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A general fixed point theorem for pairs of weakly compatible mappings in G–metric spaces

Valeriu Popa[a,*], Alina-Mihaela Patriciu[b]

[a] Department of Mathematics, Informatics and Educational Sciences, Faculty of Sciences "Vasile Alecsandri" University of Bacău, 157 Calea Mărășești, Bacău, 600115, Romania.
[b] Department of Mathematics, Informatics and Educational Sciences, Faculty of Sciences "Vasile Alecsandri" University of Bacău, 157 Calea Mărășești, Bacău, 600115, Romania.

This paper is dedicated to Professor Ljubomir Ćirić

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Abstract

In this paper a general fixed point theorem in G–metric spaces for weakly compatible mappings is proved, theorem which generalize the results from Abbas et. al. [M. Abbas and B. E. Rhoades, Appl. Math. and Computation **215** (2009), 262 - 269] and [M. Abbas, T. Nazir and S. Radanović, Appl. Math. and Computation **217** (2010), 4094 - 4099]. In the last part of this paper it is proved that the fixed point problem for these mappings is well posed.

Keywords: G–metric space, weakly compatible mappings, fixed point.
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1. Introduction

Let (X, d) be a metric space and $S, T : (X, d) \to (X, d)$ be two mappings. In 1994, Pant [22] introduced the notion of pointwise R - weakly commuting mappings. It is proved in [23] that the notion of pointwise R - weakly commutativity is equivalent to commutativity in coincidence points. Jungck [11] defined S and T to be weakly compatible if $Sx = Tx$ implies $STx = TSx$. Thus, S and T are weakly compatible if and only if S and T are pointwise R - weakly commuting.

In [9] and [10], Dhage introduced a new class of generalized metric spaces, named D - metric space. Mustafa and Sims [14], [15] proved that most of the claims concerning the fundamental topological structures

*Corresponding author
Email addresses: vpopa@ub.ro (Valeriu Popa), alina.patriciu@ub.ro (Alina-Mihaela Patriciu)

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on D - metric spaces are incorrect and introduced appropriate notion of generalized metric space, named G - metric space. In fact, Mustafa, Sims and other authors studied many fixed point results for self mappings in G - metric spaces under certain conditions [6], [16] - [21], [33] and other papers.

In [25] and [26], Popa initiated the study of fixed points for mappings satisfying implicit relations.

The notion of well posedness of a fixed point problem has generated much interest to several mathematicians, for example [8], [12], [24], [29], [30], [31]. Recently, Popa [27], [33] and Akkouchi and Popa [3], [4], [5] studied well posedness problem for mappings satisfying implicit relations in metric spaces.

The purpose of this paper is to prove a general fixed point theorem in G - metric spaces for weakly compatible pairs of mappings satisfying an implicit relation which generalize the results from [1] and [13]. In the last part of this paper we define the notion of a fixed point problem in G - metric spaces for two mappings and we prove that in G - metric space with a G - symmetric, the fixed point problem is well posed.

2. Preliminaries

Definition 2.1 ([15]). Let X be a nonempty set and $G : X^3 \to \mathbb{R}_+$ be a function satisfying the following properties:

$(G_1) : G(x, y, z) = 0$ if $x = y = z$,
$(G_2) : 0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
$(G_3) : G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,
$(G_4) : G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$ (symmetry in all three variables),
$(G_5) : G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then the function G is called a G - metric on X and the pair (X, G) is called a G - metric space.
Note that $G(x, y, z) = 0$, then $x = y = z$.

Definition 2.2 ([15]). Let (X, G) be a metric space. A sequence (x_n) in X is said to be

a) G - convergent if for $\varepsilon > 0$, there is an $x \in X$ and $k \in \mathbb{N}$ such that for all $m, n \geq k$, $G(x, x_n, x_m) < \varepsilon$.

b) G - Cauchy if for each $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for all $n, m, p \geq k$, $G(x_n, x_m, x_p) < \varepsilon$, that is $G(x_n, x_m, x_p) \to 0$ as $m, n, n \to \infty$.

c) A G - metric space is said to be G - complete if every G - Cauchy sequence is G - convergent.

Lemma 2.3 ([15]). *Let (X, G) be a G - metric space. Then, the following properties are equivalent:*

1) (x_n) is G - convergent to x;
2) $G(x_n, x_n, x) \to 0$ as $n \to \infty$;
3) $G(x_n, x, x) \to 0$ as $n \to \infty$;
4) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

Lemma 2.4 ([15]). *If (X, G) is a G - metric space, the following are equivalent:*

1) (x_n) is G - Cauchy.
2) For every $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $n, m \geq k$.

Definition 2.5 ([14]). Let (X, G) and (X', G') be two G - metric spaces. A function $f : (X, G) \to (X', G')$ is said to be G - continuous at a point $x \in X$ if for $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$ and $G(a, x, y) < \delta$, then $G'(f(a), f(x), f(y)) < \varepsilon$.

A function f is G - continuous if f is G - continuous at each $x \in X$.

Lemma 2.6 ([15]). *Let (X, G) and (X', G') be G - metric spaces. Then, a function $f : (X, G) \to (X', G')$ is G - continuous at a point $x \in X$ if and only if it is G - sequentially continuous, that is, whenever (x_n) is G - convergent to x, we have that $f(x_n)$ is G - convergent to $f(x)$.*

Lemma 2.7 ([15]). *Let (X, G) be a G - metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 2.8 ([15]). A G - metric space (X, G) is called symmetric if $G(x, y, y) = G(y, x, x$, for all $x, y \in X$.

Remark 2.9. There exists G - metric space which is not symmetric (Example 1 [15]).

3. Implicit relations

Definition 3.1. Let $\mathfrak{F}_G$ be the set of all continuous functions $F(t_1,...,t_6):\mathbb{R}_+^6 \to \mathbb{R}$ such that

(F_1) : F is nonincreasing in variable t_5,

(F_2) : There exists $h_1 \in [0,1)$ such that for all $u,v \geq 0$, $F(u,v,v,u,u+v,0) \leq 0$ implies $u \leq h_1 v$.

(F_3) : There exists $h_2 \in [0,1)$ such that for all $t,t' > 0$, $F(t,t,0,0,t,t') < 0$ implies $t \leq h_2 t'$.

Example 3.2. $F(t_1,...,t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, where $a,b,c,d,e \geq 0$ and $0 < a+b+c+2d+e < 1$.

(F_1) : Obviously.

(F_2) : Let $u,v \geq 0$ be and $F(u,v,v,u,u+v,0) = u - av - bv - cu - d(u+v) \leq 0$. Then, $u \leq h_1 v$, where $0 \leq h_1 = \dfrac{a+b+d}{1-(c+d)} < 1$.

(F_3) : Let $t,t' > 0$ and $F(t,t,0,0,t,t') = t - at - dt - et' \leq 0$. Then $t \leq h_2 t'$, where $0 \leq h_2 = \dfrac{e}{1-(a+d)} < 1$.

Example 3.3. $F(t_1,...,t_6) = t_1 - k\max\{t_2,t_3,t_4,t_5,t_6\}$, where $k \in \left[0, \frac{1}{2}\right)$.

(F_1) : Obviously.

(F_2) : Let $u,v \geq 0$ be and $F(u,v,v,u,u+v,0) = u - k\max\{u,v,u+v\} \leq 0$. Hence, $u \leq h_1 v$, where $0 \leq h_1 = \dfrac{k}{1-k} < 1$.

(F_3) : Let $t,t' > 0$ and $F(t,t,0,0,t,t') = t - k\max\{t,t'\} \leq 0$. If $t > t'$, then $t(1-k) \leq 0$, a contradiction. Hence, $t \leq t'$ which implies $t \leq h_2 t'$, where $0 \leq h_2 = k < 1$.

Example 3.4. $F(t_1,...,t_6) = t_1 - k\max\left\{t_2,t_3,t_4,\dfrac{t_5+t_6}{2},\right\}$, where $k \in [0,1)$.

(F_1) : Obviously.

(F_2) : Let $u,v \geq 0$ be and $F(u,v,v,u,u+v,0) = u - k\max\left\{u,v,\dfrac{u+v}{2}\right\} \leq 0$. If $u > v$, then $u(1-k) \leq 0$, a contradiction. Hence, $u \leq v$ which implies $u \leq h_1 v$, where $0 \leq h_1 = k < 1$.

(F_3) : Let $t,t' > 0$ and $F(t,t,0,0,t,t') = t - k\max\left\{t,\dfrac{t+t'}{2}\right\} \leq 0$. If $t > t'$, then $t(1-k) \leq 0$, a contradiction. Hence, $t \leq t'$ which implies $t \leq h_2 t'$, where $0 \leq h_2 = k < 1$.

Example 3.5. $F(t_1,...,t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5 t_6 \leq 0$, where $a,b,c,d \geq 0$ and $0 \leq a+b+c+d < 1$.

(F_1) : Obviously.

(F_2) : Let $u,v \geq 0$ be and $F(u,v,v,u,u+v,0) = u^2 - u(av+bv+cu) \leq 0$. If $u > 0$, then $u - av - bv - cu \leq 0$ which implies $u \leq h_1 v$, where $0 \leq h_1 = \dfrac{a+b}{1-c} < 1$. If $u = 0$ then $u \leq h_1 v$.

Example 3.6. $F(t_1,...,t_6) = t_1 - k\max\left\{\dfrac{t_3+t_4}{2},\dfrac{t_5+t_6}{2}\right\}$, where $k \in [0,1)$.

(F_1) : Obviously.

(F_2) : Let $u,v \geq 0$ be such that $F(u,v,v,u,u+v,0) = u - k\max\left\{v,\dfrac{u+v}{2}\right\} \leq 0$. If $u > v$, then $u(1-k) \leq 0$, a contradiction. Hence, $u \leq v$ which implies $u \leq h_1 v$, where $0 \leq h_1 = k < 1$.

(F_3) : $F(t,t,0,0,t,t') = t - k\max\left\{t,\dfrac{t+t'}{2}\right\} \leq 0$.If $t > t'$ then $t(1-k) \leq 0$, a contradiction. Hence $t \leq t'$ which implies $t \leq h_2 t'$, where $0 \leq h_2 = k < 1$.

Example 3.7. $F(t_1,...,t_6) = t_1^3 - c\dfrac{t_3^2 t_4^2 + t_5^2 t_6^2}{1+t_2+t_3+t_4}$, where $c \in [0,1)$.

(F_1) : Obviously.

(F_2) : Let $u, v \geq 0$ be and $F(u,v,v,u,u+v,0) = u^3 - c\dfrac{v^2u^2}{1+2v+u} \leq 0$. If $u > 0$, then $u \leq cv\dfrac{v}{1+2v+u} \leq cv$. Hence, $u \leq h_1 v$, where $0 \leq h_1 = c < 1$. If $u = 0$, then $u \leq h_1 v$.

(F_3) : Let $t, t' > 0$ be such that $F(t,t,0,0,t,t') = t^3 - c\dfrac{t^2t'^2}{1+t} \leq 0$, which implies $t^2 - c\dfrac{t}{1+t}t'^2 \leq ct'^2$. Hence $t \leq h_2 t'$, where $0 \leq h_2 = \sqrt{c} < 1$. If $u = 0$ then $u \leq h_1 v$.

Example 3.8. $F(t_1, ..., t_6) = t_1^2 - at_2^2 - b\dfrac{t_5t_6}{1+t_3^2+t_4^2}$, where $a, b \geq 0$ and $0 \leq a + b < 1$.

(F_1) : Obviously.

(F_2) : Let $u, v \geq 0$ be and $F(u,v,v,u,u+v,0) = u^2 - av^2 \leq 0$. Hence, $u \leq h_1 v$, where $0 \leq h_1 = \sqrt{a} < 1$.

(F_3) : Let $t, t' > 0$ be and $F(t,t,0,0,t,t') = t^2 - at^2 - btt' \leq 0$, which implies $t \leq h_2 t'$, where $0 \leq h_2 = \dfrac{b}{1-a} < 1$.

Example 3.9. $F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - c\max\{2t_4, t_5 + t_6\}$, where $a, b, c \geq 0$ and $0 \leq a + b + 2c < 1$.

(F_1) : Obviously.

(F_2) : Let $u, v \geq 0$ be and $F(u,v,v,u,u+v,0) = u - av - c\max\{2u, u+v\} \leq 0$. If $u > v$, then $u(1-(a+b+2c)) \leq 0$, a contradiction. Hence, $u \leq v$ which implies $u \leq h_1 v$, where $0 \leq h_1 = \dfrac{a+b+c}{1-c} < 1$.

(F_3) : Let $t, t' > 0$ be and $F(t,t,0,0,t,t') = t - at - c(t+t') \leq 0$, which implies $t \leq h_2 t'$, where $0 \leq h_2 = \dfrac{c}{1-(a+c)} < 1$.

Example 3.10. $F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - c\max\{t_4 + t_5, 2t_6\}$, where $a, b, c \geq 0$ and $0 \leq a + b + 3c < 1$.

(F_1) : Obviously.

(F_2) : Let $u, v \geq 0$ be and $F(u,v,v,u,u+v,0) = u - av - bv - c(2u+v) \leq 0$, which implies $u \leq h_1 v$, where $0 \leq h_1 = \dfrac{a+b+c}{1-2c} < 1$.

(F_3) : Let $t, t' > 0$ be and $F(t,t,0,0,t,t') = t - at - c\max\{t, 2t'\}$. If $t > 2t'$ then $t(1-a-c) \leq 0$, a contradiction. Hence $t \leq 2t'$ which implies $t \leq h_2 t'$, where $0 \leq h_2 = \dfrac{2c}{1-a} < 1$.

Example 3.11. $F(t_1, ..., t_6) = t_1 - c\max\{t_2, t_3, \sqrt{t_4t_6}, \sqrt{t_5t_6}\}$, where $c \in [0,1)$.

(F_1) : Obviously.

(F_2) : Let $u, v \geq 0$ be such that $F(u,v,v,u,u+v,0) = u - cv \leq 0$, which implies $u \leq h_1 v$, where $0 \leq h_1 = c < 1$.

(F_3) : Let $t, t' > 0$ be and $F(t,t,0,0,t,t') = t - c\max\{t, \sqrt{tt'}\} \leq 0$. If $t > t'$ then $t(1-c) \leq 0$, a contradiction. Hence $t \leq t'$ which implies $t \leq h_2 t'$, where $0 \leq h_2 = c < 1$.

Example 3.12. $F(t_1, ..., t_6) = t_1 - k\max\left\{t_2, t_3, t_4, \dfrac{2t_4+t_6}{3}, \dfrac{2t_4+t_3}{3}, \dfrac{t_5+t_6}{3}\right\}$, where $k \in [0,1)$.

(F_1) : Obviously.

(F_2) : Let $u, v \geq 0$ be such that

$$F(u,v,v,u,u+v,0) = u - k\max\left\{u, v, \frac{2u}{3}, \frac{2u+v}{3}, \frac{u+v}{3}\right\} \leq 0.$$

If $u > v$, then $u(1-k) \leq 0$, a contradiction. Hence $u \leq v$ which implies $u \leq h_1 v$, where $0 \leq h_1 = k < 1$.

(F_3) : Let $t, t' > 0$ be and $F(t,t,0,0,t,t') = t - k\max\left\{t, \dfrac{t'}{3}, \dfrac{t+t'}{3}\right\}$. If $t > t'$ then $t(1-k) \leq 0$, a contradiction. Hence $t \leq t'$ which implies $t \leq h_2 t'$, where $0 \leq h_2 = k < 1$.

4. General fixed point theorem

Definition 4.1. Let f and g be self maps of a nonempty set X. If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g and w is called a point of coincidence of f and g.

Lemma 4.2 ([1]). *Let f and g be weakly compatible self mappings of nonempty set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

Lemma 4.3. *Let (X, G) be a G - metric space and $f, g : (X, G) \to (X, G)$ two functions such that*

$$\begin{aligned} F(G(fx, fy, fy), G(gx, gy, gy), G(gx, fx, fx), G(gy, fy, fy), \\ G(gx, fy, fy), G(gy, fx, fx)) \leq 0 \end{aligned} \tag{4.1}$$

for all $x, y \in X$ and F satisfying property (F_3). Then, f and g have at most a point of coincidence.

Proof. Suppose that $u = fp = gp$ and $v = fq = gq$. Then by (4.1) we have

$$\begin{aligned} F(G(fq, fp, fp), G(gq, gp, gp), G(gq, fq, fq), G(gp, fp, fp), \\ G(gq, fp, fp), G(gp, fq, fq)) \leq 0, \end{aligned}$$

$$F(G(gq, gp, gp), G(gq, gp, gp), 0, 0, G(gq, gp, gp), G(gq, gp, gp)) \leq 0$$

which implies by (F_3) that

$$G(gq, gp, gp) \leq h_2 G(gp, gq, gq).$$

Similarly, we obtain that

$$G(gp, gq, gq) \leq h_2 G(gq, gp, gp)$$

which implies that $G(gq, gp, gp)(1 - h_2^2) \leq 0$. Hence $G(gq, gp, gp) = 0$, i.e. $gq = gp$. Therefore $u = fp = gp = gq = fq = v$. □

Theorem 4.4. *Let (X, G) be a G - metric space and $f, g : (X, G) \to (X, G)$ satisfying inequality (4.1) for all $x, y \in X$, where $F \in \mathfrak{F}_G$. If $f(X) \subset g(X)$ and $g(X)$ is a G - complete metric subspace of (X, G), then f and g have a unique point of coincidence. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point of X and $x_1 \in X$ such that $fx_0 = gx_1$. This can be done since $f(X) \subset g(X)$. Continuing this process, having chosen x_n in X, we obtain x_{n+1} such that $fx_n = gx_{n+1}$. Then, by (4.1) we have successively

$$\begin{aligned} F(G(fx_{n-1}, fx_n, fx_n), G(gx_{n-1}, gx_n, gx_n), G(gx_{n-1}, fx_{n-1}, fx_{n-1}), \\ G(gx_n, fx_n, fx_n), G(gx_{n-1}, fx_n, fx_n), G(gx_n, fx_{n-1}, fx_{n-1})) \leq 0, \end{aligned}$$

$$\begin{aligned} F(G(gx_n, gx_{n+1}, gx_{n+1}), G(gx_{n-1}, gx_n, gx_n), G(gx_{n-1}, gx_n, gx_n), \\ G(gx_n, gx_{n+1}, gx_{n+1}), G(gx_{n-1}, gx_{n+1}, gx_{n+1}), 0) \leq 0. \end{aligned}$$

By (F_1) and (G_5) we obtain

$$\begin{aligned} F(G(gx_n, gx_{n+1}, gx_{n+1}), G(gx_{n-1}, gx_n, gx_n), G(gx_{n-1}, gx_n, gx_n), \\ G(gx_n, gx_{n+1}, gx_{n+1}), G(gx_{n-1}, gx_n, gx_n) + G(gx_n, gx_{n+1}, gx_{n+1}), 0) \leq 0. \end{aligned}$$

By (F_2) we obtain

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq h_1 G(gx_{n-1}, gx_n, gx_n) \tag{4.2}$$

Continuing the above process we obtain

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq h_1^n G(gx_0, gx_1, gx_1). \tag{4.3}$$

Then for $m > n$

$$\begin{aligned} G(gx_n, gx_m, gx_m) &\leq G(gx_n, gx_{n+1}, gx_{n+1}) + G(gx_{n+1}, gx_{n+2}, gx_{n+2}) + \\ &\quad +... + G(gx_{m-1}, gx_m, gx_m) \\ &\leq (h_1^n + h_1^{n+1} + ... + h_1^{m-1}) G(gx_0, gx_1, gx_1) \\ &\leq \frac{h_1^n}{1 - h_1} G(gx_0, gx_1, gx_1) \end{aligned}$$

which implies that $G(gx_n, gx_m, gx_m) \to 0$ as $n, m \to \infty$.

Hence, (gx_n) is a G - Cauchy sequence. Since $g(X)$ is G - complete, there exists a point q in $g(X)$ such that $gx_n \to q$ as $n \to \infty$. Consequently, we can find a point $p \in X$ such that $gp = q$. We prove that $fp = gp$.

By (4.1) we have successively

$$F(G(fx_{n-1}, gp, gp), G(gx_{n-1}, gp, gp), G(gx_{n-1}, fx_{n-1}, fx_{n-1}), \\ G(gp, fp, fp), G(gx_{n-1}, fp, fp), G(gp, fx_{n-1}, fx_{n-1})) \leq 0,$$

$$F(G(gx_n, fp, fp), G(gx_{n-1}, gp, gp), G(gx_{n-1}, gx_n, gx_n), \\ G(gp, fp, fp), G(gx_{n-1}, fp, fp), G(gp, gx_n, gx_n)) \leq 0.$$

Letting n tend to infinity, we obtain

$$F(G(gp, fp, fp), 0, 0, G(gp, fp, fp), G(gp, fp, fp), 0) \leq 0.$$

By (F_1) it follows that $G(gp, fp, fp) = 0$ which implies $gp = fp$. Hence $w = fp = gp$ is a point of coincidence of f and g. By Lemma 4.3, w is the unique point of coincidence. Moreover, if f and g are weakly compatible, by Lemma 4.2, w is the unique common fixed point of f and g. □

Remark 4.5. 1) By Example 3.2 with $d = e = 0$ and Theorem 4.4 we obtain a partial result from Theorem 2.3 [1].

2) By Example 3.2 for $b = c = d = e = 0$ we obtain Theorem 2.1 [13].

3) By Example 3.2 for $b = c = 2$ and Theorem 4.4 we obtain a partial result from Theorem 2.6 [1].

4) By Example 3.3, for $h \in \left[0, \frac{1}{2}\right)$ we obtain a partial result of Theorems 2.4, 2.5 [1] which is a form of Ciric result [7] in G - metric space.

5) By Examples 3.4 - 3.12 we obtain new results.

5. Well posedness problem of fixed point for two mappings in G - metric spaces

Definition 5.1. Let (X, G) be a metric space and $f : (X, d) \to (X, d)$ be a mapping. The fixed point problem f is said to be well posed [8] if

1) f has a unique fixed point $x_0 \in X$,

2) for any sequence $(x_n) \in X$ with $\lim_{n\to\infty} d(x_n, fx_n) = 0$ we have

$$\lim_{n\to\infty} d(x_n, x_0) = 0.$$

Definition 5.2. A function $F : \mathbb{R}_+^6 \to \mathbb{R}$ have property (F_p) if for $u, v, w \geq 0$ and $F(u, v, 0, w, u, v) \leq 0$, there exists $p \in (0, 1)$ such that $u \leq p \max\{v, w\}$.

Example 5.3. $F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, as in Example 3.2.

Let $u, v, w \geq 0$ be and $F(u, v, 0, w, u, v) = u - av - cw - du - ev \leq 0$ which implies $u \leq p \max\{v, w\}$, where $0 < p = \dfrac{a + c + e}{1 - d} < 1$.

Example 5.4. $F(t_1, ..., t_6) = t_1 - k \max\{t_2, ..., t_6\}$, where $k \in \left[0, \frac{1}{2}\right)$.

Let $u, v, w \geq 0$ be and $F(u, v, 0, w, u, v) = u - k \max\{v, w\} \leq 0$. If $u > \max\{v, w\}$, then $u(1-k) \leq 0$, a contradiction. Hence $u \leq \max\{v, w\}$ which implies $u \leq p \max\{v, w\}$, where $0 < p = k < 1$.

Example 5.5. $F(t_1, ..., t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\right\}$, where $k \in [0, 1)$.

Let $u, v, w \geq 0$ be and $F(u, v, 0, w, u, v) = u - k \max\left\{v, w, \frac{1}{2}(u+v)\right\}$. If $u > \max\{v, w\}$, then $u > \frac{u+v}{2}$, which implies $u(1-k) \leq 0$, a contradiction, hence $u \leq \max\{v, w\}$ which implies $u \leq p \max\{v, w\}$, where $0 < p = k < 1$.

Example 5.6. $F(t_1, ..., t_6) = t_1^2 - t_2 (at_2 + bt_3 + ct_4) - dt_5 t_6$, where $a, b, c, d \geq 0$ and $0 \leq a + b + c + d < 1$.

Let $u, v, w \geq 0$ be and $F(u, v, 0, w, u, v) = u^2 - u(av + cw) - duv \leq 0$. If $u > 0$, then $u \leq p \max\{v, w\}$, where $0 \leq p = a + c + d < 1$. If $u = 0$, then $u \leq p \max\{v, w\}$.

Example 5.7. $F(t_1, ..., t_6) = t_1 - k \max\left\{t_2, \frac{t_3 + t_4}{2}, \frac{t_5 + t_6}{2}\right\}$, where $k \in [0, 1)$.

Let $u, v, w \geq 0$ be and $F(u, v, 0, w, u, v) = u - k \max\left\{v, \frac{w}{2}, \frac{u+v}{2}\right\}$ which implies $u - k \max\left\{v, \frac{w}{2}, \frac{u+v}{2}\right\} \leq 0$. If $u > \max\{v, w\}$, then $u(1-k) \leq 0$, a contradiction. Hence $u \leq \max\{v, w\}$ which implies $u \leq p \max\{v, w\}$, where $0 < p = k < 1$.

Example 5.8. $F(t_1, ..., t_6) = t_1^3 - c \frac{t_3^2 t_4^2 + t_5^2 t_6^2}{1 + t_2 + t_3 + t_4}$, where $c \in [0, 1)$.

Let $u, v, w \geq 0$ be and $F(u, v, 0, w, u, v) = u^3 - c \frac{u^2 v^2}{1 + v + w} \leq 0$. If $u > 0$, then $u \leq cv \frac{v}{1 + v + w} \leq cv \leq p \max\{v, w\}$, where $0 < p = c < 1$. If $u = 0$, then $u \leq p \max\{v, w\}$.

Example 5.9. $F(t_1, ..., t_6) = t_1^2 - at_2^2 - c \frac{t_5 t_6}{1 + t_3^2 + t_4^2}$, where $a > 0$ and $a + c < 1$.

Let $u, v, w \geq 0$ be and $F(u, v, 0, w, u, v) = u^2 - c \frac{uv}{1 + v^2} \leq 0$ which implies $u^2 - av^2 - cuv \leq 0$. Let $v > 0$, then $f(t) = t^2 - ct - a$, where $t = \frac{u}{v}$. Then $f(0) < 0$ and $f(1) > 0$ and hence there exists $p \in (0, 1)$ such that $f(t) \leq 0$ for $t \leq p$. Hence $u \leq pv \leq p \max\{v, w\}$. If $v = 0$, then $u = 0$ and $u \leq p \max\{v, w\}$.

Example 5.10. $F(t_1, ..., t_6) = t_1 - at_2 - c \max\{2t_4, t_5 + t_6\}$, where $0 \leq a + 2c < 1$.
Let $u, v, w \geq 0$ be and $F(u, v, 0, w, u, v) = u - av - c \max\{2w, u+v\}$. If $u > \max\{v, w\}$ then $u(1-a-2c) \leq 0$, a contradiction. Hence $u \leq \max\{v, w\}$ which implies $u \leq p \max\{v, w\}$, where $0 < p = a + 2c < 1$.

Example 5.11. $F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - c \max\{t_4 + t_5, 2t_6\} \leq 0$, where $0 < p = a + 3c < 1$. The proof is similar to the proof of Example 5.8.

Example 5.12. $F(t_1, ..., t_6) = t_1 - c \max\left\{t_2, t_3, \sqrt{t_4 t_6}, \sqrt{t_5 t_6}\right\}$, where $c \in [0, 1)$.

Let $u, v, w \geq 0$ be and $F(u, v, 0, w, u, v) = u - c \max\{v, \sqrt{vw}, \sqrt{uv}\} \leq 0$. If $u > \max\{v, w\}$ then $u(1-c) \leq 0$, a contradiction. Hence $u \leq \max\{v, w\}$ which implies $u \leq p \max\{v, w\}$, where $0 < p = c < 1$.

Example 5.13. $F(t_1, ..., t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \frac{2t_4 + t_6}{3}, \frac{2t_4 + t_5}{3}, \frac{t_5 + t_6}{3}\right\}$, where $k \in [0, 1)$.

Let $u, v, w \geq 0$ be and $F(u, v, 0, w, u, v) = u - k \max\left\{v, w, \frac{2w + v}{3}, \frac{2w}{3}, \frac{u + v}{3}\right\} \leq 0$. If $u > \max\{v, w\}$ then $u(1-k) \leq 0$, a contradiction. Hence $u \leq \max\{v, w\}$ which implies $u \leq p \max\{v, w\}$, where $0 < p = k < 1$.

Definition 5.14. Let (X, G) be a G - metric space and $f, g : (X, G) \to (X, G)$. The common fixed problem of f and g is said to be well posed if:

1) f and g have a unique common fixed point,
2) for any sequence (x_n) in X with

$$\lim_{n\to\infty} G(x_n, fx_n, fx_n) = 0$$

and

$$\lim_{n\to\infty} G(x_n, gx_n, gx_n) = 0,$$

then

$$\lim_{n\to\infty} G(x, x_n, x_n) = 0.$$

Theorem 5.15. *Let (X, G) be a symmetric G - metric space. For mappings $f, g : (X, G) \to (X, G)$ satisfying Theorem 4.4 and F having property (F_p), the fixed point problem of f and g is well posed.*

Proof. By Theorem 4.4 f and g have a unique common fixed point x. Let (x_n) be a sequence in (X, G) such that $\lim_{n\to\infty} G(x_n, fx_n, fx_n) = 0$ and $\lim_{n\to\infty} G(x_n, gx_n, gx_n) = 0$. By (4.1) we have successively

$$F(G(fx, fx_n, fx_n), G(gx, gx_n, gx_n), G(gx, fx, fx),$$
$$G(gx_n, fx_n, fx_n), G(gx, fx_n, fx_n), G(gx_n, fx, fx)) \le 0,$$

$$F(G(x, fx_n, fx_n), G(x, gx_n, gx_n), 0, G(gx_n, fx_n, fx_n),$$
$$G(x, fx_n, fx_n), G(gx_n, x, x)) \le 0.$$

Since G is a symmetric G - metric, $G(gx_n, x, x) = G(x, gx_n, gx_n)$ and

$$F(G(x, fx_n, fx_n), G(x, gx_n, gx_n), 0, G(gx_n, fx_n, fx_n),$$
$$G(x, fx_n, fx_n), G(x, gx_n, gx_n)) \le 0.$$

By (F_p) we have

$$\begin{aligned} G(x, fx_n, fx_n) &\le p \max\{G(x, gx_n, gx_n), G(gx_n, fx_n, fx_n)\} \\ &\le p(G(x, gx_n, gx_n) + G(gx_n, fx_n, fx_n)). \end{aligned}$$

Then by (G_5) and the fact that (X, G) is a symmetric G - metric space we have

$$\begin{aligned} G(x, x_n, x_n) &\le G(x, fx_n, fx_n) + G(fx_n, x_n, x_n) \\ &\le p(G(x, gx_n, gx_n) + G(gx_n, fx_n, fx_n)) + G(fx_n, x_n, x_n) \\ &\le p(G(x, x_n, x_n) + G(x_n, gx_n, gx_n) + G(gx_n, x_n, x_n) + \\ &\quad +G(x_n, fx_n, fx_n)) + G(fx_n, x_n, x_n) \\ &= p(G(x, x_n, x_n) + 2G(x_n, gx_n, gx_n) + \\ &\quad +G(x_n, fx_n, fx_n)) + G(fx_n, x_n, x_n). \end{aligned}$$

Hence $G(x, x_n, x_n) \le \dfrac{p+1}{1-p} G(x_n, fx_n, fx_n) + \dfrac{2p}{1-p} G(x_n, gx_n, gx_n)$. Letting n tend to infinity we obtain $\lim_{n\to\infty} G(x, x_n, x_n) = 0$. Hence the common fixed point problem of f and g is well posed. □

Remark 5.16. By Theorem 4.4 and Examples 5.3 - 5.13 we obtain new results.

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