisfying (2.7). Then there exists a unique orthogonally cubic mapping $C : X \to Y$ such that*

$$\|f(x) - C(x)\|_Y \le \frac{\theta}{2(2^p-8)}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x,y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$. Then we can choose $\alpha = 2^{3-p}$ and we get the desired result. □

3. Stability of the orthogonally cubic-quartic functional equation: an even mapping case

In this section, applying some ideas from [16, 19], we deal with the stability problem for the orthogonally cubic-quartic functional equation given in the previous section: an even mapping case.

Definition 3.1. A mapping $f : X \to Y$ is called an *orthogonally quartic mapping* if

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y)$$

for all $x, y \in X$ with $x \perp y$.

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x,y) \le 16\alpha\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x, y \in X$ with $x \perp y$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2). Then there exists a unique orthogonally quartic mapping $P : X \to Y$ such that

$$\|f(x)-P(x)\|_Y \le \frac{1}{32-32\alpha}\varphi(x,0)$$

for all $x \in X$.

Proof. Putting $y = 0$ in (2.2), we get

$$\|2f(2x)-32f(x)\|_Y \le \varphi(x,0) \tag{3.1}$$

for all $x \in X$, since $x \perp 0$. So

$$\left\|f(x)-\frac{1}{16}f(2x)\right\|_Y \le \frac{1}{32}\varphi(x,0)$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{16}g(2x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.3. *Let θ be a positive real number and p a real number with $0 < p < 4$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.7). Then there exists a unique orthogonally quartic mapping $P : X \to Y$ such that*

$$\|f(x)-P(x)\|_Y \le \frac{\theta}{2(16-2^p)}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x,y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$. Then we can choose $\alpha = 2^{p-4}$ and we get the desired result. □

Theorem 3.4. *Let $f : X \to Y$ be an even mapping satisfying (2.2) and $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0,\infty)$ such that*

$$\varphi(x,y) \le \frac{\alpha}{16}\varphi(2x,2y)$$

for all $x, y \in X$ with $x \perp y$. There exists a unique orthogonally quartic mapping $P : X \to Y$ such that

$$\|f(x)-P(x)\|_Y \le \frac{\alpha}{32-32\alpha}\varphi(x,0) \tag{3.2}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 16g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (3.1) that $d(f, Jf) \leq \frac{\alpha}{32}$. So we obtain the inequality (3.2).

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.5. *Let θ be a positive real number and p a real number with $p > 4$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.7). Then there exists a unique orthogonally quartic mapping $P : X \to Y$ such that*

$$\|f(x) - P(x)\|_Y \leq \frac{\theta}{2(2^p - 16)}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$. Then we can choose $\alpha = 2^{4-p}$ and we get the desired result. □

Let $f_o(x) = \frac{f(x)-f(-x)}{2}$ and $f_e(x) = \frac{f(x)+f(-x)}{2}$. Then f_o is an odd mapping and f_e is an even mapping such that $f = f_o + f_e$.

The above corollaries can be summarized as follows:

Theorem 3.6. *Assume that $(X, \perp)$ is an orthogonality normed space. Let θ be a positive real number and p a real number with $0 < p < 3$ or $p > 4$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.7). Then there exist an orthogonally cubic mapping $C : X \to Y$ and an orthogonally quartic mapping $P : X \to Y$ such that*

$$\|f(x) - C(x) - P(x)\|_Y \leq \left(\frac{1}{|8 - 2^p|} + \frac{1}{|16 - 2^p|}\right)\frac{\theta}{2}\|x\|^p$$

for all $x \in X$.

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