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Existence of mild solutions of random impulsive functional differential equations with almost sectorial operators

A. Anguraj*, M.C. Ranjini

Department of Mathematics, P.S.G. College of Arts and Science, Coimbatore-641 014, Tamil Nadu, India.

This paper is dedicated to Professor Ljubomir Ćirić

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Abstract

By using the theory of semigroups of growth α, we prove the existence and uniqueness of the mild solution for the random impulsive functional differential equations involving almost sectorial operators. An example is given to illustrate the theory.

Keywords: Impusive differential equations, random impulses, almost sectorial operator, semigroup of growth α, mild solution
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1. Introduction

Sectorial operators, that is, linear operators A defined in Banach spaces, whose spectrum lies in a sector

$$S_w = \{\lambda \in C/\{0\} \mid |arg\lambda| \leq w\} \cup \{0\} \quad for\ some \quad 0 \leq w \leq \pi$$

and whose resolvent satisfies an estimate

$$||(\lambda - A)^{-1}|| \quad \leq \quad M|\lambda|^{-1}, \quad \forall \quad \lambda \in \mathbb{C}\backslash S_w, \tag{1.1}$$

*Corresponding author
Email addresses: angurajpsg@yahoo.com (A. Anguraj), ranjiniprasad@gmail.com (M.C. Ranjini)

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have been studied extensively during the last 40 years, both in abstract settings and for their applications to partial differential equations. Many important elliptic differential operators belong to the class of sectorial operators, especially when they are considered in the Lebesgue spaces or in spaces of continuous functions (see [1] and [[2], chapter 3]). However, if we look at spaces of more regular functions such as the spaces of Holder continuous functions, we find that these elliptic operators do no longer satisfy the estimate 1.1 and therefore are not sectorial as was pointed out by Von Wahl (see [[3], Ex.3.1.33], see [4]).

Neverthless, for these operators estimates such as

$$||(\lambda - A)^{-1}|| \leq \frac{M}{|\lambda|^{1-\alpha}} \qquad , \lambda \in \sum_{w,v} = \left\{ \lambda \in \mathbb{C} : |arg(\lambda - w)| < v \right\} \tag{1.2}$$

where $\alpha \in (0, 1)$, $w \in \mathbb{R}$ and $v \in (\frac{\pi}{2}, \pi)$, can be obtained, (see[4]) which allows to define an associated "analytic semigroup" by means of the Dunford Integral

$$T(t) \;=\; \frac{1}{2\pi i} \int_{\Gamma_\theta} e^{\lambda t} (\lambda - A)^{-1} d\lambda, \qquad t > 0 \tag{1.3}$$

where $\Gamma_\theta = \left\{ te^{i\theta} : t \in R \backslash \{0\} \right\}$, $\theta \in (v, \frac{\pi}{2})$.

In the literature, a linear operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ which satisfy the condition 1.2 is called almost sectorial and the operator family $\left\{ T(t), T(0) = I, t \geq 0 \right\}$ is said the "semigroup of growth α" generated by A. The operator family $T(t)_{t \geq 0}$ has properties similar at those of analytic semigroup which allow to study some classes of partial differential equations via the usual methods of semigroup theory. Concerning almost sectorial operators, semigroups of growth α and applications to partial differential equations, we refer the reader to [4, 5, 6, 7, 8] and the references there in.

Also, many evolution processes from fields as diverse as physics, population dynamics, aeronautics, economics, telecommunications and engineering etc., are characterized by the fact that they undergo an abrupt change of state at certain moments of time between intervals of continuous evolution. The duration of these changes are often negligible compared to the total duration of the process acting instantaneously in the form of impulses. The impulses may be deterministic or random. There are lot of papers which investigate the qualitative properties of deterministic impulses see for example [9, 10, 11, 12, 13, 14] and the references there in.

When the impulses exist at random points, solutions of the differential systems are stochastic process. Random impulsive systems are more realistic than deterministic impulsive systems. The study of random impulsive differential equations is a new area of research. So far there are few results have been discussed in random impulsive systems. In [15, 16], the authors proved the existence and uniqueness of differential system with random impulses. In [17], Wu and Duan discussed the oscillation, stability and boundedness of second-order differential systems with random impulses, and in [18, 19], the authors studied the existence and stability results of random impulsive semilinear differential systems.

To the best of our knowledge, the study of the existence of solutions of abstract system as 2.1 for which the operator A is almost sectorial is an untreated topic in the literature. In [20], Hernandez proved the existence of mild solutions for a class of abstract functional differential equations with almost sectorial operators and in [19], A. Anguraj et al. proved the existence and exponential stability of semilinear functional differential equations with random impulses under non-uniqueness. By the motivation of the above papers, we present a new idea of research to prove the existence and uniqueness of mild solutions of functional differential equations with random impulses involving almost sectorial operators.

2. Preliminaries

Here, we introduce some notations and technicalities. Let $(Z, ||.||_z)$ be a Banach space. In this paper, $\mathcal{L}(Z, W)$ represents the space of bounded linear operators from Z into W endowed with the norm of operators denoted $||.||_{\mathcal{L}(Z,W)}$, and we write $\mathcal{L}(Z)$ and $||.||_{\mathcal{L}(Z)}$ when $Z = W$. In addition, $B_l(z, Z)$ denotes the closed ball with center at $z \in Z$ and radius $l > 0$ in Z. As usual, $C([c, d], Z)$ represents the space formed by all the continuous functions from $[c, d]$ into Z endowed with the sup-norm denoted by $||.||_{C([c,d],Z)}$ and $L^p([c, d], \mathbb{X})$, $p \geq 1$, denotes the space formed by all the classes of Lebesgue-integrable functions from $[c, d]$ into $\mathbb{X}$ endowed with the norm

$$||h||_{L^p([c,d],\mathbb{X})} \;=\; \Big(\int_{[c,d]} ||h(s)||^p ds\Big)^{\frac{1}{p}}.$$

Throughout this paper, $(\mathbb{X}, ||.||)$ is a Banach space, $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is an almost sectorial operator and $(T(t))_{t\geq 0}$ is the semigroup of growth α generated by A. For simplicity, next we assume $w = 0$. The next lemma consider some properties of the operator family $(T(t))_{t\geq 0}$.

Lemma 2.1. *([5, 8]). Under the above conditions, the followings properties are satisfied.*

(a) *The operator A is closed, $T(t + s) = T(t)T(s)$ and $AT(t)x = T(t)Ax$ for all $t, s \in [0, \infty)$ and each $x \in D(A)$.*

(b) *$T(.) \in C((0, \infty), \mathbb{X}) \cap C^1((0, \infty), \mathbb{X})$ and $\frac{d}{dt}T(t) = AT(t)$ for all $t > 0$.*

(c) *For $n \in \mathbb{N} \cup \{0\}$, $A^nT(.) \in C((0, \infty), \mathbb{X})$ and there exists $D_n > 0$ and a constant $\gamma > 0$, which is independent of n, such that $||A^nT(t)||_{\mathcal{L}(\mathbb{X})} \leq D_n e^{\gamma t} t^{-(n+\alpha)}$ for all $t > 0$.*

Let $\mathbb{X}$ be the Banach space and Ω a non - empty set. Assume that τ_k is a random variable defined from Ω to $D_k \overset{def.}{=} (0, d_k)$ for all $k = 1, 2, ...$ where $0 < d_k < \infty$. Furthermore, assume that τ_i and τ_j are independent of each other as $i \neq j$ for $i, j = 1, 2, ...$ For the sake of simplicity, we denote $R^+ = [0, \infty)$.

We consider the functional differential equations with random impulses of the form,

$$\begin{cases} x'(t) = Ax(t) + f(t, x_t), \quad t \geq 0, \quad t \neq \xi_k \\ x(\xi_k) = b_k(\tau_k)x(\xi_k^-), \qquad\qquad k = 1, 2, ... \\ x_0 = \phi \end{cases} \tag{2.1}$$

where $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is an almost sectorial operator, the function $f : R^+ \times \widehat{C} \to \mathbb{X}$, $\widehat{C} = C([-r, 0], \mathbb{X})$ is the set of piecewise continuous functions mapping $[-r, 0]$ into $\mathbb{X}$ with some given $r > 0$; x_t is a function where t is fixed, defined by $x_t(s) = x(t + s)$ for all $s \in [-r, 0]$; $\xi_0 = t_0$ and $\xi_k = \xi_{k-1} + \tau_k$ for $k = 1, 2, ...$ Here $t_0 = 0 = \xi_0 < \xi_1 < \xi_2 < ... < \xi_k < ...,$ $b_k : D_k \to \mathbb{X}$ for each $k = 1, 2, ...,$ $x(\xi_k^-) = \lim_{t\to\xi_k} x(t)$ with the norm $||x||_t = \sup_{t-r<s<t} |x(s)|$ for each t satisfying $0 \leq t \leq T$ and $T \in R^+$ is a given number, $||.||$ is any given norm in $\mathbb{X}$; ϕ is a function defined from $[-r, 0]$ to $\mathbb{X}$.

Denote $\{B_t, t \geq 0\}$ the simple counting process generated by ξ_n, that is, $\{B_t \geq n\} = \{\xi_n \leq t\}$, and denote $\mathcal{F}_t$ the σ - algebra generated by $\{B_t, t \geq 0\}$. Then $(\Omega, P, \{\mathcal{F}_t\})$ is a probability space.

Definition 2.2. For a given $T \in (0, \infty)$, a stochastic process $\{x(t), -r \leq t \leq T\}$ is called a mild solution to the equation 2.1 in $(\Omega, P, \{\mathcal{F}_t\})$, if
(i) x(t) is $\mathcal{F}_t$-adapted.
(ii)$x(s) = \phi(s)$ when $s \in [-r, 0]$, and

$$\begin{aligned} x(t) \;=\; & \sum_{k=0}^{\infty} \Big[\prod_{i=1}^{k} b_i(\tau_i)T(t)\phi(0) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} T(t-s)\ f(s, x_s)ds \\ & + \int_{\xi_k}^{t} T(t-s)\ f(s, x_s)ds\Big] I_{[\xi_k,\xi_{k+1})}(t), \qquad t \in [0, T] \end{aligned}$$

where $\prod_{j=m}^{n}(.) = 1$ as $m > n$, $\prod_{j=i}^{k} b_j(\tau_j) = b_k(\tau_k)b_{k-1}(\tau_{k-1})...b_i(\tau_i)$, and $I_A(.)$ is the index function, i.e.,

$$I_A(t) = \begin{cases} 1, & if \ \ t \in A \\ 0, & if \ \ t \notin A \end{cases}$$

3. Existence Results

In this section, we discuss the existence and uniqueness of the solution of the system 2.1.

Remark 3.1. In the remainder of this paper, $\phi : [-r, 0] \to \mathbb{X}$ is a given function and $y : [-r, T] \to \mathbb{X}$ is the function defined by $y(\theta) = \phi(\theta)$ for $\theta \leq 0$ and $y(t) = \sum_{k=0}^{\infty} \left[\prod_{i=1}^{k} b_i(\tau_i) T(t)\phi(0) \right] I_{[\xi_k, \xi_{k+1})}(t)$ for $t > 0$.

In addition, $C_n, n \in N$, are positive constants such that

$$||A^n T(t)||_{\mathcal{L}(\mathbb{X})} \leq C_n t^{-(n+\alpha)}, \forall\ t \in (0, T],$$

and for a bounded set $B \subset \mathbb{X}$, we use the notation $Diam_{\mathbb{X}}(B)$ for

$$Diam_{\mathbb{X}}(B) = \sup_{a,b \in B} ||a - b||.$$

To prove our results, we introduce the following hypotheses. In the next assumptions, $q \in \left(\frac{1}{1-\alpha}, \infty\right)$ or $q = \infty$ and $q' = \frac{p}{p-1}$ for $q < \infty$ and $q' = 1$ if $q = \infty$.

(H_1) The function $f(.\ , \psi)$ is strongly measurable on $[0, T]$ for all $\psi \in \widehat{C}$ and $f(t\ , .) \in C(\widehat{C}, \mathbb{X})$ for each $t \in [0, T]$. There exists a non-decreasing function $W_f \in C(R^+, (0, \infty))$ and $m_f \in L^q([0, T], R^+)$ such that

$$E||f(t, \psi_t)||^2 \leq m_f(t) W_f(E||\psi||_t^2), \qquad \forall\ \ (t, \psi) \in [0, T] \times \widehat{C}$$

(H_2) The function f is continuous and for all $l > 0$ with $[0, l] \times B_l(\phi, \widehat{C}) \subset [0, T] \times \widehat{C}$, there exists $L_{f,l} \in L^q([0, T], R^+)$ such that

$$E||f(s, \psi_1) - f(s, \psi_2)||^2 \leq L_{f,l}(s)\ E||\psi_1 - \psi_2||^2, \qquad \forall\ \ (s, \psi_i) \in [0, l] \times B_l(\phi, \widehat{C})$$

(H_3) The condition $\max_{i,k} \left\{ \prod_{j=i}^{k} ||b_j(\tau_j)|| \right\}$ is uniformly bounded if there is a constant $B > 0$ such that

$$\max_{i,k} \left\{ \prod_{j=i}^{k} ||b_j(\tau_j)|| \right\} \quad \leq \quad B, \qquad \forall \tau_j \in D_j, j = 1, 2, ...$$

Theorem 3.2. *If the hypotheses (H_1), (H_3) are satisfied, $T(.)\phi(0) \in C([0, T], \mathbb{X})$ and $T(t)$ is compact for all $t > 0$, then there exists a mild solution of 2.1 on $[-r, T]$.*

Proof 1. Let T be an arbitrary number $0 < T < \infty$ where $T < b_1 < \infty$ and for $C > 0$ let $W_f(||\psi||^2) \leq C$, for all $\psi \in B_{b_1}(\phi, \widehat{C})$.

Also, let

$$\sup_{s \in [0,T]} ||y_s - \phi||^2 \leq \frac{b_1^2}{4}$$

and

$$\frac{CC_0^2\ ||m_f||_{L^q([0,T])}\ \max\{1, B^2\}\ T^{\frac{1}{q'} - 2\alpha + 1}}{(1 - 2q'\alpha)^{\frac{1}{q'}}} \leq \frac{b_1^2}{4}$$

For the simplification, on the space

$$B_{\frac{b_1}{2}}(0, S(T)) = \left\{ u \in C([-r, T], \mathbb{X}) : u_0 = 0,\ ||u||^2_{C([0,T],\mathbb{X})} \leq \frac{b_1^2}{4} \right\}$$

we define the map

$$\Gamma : B_{\frac{b_1}{2}}(0, S(T)) \to C([-r,T],\mathbb{X})$$

by $(\Gamma u)_0 = 0$ and

$$\Gamma u(t) = \begin{cases} \phi(t) & t \in [-r,0] \\ \sum_{k=0}^{\infty} \Big[\sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} T(t-s)\ f(s, u_s + y_s) ds \\ + \int_{\xi_k}^{t} T(t-s)\ f(s, u_s + y_s) ds \Big] I_{[\xi_k,\xi_{k+1})}(t), & t \in [0,T] \end{cases}$$

with the norm defined as

$$||\chi||_{\Gamma}^2 = \sup_{0 \le t \le T} E||\chi||_t^2$$

Now, we prove that Γ is completely continuous from $B_{\frac{b_1}{2}}(0, S(T))$ into $B_{\frac{b_1}{2}}(0, S(T))$.

For that, $(s,u) \in [0,T] \times B_{\frac{b_1}{2}}(0, S(T))$,

$$\begin{aligned} ||u_s + y_s - \phi||^2 &\le 2\Big[\sup_{\theta \in [0,s]} ||u(\theta)||^2 + ||y_s - \phi||^2 \Big] \\ &\le 2\Big[\frac{b_1^2}{4} + \frac{b_1^2}{4} \Big] \\ &\le b_1^2 \end{aligned}$$

which implies that $u_s + y_s \in B_{b_1}(\phi, \widehat{C})$ and $W_f(||u_s + y_s||^2) \le C$.

Now, from the properties of $(T(t))_{t \ge 0}$ and f, the Bochner's criterion for integrable functions and the inequality,

$$\begin{aligned} ||T(t-s) f(s, u_s + y_s)||^2 &\le \frac{C_0^2\ m_f(s)\ W_f(||u_s + y_s||^2)}{(t-s)^{2\alpha}} \\ &\le \frac{C_0^2 C m_f(s)}{(t-s)^{2\alpha}} \end{aligned}$$

Therefore, the function $s \to T(t-s) f(s, u_s + y_s)$ is integrable on $[0,t]$ for all $t \in [0,T]$, which implies that $\Gamma u \in C([-r,T], X)$ and Γ is well defined.

Next, we show that $\Gamma B_{\frac{b_1}{2}}(0, S(T)) \subset B_{\frac{b_1}{2}}(0, S(T))$.

Consider,

$$\begin{aligned} ||\Gamma u(t)||^2 &= ||\sum_{k=0}^{\infty} \Big[\sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} T(t-s)\ f(s, u_s + y_s) ds \\ &\quad + \int_{\xi_k}^{t} T(t-s)\ f(s, u_s + y_s) ds \Big] I_{[\xi_k,\xi_{k+1})}(t)||^2 \\ &\le \Big[\max_{i,k} \Big\{ 1, \prod_{j=i}^{k} ||b_j(\tau_j)|| \Big\} \Big]^2 \\ &\quad \Big(\int_0^t ||T(t-s)||\ ||f(s, u_s + y_s)|| ds\ I_{[\xi_k,\xi_{k+1})}(t) \Big)^2 \\ E||\Gamma u(t)||^2 &\le \max \Big\{ 1, B^2 \Big\}\ t\ \Big(\int_0^t ||T(t-s)||^2 E||f(s, u_s + y_s)||^2 ds \Big) \end{aligned}$$

Taking supremum over t, we get

$$\begin{aligned}\sup_{t\in[0,T]} E||\Gamma u(t)||^2 &\leq \max\Big\{1, B^2\Big\}\ T\ C_0^2 \int_0^t \frac{m_f(s)W_f(\sup_{t\in[0,T]} E||u_s+y_s||^2)}{(t-s)^{2\alpha}}ds \\ &\leq CC_0^2 T \max\Big\{1, B^2\Big\} \int_0^t \frac{m_f(s)}{(t-s)^{2\alpha}}ds \\ &\leq CC_0^2 \max\Big\{1, B^2\Big\} ||m_f||_{L^q([0,T])} \frac{T^{\frac{1}{q'}-2\alpha+1}}{(1-2q'\alpha)^{\frac{1}{q'}}}\end{aligned}$$

Thus,

$$||\Gamma u(t)||^2 \leq \frac{b_1^2}{4}$$

which implies that $\Gamma u \in B_{\frac{b_1}{2}}(0, S(T))$ and therefore $\Gamma B_{\frac{b_1}{2}}(0, S(T)) \subset B_{\frac{b_1}{2}}(0, S(T))$. Moreover, a standard application of the Lebesgue dominated convergence theorem proves that Γ is continuous.

Now, we prove the compactness of the operator Γ.

Step 1: The set $\Gamma B_{\frac{b_1}{2}}(0, S(T)) = \Big\{\Gamma u(t) : u \in B_{\frac{b_1}{2}}(0, S(T))\Big\}$ is relatively compact for all $t \in [-r, T]$.

The case $t \leq 0$ is trivial. Let $0 < t \leq T$ be fixed and let ϵ be a real number with $0 < \epsilon < t$. For $u \in B_{\frac{b_1}{2}}(0, S(T))$, we define

$$\begin{aligned}\Gamma_\epsilon u(t) &= \sum_{k=0}^{\infty} \Big[\sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} T(t-s)\ f(s, u_s+y_s)ds \\ &\quad + \int_{\xi_k}^{t-\epsilon} T(t-s)\ f(s, u_s+y_s)ds\Big] I_{[\xi_k,\xi_{k+1})}(t)\end{aligned}$$

Since, $T(t)$ is compact, the set $\Gamma_\epsilon u(t) = \Big\{u(t) : u \in B_{\frac{b_1}{2}}(0, S(T))\Big\}$ is relatively compact in $\mathbb{X}$ for every $\epsilon \in (0, t)$.

Moreover, for every $u \in B_{\frac{b_1}{2}}(0, S(T))$, we have

$$\begin{aligned}\Gamma u(t) - \Gamma_\epsilon u(t) &= \sum_{k=0}^{\infty} \Big[\sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} T(t-s)\ f(s, u_s+y_s)ds \\ &\quad + \int_{\xi_k}^{t} T(t-s)\ f(s, u_s+y_s)ds\Big] I_{[\xi_k,\xi_{k+1})}(t) \\ &\quad - \sum_{k=0}^{\infty} \Big[\sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} T(t-s)\ f(s, u_s+y_s)ds \\ &\quad + \int_{\xi_k}^{t-\epsilon} T(t-s)\ f(s, u_s+y_s)ds\Big] I_{[\xi_k,\xi_{k+1})}(t) \\ ||\Gamma u(t) - \Gamma_\epsilon u(t)||^2 &\leq \max\Big\{1, B^2\Big\}\ t \int_{t-\epsilon}^{t} ||T(t-s)||^2 E||f(s, u_s+y_s)||^2 ds \\ E||\Gamma u(t) - \Gamma_\epsilon u(t)||^2 &\leq \max\Big\{1, B^2\Big\}\ T\ C_0^2 \int_{t-\epsilon}^{t} \frac{m_f(s)W_f(E||u_s+y_s||^2)}{(t-s)^{2\alpha}}ds \\ \sup_{t\in[0,T]} E||\Gamma u(t) - \Gamma_\epsilon u(t)||^2 &\leq CC_0^2 T \max\Big\{1, B^2\Big\} \int_{t-\epsilon}^{t} \frac{m_f(s)}{(t-s)^{2\alpha}}ds\end{aligned}$$

Thus,

$$||\Gamma u(t) - \Gamma_\epsilon u(t)||^2 \leq CC_0^2 T \max\Big\{1, B^2\Big\} \int_{t-\epsilon}^{t} \frac{m_f(s)}{(t-s)^{2\alpha}} ds$$

Therefore, letting $\epsilon \to 0$, we see that there are relatively compact sets arbitrary close to the set $\Big\{\Gamma u(t) : u \in B_{\frac{b_1}{2}}(0, S(T))\Big\}$. Hence, the set $\Big\{\Gamma u(t) : u \in B_{\frac{b_1}{2}}(0, S(T))\Big\}$ is relatively compact in $\mathbb{X}$.

Step 2: Γ is equicontinuous.

For any $0 \leq t_1 < t_2 \leq T$ and for $u \in B_{\frac{b_1}{2}}(0, S(T))$, we have

$$\begin{aligned}
\Gamma u(t_2) - \Gamma u(t_1) &= \sum_{k=0}^{\infty} \Big[\sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} T(t_2 - s)\ f(s, u_s + y_s) ds \\
&\quad + \int_{\xi_k}^{t_2} T(t_2 - s)\ f(s, u_s + y_s) ds\Big] I_{[\xi_k, \xi_{k+1})}(t_2) \\
&\quad - \sum_{k=0}^{\infty} \Big[\sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} T(t_1 - s)\ f(s, u_s + y_s) ds \\
&\quad + \int_{\xi_k}^{t_1} T(t_1 - s)\ f(s, u_s + y_s) ds\Big] I_{[\xi_k, \xi_{k+1})}(t_1) \\
&= \sum_{k=0}^{\infty} \Big[\sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} T(t_2 - s)\ f(s, u_s + y_s) ds \\
&\quad + \int_{\xi_k}^{t_2} T(t_2 - s)\ f(s, u_s + y_s) ds\Big] \Big(I_{[\xi_k, \xi_{k+1})}(t_2) - I_{[\xi_k, \xi_{k+1})}(t_1)\Big) \\
&\quad + \sum_{k=0}^{\infty} \Big[\sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} [T(t_2 - s) - T(t_1 - s)]\ f(s, u_s + y_s) ds \\
&\quad + \int_{\xi_k}^{t_1} [T(t_2 - s) - T(t_1 - s)]\ f(s, u_s + y_s) ds + \int_{t_1}^{t_2} T(t_2 - s)\ f(s, u_s + y_s) ds\Big] I_{[\xi_k, \xi_{k+1})}(t_1)
\end{aligned}$$

Then

$$E\|\Gamma u(t_2) - \Gamma u(t_1)\|^2 \leq 2E\|I_1\|^2 + 2E\|I_2\|^2 \tag{3.1}$$

where,

$$\begin{aligned}
I_1 &= \sum_{k=0}^{\infty} \Big[\sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} T(t_2 - s)\ f(s, u_s + y_s) ds \\
&\quad + \int_{\xi_k}^{t_2} T(t_2 - s)\ f(s, u_s + y_s) ds\Big] \Big(I_{[\xi_k, \xi_{k+1})}(t_2) - I_{[\xi_k, \xi_{k+1})}(t_1)\Big)
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \sum_{k=0}^{\infty} \Big[\sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} [T(t_2 - s) - T(t_1 - s)]\ f(s, u_s + y_s) ds \\
&\quad + \int_{\xi_k}^{t_1} [T(t_2 - s) - T(t_1 - s)]\ f(s, u_s + y_s) ds \\
&\quad + \int_{t_1}^{t_2} T(t_2 - s)\ f(s, u_s + y_s) ds\Big] I_{[\xi_k, \xi_{k+1})}(t_1)
\end{aligned}$$

Consider,

$$
\begin{aligned}
E||I_1||^2 \;\le\;& \Big[\max\Big\{1, ||\prod_{j=i}^{k}||b_j(\tau_j)||\Big\}\Big]^2 \\
& E\Big[\int_0^{t_2} ||T(t_2-s)||\;||f(s,u_s+y_s)||ds I_{[\xi_k,\xi_{k+1})}(t_2) - I_{[\xi_k,\xi_{k+1})}(t_1)\Big]^2 \\
\le\;& \max\Big\{1,B^2\Big\}\;T\;\Big[\int_0^{t_2}||T(t_2-s)||^2\;E||f(s,u_s+y_s)||^2ds\Big] \\
& E\Big(I_{[\xi_k,\xi_{k+1})}(t_2) - I_{[\xi_k,\xi_{k+1})}(t_1)\Big) \\
\le\;& \max\Big\{1,B^2\Big\}\;T\;C_0^2\;\Big[\int_0^{t_2}\frac{m_f(s)W_f(E||u_s+y_s||^2)}{(t-s)^{2\alpha}}ds\Big] \\
& E\Big(I_{[\xi_k,\xi_{k+1})}(t_2) - I_{[\xi_k,\xi_{k+1})}(t_1)\Big) \\
\le\;& CC_0^2T\max\Big\{1,B^2\Big\}\Big[\int_0^{t_2}\frac{m_f(s)}{(t-s)^{2\alpha}}ds\Big]E\Big(I_{[\xi_k,\xi_{k+1})}(t_2) - I_{[\xi_k,\xi_{k+1})}(t_1)\Big) \\
\to\;& 0 \text{ as } t_2\to t_1 \qquad (3.2)
\end{aligned}
$$

$$
\begin{aligned}
E||I_2||^2 \;\le\;& \Big[\sum_{k=0}^{\infty}\Big[\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}[T(t_2-s)-T(t_1-s)]\;f(s,u_s+y_s)ds \\
& + \int_{\xi_k}^{t_1}[T(t_2-s)-T(t_1-s)]\;f(s,u_s+y_s)ds \\
& + \int_{t_1}^{t_2}T(t_2-s)\;f(s,u_s+y_s)ds\Big]I_{[\xi_k,\xi_{k+1})}(t_1)\Big]^2 \\
\le\;& 2\max\Big\{1,B^2\Big\}\;t_1\;\int_0^{t_1}||T(t_2-s)-T(t_1-s)||^2\;E||f(s,u_s+y_s)||^2ds \\
& +2(t_2-t_1)\int_{t_1}^{t_2}||T(t_2-s)||^2\;E||f(s,u_s+y_s)||^2ds
\end{aligned}
$$

$$
\begin{aligned}
\le\;& 2\max\Big\{1,B^2\Big\}\;T\;\int_0^{t_1}||T(t_2-s)-T(t_1-s)||^2\;m_f(s)\;W_f(E||f(s,u_s+y_s)||^2)ds \\
& +2(t_2-t_1)\int_{t_1}^{t_2}||T(t_2-s)||^2\;m_f(s)\;W_f(E||f(s,u_s+y_s)||^2)ds \\
\le\;& 2\max\Big\{1,B^2\Big\}T\;C\;\int_0^{t_1}||T(t_2-s)-T(t_1-s)||^2\;m_f(s)ds \\
& +2(t_2-t_1)\;C\;\int_{t_1}^{t_2}||T(t_2-s)||^2\;m_f(s)ds \\
& \to 0 \text{ as } t_2\to t_1 \qquad (3.3)
\end{aligned}
$$

From the equations (3.2) and (3.3), it follows that the right hand side of 3.1 tends to zero as $t_2 \to t_1$. Thus Γ maps $B_{\frac{b_1}{2}}(0, S(T))$ into an equicontinuous family of functions.

Finally, from the Schauder's fixed point theorem, Γ has a fixed point $x = u + y$ which is a mild solution of the problem 2.1.

Theorem 3.3. *Let the hypothesis (H_2) and (H_3) hold. If the following inequality*

$$C_0^2\ ||L_{f,b_1}||_{L^q([0,T])}\ \max\{1,B^2\}\ \frac{T^{\frac{1}{q'}-2\alpha+1}}{(1-2q'\alpha)^{\frac{1}{q'}}}<1$$

is satisfied, then the system 2.1 has a unique mild solution on $[-r,T]$.

Proof 2. Let $T<b_1<\infty$ and $C>0$ such that $||f(t,\psi)||\leq C,\ \ \forall(t,\psi)\in[0,b_1]\times B_{b_1}(\phi,\widehat{C})$. Also, let

$$\sup_{s\in[0,T]}||y_s-\phi||^2\leq\frac{b_1^2}{4}$$

and

$$2C_0^2\max\{1,B^2\}\Big[b_1^2\ ||L_{f,b_1}||_{L^q([0,T])}\ \frac{T^{\frac{1}{q'}-2\alpha+1}}{(1-2q'\alpha)^{\frac{1}{q'}}}+||f(s,\phi)||^2\ \frac{T^{2(1-\alpha)}}{1-2\alpha}\Big]\leq\frac{b_1^2}{4}$$

Let $\Gamma:B_{\frac{b_1}{2}}(0,S(T))\to C([-r,T],\mathbb{X})$ be the operator introduced as in the Theorem 3.2. Proceeding as in the proof of the previous theorem, we can prove that Γ is well-defined.

Next, we have to prove that Γ is a contraction mapping.

Before that, for $(s,u)\in[0,T]\times B_{\frac{b_1}{2}}(0,S(T))$,

$$\begin{aligned}||u_s+y_s-\phi||^2 &\leq 2\Big[\sup_{\theta\in[0,s]}||u(\theta)||^2+||y_s-\phi||^2\Big]\\ &\leq 2\Big[\frac{b_1^2}{4}+\frac{b_1^2}{4}\Big]\\ &\leq b_1^2\end{aligned}$$

which implies that $u_s+y_s\in B_{b_1}(\phi,\widehat{C})$ and therefore $||f(s,u_s+y_s)||\leq C$.

Using this fact, for $u\in B_{\frac{b_1}{2}}(0,S(T))$,

$$\begin{aligned}||\Gamma u(t)||^2 &= ||\sum_{k=0}^{\infty}\Big[\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}T(t-s)\ f(s,u_s+y_s)ds\\ &\quad+\int_{\xi_k}^{t}T(t-s)\ f(s,u_s+y_s)ds\Big]I_{[\xi_k,\xi_{k+1})}(t)||^2\\ &\leq \Big[\max_{i,k}\Big\{1,\prod_{j=i}^{k}||b_j(\tau_j)||\Big\}\Big]^2\\ &\quad\Big(\int_0^t||T(t-s)||\ ||f(s,u_s+y_s)||ds\ I_{[\xi_k,\xi_{k+1})}(t)\Big)^2\\ &\leq \max\Big\{1,B^2\Big\}\ t\ \Big(\int_0^t||T(t-s)||^2\ ||f(s,u_s+y_s)-f(s,\phi)+f(s,\phi)||^2ds\Big)\\ E||\Gamma u(t)||^2 &\leq 2\max\Big\{1,B^2\Big\}\ T\ C_0^2\ \Big[\int_0^t\frac{E||f(s,u_s+y_s)-f(s,\phi)||^2}{(t-s)^{2\alpha}}ds\\ &\quad+\int_0^t\frac{E||f(s,\phi)||^2}{(t-s)^{2\alpha}}ds\Big]\end{aligned}$$

Taking supremum, we get

$$\begin{aligned}
||\Gamma u(t)||^2 &\leq 2C_0^2 \max\{1, B^2\}\Big[b_1^2\ ||L_{f,b_1}||_{L^q([0,T])}\ \frac{T^{\frac{1}{q'}-2\alpha+1}}{(1-2q'\alpha)^{\frac{1}{q'}}} + ||f(s,\phi)||^2\ \frac{T^{2(1-\alpha)}}{1-2\alpha}\Big] \\
&\leq \frac{b_1^2}{4}
\end{aligned}$$

which implies that $\Gamma u \in B_{\frac{b_1}{2}}(0, S(T))$ and $\Gamma B_{\frac{b_1}{2}}(0, S(T)) \subset B_{\frac{b_1}{2}}(0, S(T))$.

Moreover,

$$\begin{aligned}
\|\Gamma u(t) - \Gamma v(t)\|^2 &\leq \Big[\sum_{k=0}^{\infty}\Big[\sum_{i=1}^{k}\prod_{j=i}^{k}||b_j(\tau_j)||\int_{\xi_{i-1}}^{\xi_i}||T(t-s)||\ ||f(s, u_s + y_s) - f(s, v_s + y_s)||ds \\
&\quad + \int_{\xi_k}^{t}||T(t-s)||\ ||f(s, u_s + y_s)ds - f(s, v_s + y_s)||ds\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big]^2 \\
E\|\Gamma u(t) - \Gamma v(t)\|^2 &\leq \max\Big\{1, B^2\Big\}\ T\ \int_0^t \frac{C_0^2}{(t-s)^{2\alpha}}\ E||f(s, u_s + y_s) - f(s, v_s + y_s)||^2 ds \\
&\leq \max\Big\{1, B^2\Big\}\ T\ C_0^2\ \int_0^t \frac{L_{f,b_1}\ E||u_s - v_s||^2}{(t-s)^{2\alpha}}ds
\end{aligned}$$

Taking supremum over t,

$$||\Gamma u(t) - \Gamma v(t)\|^2 \leq C_0^2\ ||L_{f,b_1}||_{L^q([0,T])}\ \max\{1, B^2\}\ \frac{T^{\frac{1}{q'}-2\alpha+1}}{(1-2q'\alpha)^{\frac{1}{q'}}}||u - v||_{C([0,T],\mathbb{X})}$$

It follows that Γ is contraction on $B_{\frac{b_1}{2}}(0, S(T))$ and there exists a unique fixed point x of Γ which is defined as $x = u + y$ and is a mild solution of 2.1.

This completes the proof.

4. Application

In this section, we apply our abstract results to random impulsive partial differential equation. To apply our results, we need to introduce the required technical tools. Let $\mathcal{U} \subset \mathbb{R}^n$ is a open bounded set with smooth boundary $\partial\mathcal{U}, \eta \in (0, 1)$ and $X = C^\eta(\overline{\mathcal{U}}, \mathbb{R}^n)$ is the space formed by all the η - Hlder continuous functions from $\overline{\mathcal{U}}$ into $\mathbb{R}^n$ endowed with the norm

$$||\xi||_{C^\eta(\overline{\mathcal{U}},\mathbb{R}^n)} = ||\xi||_{C(\overline{\mathcal{U}},\mathbb{R}^n)} + [|\xi|]_{C^\eta(\overline{\mathcal{U}},\mathbb{R}^n)}$$

where $||.||_{C(\overline{\mathcal{U}},\mathbb{R}^n)}$ is the sup-norm on $\overline{\mathcal{U}}$,

$$[|\xi|]_{C^\eta(\overline{\mathcal{U}},\mathbb{R}^n)} = \sup_{x,y\in\overline{\mathcal{U}},x\neq y}\frac{|\xi(x) - \xi(y)|}{|x-y|^\eta}$$

and $|.|$ is the Euclidean norm in $\mathbb{R}^n$.

On the space $\mathbb{X}$, we consider the operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ given $Au = \Delta u$ with domain

$$D(A) = \big\{u \in C^{2+\eta}(\overline{\mathcal{U}}, \mathbb{R}^n) : u_{|\partial\mathcal{U}} = 0\big\}$$

From[4], we know that A is an almost sectorial operator which verifies 1.2 with $\alpha = \frac{\eta}{2}$ and A is not sectorial. In the remainder of this section, $(T(t))_{t\geq0}$ represents the analytic semigroup of growth α generated by A.

Let $\Omega \subset \mathbb{X}$ be a bounded domain with smooth boundary $\partial\Omega$.

$$\begin{cases} \frac{\partial}{\partial t}u(t,\xi) & = & \Delta u(t,\xi) + a_1 u(x,t-r)u^3(x,t), & t \neq \xi_k,\ t \geq 0, \\ u(t,\xi_k) & = & q(k)\tau_k u(t,\xi_k^-) \quad \text{a.s.} & x \in \Omega, \\ u(t,\xi) & = & \Phi(t,\xi) \quad \text{a.s.} & \xi \in \Omega,\ -r \leq t \leq 0, \\ u(t,\xi) & = & 0 \quad \text{a.s.} & \xi \in \partial\Omega. \end{cases} \tag{4.1}$$

Let τ_k be a random variable defined in $D_k \equiv (0, d_k)$ for all $k = 1, 2, ...$ where $0 < d_k < \infty$. Furthermore, assume that τ_i and τ_j be independent with each other as $i \neq j$ for $i, j = 1, 2, ...$ and
$E\Big[\max_{i,k}\Big\{\prod_{j=i}^{k} \|q(j)(\tau_j)\|^2\Big\}\Big] < \infty.$

(H_1) The function $f(.\ ,\psi)$ is strongly measurable on $[0,T]$ for all $\psi \in \widehat{C}$ and $f(t\ ,.) \in C(\widehat{C}, \mathbb{X})$ for each $t \in [0,T]$. There exists a non-decreasing function $W_{1f} \in C(R^+,(0,\infty))$ and $m_{1f} \in L^q([0,T], R^+)$ such that

$$E||f(t,\psi_t)||^2 \leq m_{1f}(t)W_{1f}(E||\psi||_t^2), \forall\ (t,\psi) \in [0,T] \times \widehat{C}$$

(H_2) The function f is continuous and for all $l > 0$ with $[0,l] \times B_l(\phi, \widehat{C}) \subset [0,T] \times \widehat{C}$, there exists $L_{1f,l} \in L^q([0,T], R^+)$ such that

$$E||f(s,\psi_1) - f(s,\psi_2)||^2 \leq L_{1f,l}(s)\ E||\psi_1 - \psi_2||^2, \forall\ (s,\psi_i) \in [0,l] \times B_l(\phi, \widehat{C})$$

(H_3) The condition $\max_{i,k}\Big\{\prod_{j=i}^{k} ||b_j(\tau_j)||\Big\}$ is uniformly bounded if there is a constant $B_1 > 0$ such that

$$\max_{i,k}\Big\{\prod_{j=i}^{k} ||b_j(\tau_j)||\Big\} \quad \leq \quad B_1, \forall \tau_j \in D_j, j = 1,2,...$$

If the inequality,

$$\frac{C\ C_0^2\ ||m_{1f}||_{L^q([0,T])}\ \max\{1, {B_1}^2\}\ T^{\frac{1}{q'}-2\alpha+1}}{(1-2q'\alpha)^{\frac{1}{q'}}} \leq \frac{b_1^2}{4}$$

holds, then it is easy to check that all the hypotheses of the Theorem 3.2are satisfied and therefore, Theorem 3.2 guarantees the existence of mild solution of the partial differential equation 4.1.

Furthermore, if the following condition holds

$$C_0^2\ ||L_{1f,b_1}||_{L^q([0,T])}\ \max\{1, {B_1}^2\}\ \frac{T^{\frac{1}{q'}-2\alpha+1}}{(1-2q'\alpha)^{\frac{1}{q'}}} < 1$$

then by Theorem 3.3 we know that the solution to 4.1 is unique.

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