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A viscosity type iteration by weak contraction for approximating solutions of generalized equilibrium problem

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This paper is dedicated to Professor Ljubomir Ćirić

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Abstract

Viscosity iterations which include contraction mapping have been widely used to find solutions of equilibrium problems. Here we introduce a modification of the viscosity iteration scheme by replacing the contraction with a weak contraction. Weakly contractive mappings are intermediate to contractive and nonexpansive mappings and are known to have unique fixed points in complete metric spaces. We apply this iteration to the case of a generalized equilibrium problem. The special case where the weak contraction is a contraction has also been discussed. ©2012 NGA. All rights reserved.

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1. Introduction and Preliminaries

An equilibrium problem in a real Hilbert space is the following:
Let $F : C\times C\to R$ be a bifunction where R is a the set of real numbers and C is a nonempty closed convex subset of a real Hilbert space H. Let A be an α- inverse strongly monotone mapping from C to H, that is,

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a mapping $A : C \to H$ such that
$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2$ for all $x, y \in C$ where $\alpha > 0$. Then the generalized equilibrium problem is to find $x \in C$ such that

$$F(x, y) + \langle Ax, y - x\rangle \geq 0, \quad \text{for all } y \in C. \tag{1.1}$$

The above mentioned problem is a very general problem which includes as special cases several optimization problems, variational inequalities, minimax problems etc [2, 9].
The set of solutions of (1.1) is denoted by $GEP(F)$, that is,

$$GEP(F) = \{x \in C : F(x, y) + \langle Ax, y - x\rangle \geq 0 \text{ for all } y \in C\}.$$

Several authors have developed iterative algorithms for finding common elements of $GEP(F)$ and the set of fixed points of nonexpansive mappings in Hilbert spaces [3, 4, 21, 23, 25]. In particular, viscosity approximation methods were applied to this problem in a number of works like [8, 16, 22]. Further the problem of finding common elements of $GEP(F)$ and the fixed point set of mappings like k- strict pseudocontraction and asymptotically k-strict pseudocontraction mappings have applied in works like [5, 11, 12, 24]. Viscosity approximation method was introduced by Moudafi [14] for approximating fixed point of nonexpansive mappings. This iteration involves a contraction mapping. Here in this paper we have approximated common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping by a generalized two step viscosity approximation method where we have replaced the contraction mapping by a weak contraction. Weak contractions are mappings which are more general than contractions and more restrictive than nonexpasive mappings. These mappings are somewhat in between contraction and nonexpansion. Weak contraction principle was first introduced by Alber et al [1] in Hilbert spaces and later established in metric spaces by Rhoades [18]. This and similar types of results have been discussed in a large number of works in recent times [6, 7, 10, 15, 17, 19, 26]. We have also shown that our result can be improved if we consider the special case where weak contraction is a contraction.
For any $x \in H$, the metric projection P_C from H into C is defined as $P_C x = \operatorname{Inf}\{\|y - x\| : y \in C\}$. Obviously, $\|x - P_C x\| \leq \|x - y\|$. It is well known that P_C is a firmly nonexpansive mapping from H onto C, that is, $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$ for all $x, y \in H$.
Also the Hilbert space H satisfies Opial's condition,that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ the inequality $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$ holds for every $y \in H$ with $y \neq x$.
A mapping $T : C \to C$ is said to be a nonexpansive mapping if for all $x, y \in C$

$$\|Tx - Ty\| \leq \|x - y\|.$$

A mapping $f : C \to C$ is said to be a weakly contractive mapping if for each $x, y \in C$,

$$\|fx - fy\| \leq \|x - y\| - \phi(\|x - y\|) \tag{1.2}$$

where $\phi : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing such that ϕ is positive on$(0,\infty)$, $\phi(0) = 0$, $\lim_{t\to\infty} \phi(t) = \infty$ and $\phi(t) < t$, for all $t > 0$.

Theorem 1.1. *[18] Let (X, d) be a complete metric space, T a weakly contractive map. Then T has a unique fixed point p in X.*

We will require the results noted in the following lemmas.

Lemma 1.2. *[20]. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in [0,1] with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 1$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*
In the equilibrium problem for the bifunction F from $C \times C \to R$, we assume that F satisfies following conditions:
(C1) $F(x, x) = 0$ for all $x \in C$,
(C2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$,
(C3) for each $x, y, z \in C$,

$\lim_{t\to 0^+} F(tz+(1-t)x,y) \le F(x,y)$,

(C4) for each $x \in C$, $y \to F(x,y)$ *is convex and lower semicontinuous.*

Lemma 1.3. *[9]. Let* C *be a nonempty closed convex subset of* H *and let* F *be a bifunction from* $C\times C$ *into* R *satisfying conditions (C1)- (C4). Then for any* $r > 0$ *and* $x \in H$ *there exists* $z \in C$ *such that*

$F(z,y) + (1/r)\langle y-z, z-x\rangle \ge 0$ *for all* $y \in C$.

Further, if $T_r x = \{z \in C : F(z,y) + (1/r)\langle y-z, z-x\rangle \ge 0 \text{ for all } y \in C\}$ *then the following hold:*

(1) T_r *is single valued,*

(2) T_r *is firmly nonexpansive, that is, for any* $x,y \in H$

$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x-y\rangle$,

(3) $F(T_r) = GEP(F)$,

(4) $GEP(F)$ *is closed and convex.*

Lemma 1.4. *[23]. Let* C, H, F *and* $T_r x$ *be described as in Lemma 1.2. Then the following holds :*

$\|T_s x - T_t y\|^2 \le ((s-t)/s)\langle T_s x - T_t y, T_s x - x\rangle$

Lemma 1.5. *[13]. Let* $\{a_n\}$,$\{b_n\}$ *and* $\{c_n\}$ *be three sequences of nonnegative real numbers satisfying*

$a_{n+1} \le (1-\lambda_n)a_n + b_n + c_n, \qquad n \ge n_0$

where n_0 *is some nonnegative integer,* $\lambda_n \in [0,1]$, $\sum_{n=1}^{\infty}\lambda_n = \infty$, $b_n = o(\lambda_n)$, *and* $\sum_{n=1}^{\infty} c_n < \infty$.

Then $a_n \to 0$ *as* $n \to \infty$.

Lemma 1.6. *[16]. Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* A *be an* α*-inverse strongly monotone mapping from* C *to* H. *Then for any real number* $r > 0$, $I - rA$ *is nonexpansive.*

Lemma 1.7. *[16]. Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Given* $z \in H$ *and* $x \in C$, *the inequality* $\langle x-z, y-x\rangle \ge 0$, *for all* $y \in C$ *holds if and only if* $x = P_C z$, *where* P_C *denotes the metric projection from* H *onto* C.

The following lemma is a well known result of functional analysis.

Lemma 1.8. *Let* X *be a reflexive Banach space. Then every bounded sequence in* X *has a weakly convergent subsequence.*

2. Main results

Theorem 2.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H, A *be an* α*-inverse strongly monotone mapping from* C *into* H , S *be a nonexpansive mapping of* C *into itself and* f *be a weakly contractive mapping of* C *into itself. Let* F *be a bifunction from* $C\times C \to R$ *satisfying (C1)-(C4). Suppose that* $F(S) \cap GEP(F) \ne \phi$. *Let* $x_0 \in C$. *The sequences* $\{z_n\} \subset C$ *and* $\{x_n\} \subset C$ *are constructed iteratively as follows:*

$$\text{For } n \ge 0,\ F(z_n, y) + \langle Ax_n, y-z_n\rangle + (1/r_n)\langle y-z_n, z_n-x_n\rangle \ge 0 \text{ for all } y \in C, \tag{2.1}$$

$$x_{n+1} = \beta_n x_n + (1-\beta_n)Sy_n, \tag{2.2}$$

$$y_n = \alpha_n f(x_n) + (1-\alpha_n)z_n, \tag{2.3}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\} \subset [0,1]$. *Assume that,*

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $\lim_{n\to\infty}(r_{n+1}-r_n)=0$,
(iv) $0< a \leq r_n \leq b < 2\alpha$,
(v) $0< c \leq \beta_n \leq d < 1$.

If $\{x_n\}$ is bounded, then there exists $z \in F(S)\cap GEP(F)$ such that $\liminf_{n\to\infty}\|x_n - z\|=0$.

Proof. The bifunction F satisfies conditions (C1)- (C4). Hence, by Lemma (1.3), for given $r > 0$ and $x \in C$, the mapping T_r given by:
$T_r x = \{z \in C : F(z, y) + (1/r)\langle y - z, z - x\rangle \geq 0 \text{ for all } y \in C\}$
is a single valued mapping from $H \to C$.
Again $z \in GEP(F) \Leftrightarrow z \in C$ such that $F(z, y) + \langle Az, y - z\rangle \geq 0$ for all $y \in C$
$\Leftrightarrow z \in C$ such that $F(z, y) + (1/r_n)\langle z - (I - r_nA)z, y - z\rangle \geq 0$ for all $y \in C$
$\Leftrightarrow z = T_{r_n}(I - r_nA)z \Leftrightarrow z \in F(T_{r_n}(I - r_nA))$.
By an application of Lemma (1.3), we conclude that $GEP(F) = F(T_{r_n}(I - r_nA))$. Again by Lemma (1.6) , $I - r_nA$ is non expansive so that $T_{r_n}(I - r_nA)$ is firmly nonexpansive for each $n \geq 0$. Thus, $GEP(F)$ is closed and convex so that the mapping $P_{F(S)\cap GEP(F)}$ is well defined.
Now $P_{F(S)\cap GEP(F)}\, f$ is a mapping of C into $F(S) \cap GEP(F) \subset C$ such that
$\|P_{F(S)\cap GEP(F)}f(x) - P_{F(S)\cap GEP(F)}f(y)\|$
$\leq \|f(x) - f(y)\| \leq \|x - y\| - \phi(\|x - y\|)$.
Therefore $P_{F(S)\cap GEP(F)}f$ is a weakly contractive mapping. Hence, by Theorem (1.1), $P_{F(S)\cap GEP(F)}f$ possesses a unique fixed point. Therefore, there exists a unique element

$$z \in F(S) \cap GEP(F) \subset C \text{ such that } z = P_{F(S)\cap GEP(F)}f(z). \tag{2.4}$$

By Lemma (1.3) and (2.1), we have

$$\|z_n - z\| = \|T_{r_n}(I - r_nA)x_n - T_{r_n}(I - r_nA)z\| \leq \|x_n - z\|. \tag{2.5}$$

(since $z_n = T_{r_n}(I - r_nA)x_n, z = T_{r_n}(I - r_nA)z$)
Since $\{x_n\}$ is bounded , $\{y_n\}, \{z_n\}, \{f(x_n)\}, \{Ax_n\}$ and $\{T_{r_n}x_n\}$ are all bounded.
Next we prove that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.
Putting $u_n = (I - r_nA)x_n$, it follows from lemma (1.4) that there exists a constant $M > 0$ such that

$$\begin{aligned}\|T_{r_{n+1}}u_n - T_{r_n}u_n\|^2 &\leq ((r_{n+1} - r_n)/r_{n+1})\langle T_{r_{n+1}}u_n - T_{r_n}u_n, T_{r_{n+1}}u_n - u_n\rangle\\ &\leq (|r_{n+1} - r_n|/r_{n+1})(\|T_{r_{n+1}}u_n - T_{r_n}u_n\|.\|T_{r_{n+1}}u_n - u_n\|)\\ &\leq \{|r_{n+1} - r_n|/a\}M.\end{aligned}$$

Then we have , for all $n \geq 0$,

$$\begin{aligned}\|z_{n+1} - z_n\| &= \|T_{r_{n+1}}(I - r_{n+1}A)x_{n+1} - T_{r_n}(I - r_nA)x_n\|\\ &\leq \|T_{r_{n+1}}(I - r_{n+1}A)x_{n+1} - T_{r_{n+1}}(I - r_nA)x_n\| + \|T_{r_{n+1}}u_n - T_{r_n}u_n\|\\ &\leq \|x_{n+1} - x_n\| + |r_{n+1} - r_n|.\|Ax_n\| + \sqrt{\{|r_{n+1} - r_n|/a\}M}.\end{aligned}$$

Therefore, for all $n \geq 0$,

$$\begin{aligned}&\|Sy_{n+1} - Sy_n\|\\ &\leq \|y_{n+1} - y_n\|\\ &= \|\alpha_{n+1}(f(x_{n+1}) - f(x_n)) + (\alpha_{n+1} - \alpha_n)(f(x_n) - z_n) + (1 - \alpha_{n+1})(z_{n+1} - z_n)\|\\ &\leq \alpha_{n+1}\|f(x_{n+1}) - f(x_n)\| + |\alpha_{n+1} - \alpha_n|.\|f(x_n) - z_n\|\\ &+ \|x_{n+1} - x_n\| + |r_{n+1} - r_n|\|Ax_n\| + \sqrt{\{|r_{n+1} - r_n|/a\}M}.\end{aligned}$$

By the assumption imposed on α_n and r_n we have
$\limsup_{n\to\infty}(\|Sy_{n+1} - Sy_n\| - \|x_{n+1} - x_n\|) \leq 0$.
From lemma (1.2)we have,

$$Sy_n - x_n \to 0. \tag{2.6}$$

$$\text{Therefore } \lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|Sy_n-x_n\| = 0. \tag{2.7}$$

For each $z \in F(S)\cap GEP(F)$, since $z_n = T_{r_n}(I-r_nA)x_n$,for all $n \geq 0$, we have ,

$$\begin{aligned}\|z_n - z\|^2 &= \|T_{r_n}(I-r_nA)x_n - T_{r_n}(I-r_nA)z\|^2\\ &\leq \|(x_n-z)-(r_n(Ax_n-Az)\|^2\\ &= \|x_n-z\|^2 - 2r_n\langle x_n-z, Ax_n-Az\rangle + r_n^2\|Ax_n-Az\|^2\\ &\leq \|x_n-z\|^2 + r_n(r_n-2\alpha)\|Ax_n-Az\|^2.\end{aligned} \tag{2.8}$$

It follows from (2.2) and (2.8), for all $n \geq 0$,

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \|\beta_n(x_n-z)+(1-\beta_n)(Sy_n-z)\|^2\\ &\leq \beta_n\|x_n-z\|^2 + (1-\beta_n)\|y_n-z\|^2\\ &\leq \beta_n\|x_n-z\|^2 + (1-\beta_n)\{\alpha_n\|f(x_n)-z\|^2 + (1-\alpha_n)\|z_n-z\|^2\}\\ &\leq \beta_n\|x_n-z\|^2 + (1-\beta_n)\{\alpha_n\|f(x_n)-z\|^2 + (1-\alpha_n)\|x_n-z\|^2\}\\ &\quad + (1-\beta_n)(1-\alpha_n)r_n(r_n-2\alpha)\|Ax_n-Az\|^2\\ &\leq \|x_n-z\|^2 + \alpha_n(1-\beta_n)\|f(x_n)-z\|^2\\ &\quad + (1-\alpha_n)(1-\beta_n)r_n(r_n-2\alpha)\|Ax_n-Az\|^2.\end{aligned} \tag{2.9}$$

By the fact that $0 < c \leq \beta_n \leq d < 1$, $\alpha_n \to 0$ as $n \to \infty$ and (2.7) and (2.9) we have $\|Ax_n - Az\| \to 0$. (2.10)

Using lemma (1.3) and (2.5), for all $n \geq 0$, we have,

$$\begin{aligned}\|z_n-z\|^2 &= \|T_{r_n}(I-r_nA)x_n - T_{r_n}(I-r_nA)z\|^2\\ &\leq \langle (I-r_nA)x_n-(I-r_nA)z, z_n-z\rangle\\ &= (1/2)(\|(I-r_nA)x_n-(I-r_nA)z\|^2 + \|z_n-z\|^2)\\ &\quad - (1/2)(\|(I-r_nA)x_n-(I-r_nA)z-(z_n-z)\|^2)\\ &\leq (1/2)(\|x_n-z\|^2 + \|z_n-z|^2 - \|(x_n-z_n)-r_n(Ax_n-Az)\|^2)\\ &= (1/2)(\|x_n-z\|^2 + \|z_n-z\|^2 - \|x_n-z_n\|^2)\\ &\quad - (1/2)(r_n^2\|Ax_n-Az\|^2 - 2r_n\langle x_n-z_n, Ax_n-Az\rangle)\end{aligned}$$

$$\text{or, } \|z_n-z\|^2 \leq \|x_n-z\|^2 - \|x_n-z_n\|^2 - r_n^2\|Ax_n-Az\|^2 + 2r_n\langle x_n-z_n, Ax_n-Az\rangle. \tag{2.11}$$

It follows from (2.2) and (2.11), for all $n \geq 0$

$$\begin{aligned}\|x_{n+1}-z\|^2 &\leq \beta_n\|x_n-z\|^2 + (1-\beta_n)\|y_n-z\|^2\\ &\leq \beta_n\|x_n-z\|^2 + (1-\beta_n)[\alpha_n\|f(x_n)-z\|^2 + (1-\alpha_n)\|z_n-z\|^2]\\ &\leq \beta_n\|x_n-z\|^2 + \alpha_n\|f(x_n)-z\|^2 + (1-\beta_n)\|z_n-z\|^2\\ &\leq \|x_n-z\|^2 + \alpha_n\|f(x_n)-z\|^2 - (1-\beta_n)\|x_n-z_n\|^2\\ &\quad + 2(1-\beta_n)r_n\|x_n-z_n\|\|Ax_n-Az\|.\end{aligned}$$

Hence , $(1-d)\|x_n-z_n\|^2 \leq \|x_n-z\|^2 - \|x_{n+1}-z\|^2 + \alpha_n\|f(x_n)-z\|^2 + 2(1-\beta_n)r_n\|x_n-z_n\|\|Ax_n-Az\|.$

By virtue of (2.7), (2.10) and the fact $\alpha_n \to 0$ as $n \to \infty$, $\|x_n - z_n\| \to 0$ as $n \to \infty$. (2.12)

Since $\{f(x_n)\}$ and $\{z_n\}$ are bounded, $y_n = \alpha_n f(x_n) + (1-\alpha_n)z_n$ and $\alpha_n \to 0$ as $n \to \infty$ we have

$$\|y_n - z_n\| = \alpha_n\|f(x_n) - z_n\| \to 0 \text{ as } n \to \infty. \tag{2.13}$$

From (2.12) and (2.13), we have

$$\|y_n - x_n\| \to 0. \tag{2.14}$$

Since $\|Sy_n - y_n\| \leq \|Sy_n - x_n\| + \|x_n - y_n\|$, from (2.6) and (2.14) we have

$$\|Sy_n - y_n\| \to 0. \tag{2.15}$$

By (2.4), $z = P_{F(S)\cap GEP(F)}f(z)$, we shall prove that $\limsup_{n\to\infty}\langle f(z) - z, y_n - z\rangle = 0$.
We take a subsequence $\{y_{n_i}\}$ of $\{ y_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z) - z, y_n - z\rangle$$
$$= \lim_{i\to\infty}\langle f(z) - z, y_{n_i} - z\rangle. \quad (2.16)$$

Since $\{y_n\}$ is bounded and the Hilbert space H is reflexive, by lemma (1.8) there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ which converges weakly to w. C is closed and bounded, hence it is weakly closed and hence $w \in C$.
Next we show that $w \in F(S) \cap GEP(F)$. Since $z_n = T_{r_n}(I - r_nA)x_n$ for any $y \in C$ we have,
$F(z_n, y) + \langle Ax_n, y - z_n\rangle + (1/r_n)\langle y - z_n, z_n - x_n\rangle \geq 0$.
From (C2) we have , $\langle Ax_n, y - z_n\rangle + (1/r_n)\langle y - z_n, z_n - x_n\rangle \geq F(y, z_n\ )$.
Replacing n by n_i we have,

$$\langle Ax_{n_i}, y - z_{n_i}\rangle + \langle y - z_{n_i}, (z_{n_i} - x_{n_i})/r_{n_i}\rangle \geq F(y, z_{n_i}). \quad (2.17)$$

For $t \in (0, 1]$ and $y \in C$, we define $z_t = ty + (1 - t)w$. Since C is convex have $z_t \in C$.
Therefore , from (2.17) we have,
$\langle z_t - z_{n_i}, Az_t\rangle \geq \langle z_t - z_{n_i}, Az_t\rangle - \langle Ax_{n_i}, z_t - z_{n_i}\rangle - \langle z_t - z_{n_i}, (z_{n_i} - x_{n_i})/r_{n_i}\rangle + F(z_t, z_{n_i})$
$= \langle z_t - z_{n_i}, Az_t - Ax_{n_i}\rangle - \langle z_t - z_{n_i}, (z_{n_i} - x_{n_i})/r_{n_i}\rangle + F(z_t, z_{n_i})$.
Since $\|z_{n_i} - x_{n_i}\| \to 0$,we have , $\|Az_{n_i} - Ax_{n_i}\| \to 0$. From the monotonicity of A , we have , $\langle z_t - z_{n_i}, Az_t - Ax_{n_i}\rangle \geq 0$. Therefore, from (C4), we have,

$$\langle z_t - w, Az_t\rangle \geq F(z_t, w) \ \text{ as } i \to \infty. \quad (2.18)$$

From (C1),(C4) and (2.18) we also have,
$0 = F(z_t, z_t) \leq tF(z_t, y) + (1 - t)F(z_t, w)$
$\leq tF(z_t, y) + (1 - t)\langle z_t - w, Az_t\rangle$
$=tF(z_t, y) + (1 - t)t\langle y - w, Az_t\rangle$.
Hence, $0 \leq F(z_t, y) + (1 - t)\langle y - w, Az_t\rangle$.Taking $t \to 0^+$, for each $y \in C$, we have
$0 \leq F(w, y) + \langle y - w, Aw\rangle$ which holds for any $y \in C$.
This implies that $w \in GEP(F)$.
Next we show that $w \in F(S)$. If $w \notin F(S)$ we have $w \neq Sw$. From Opial's condition and (2.15)
$\liminf_{i\to\infty} \|y_{n_i} - w\| < \liminf_{i\to\infty} \|y_{n_i} - Sw\|$
$= \liminf_{i\to\infty} \|y_{n_i} - Sy_{n_i} + Sy_{n_i} - Sw\|$
$\leq \liminf_{i\to\infty}\|y_{n_i} - w\|$.
This is a contradiction. Therefore, we have $w \in F(S)$. Since $w \in F(S) \cap GEP(F)$, from Lemma (1.7), we have,

$$\lim_{n\to\infty}\langle f(z) - z, y_n - z\rangle$$
$$= \lim_{i\to\infty}\langle f(z) - z, y_{n_i} - z\rangle \text{ (by using(2.16))}$$
$$= \langle f(z) - z, w - z\rangle \leq 0. \quad (2.19)$$

From (2.19) we can consider for some positive integer $n \geq n_l$,

$$\langle f(z) - z, y_n - z\rangle \leq 0. \quad (2.20)$$

For all $n \geq n_l$, $\|x_{n+1} - z\|^2 = \|\beta_n(x_n - z) + (1 - \beta_n)(Sy_n - z)\,\|^2$
$\leq \beta_n\|x_n - z\|^2 + (1 - \beta_n)\|Sy_n - z\|^2$
$\leq \beta_n\|x_n - z\|^2 + (1 - \beta_n)\|y_n - z\|^2$
$\leq \beta_n\|x_n - z\|^2 + (1 - \beta_n)\{(1 - \alpha_n)^2\|z_n - z\|^2 + 2\alpha_n\langle f(x_n) - z, y_n - z\rangle\}$

$$\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)(1-2\alpha_n+\alpha_n^2)\|x_n - z\|^2$$
$$+2\alpha_n(1-\beta_n)\langle f(x_n) - z, y_n - z\rangle$$
$$\leq (1-2(1-\beta_n)\alpha_n)\|x_n - z\|^2 + \alpha_n^2\|x_n - z\|^2$$
$$+ 2\alpha_n(1-\beta_n)\{\langle f(x_n) - f(z), y_n - z\rangle + \langle f(z) - z, y_n - z\rangle\}$$
$$\leq (1-2(1-\beta_n)\alpha_n)\|x_n - z\|^2 + \alpha_n^2 M_0$$
$$+ 2\alpha_n(1-\beta_n)[\{\|x_n - z\| - \phi(\|x_n - z\|)\}\|y_n - z\|] \text{ (using(2.20)}$$
$$\leq (1-2(1-\beta_n)\|x_n - z\|^2 + \alpha_n^2 M_0$$
$$+ 2\alpha_n(1-\beta_n)[\|x_n - z\| - \{\phi(\|x_n - z\|)/\|x_n - z\|\}\|x_n - z\|]\|y_n - z\|$$

where $M_0 = \sup_{n\geq 0}\{\|x_n - z\|^2 + \|f(x_n) - z\|^2\}$ (M_0 finitely exists since $\{x_n\}$ and $\{f(x_n\}$ are bounded sequence).

Now, $2\alpha_n(1-\beta_n)[\{1 - \phi(\|x_n - z\|)/\|x_n - z\|\}\|y_n - z\|\|x_n - z\|]$
$\leq \alpha_n(1-\beta_n)[1 - \phi(\|x_n - z\|)/\|x_n - z\|][\|x_n - z\|^2 + \|y_n - z\|^2]$

Now, $\|y_n - z\|^2 \leq \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)\|z_n - z\|^2$
$\leq \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)\|x_n - z\|^2$
$\leq \alpha_n M_0 + \|x_n - z\|^2$.

Therefore, $\alpha_n(1-\beta_n)[1 - \phi(\|x_n - z\|)/\|x_n - z\|][\|x_n - z\|^2 + \|y_n - z\|^2]$
$\leq 2\alpha_n(1-\beta_n)[1 - \phi(\|x_n - z\|)/\|x_n - z\|]\|x_n - z\|^2 + \alpha_n^2 M_0(1-\beta_n)[1 - \phi(\|x_n - z\|)/\|x_n - z\|]$
$\leq 2\alpha_n(1-\beta_n)[1 - \phi(\|x_n - z\|)/\|x_n - z\|]\|x_n - z\|^2 + \alpha_n^2 M_0[1 - \phi(\|x_n - z\|)/\|x_n - z\|]$

Therefore, $\|x_{n+1} - z\|^2 \leq [1 - 2\alpha_n(1-\beta_n) + 2\alpha_n(1-\beta_n)\{1 - \phi(\|x_n - z\|)/\|x_n - z\|\}]\|x_n - z\|^2$
$+\alpha_n^2 M_0[2 - \phi(\|x_n - z\|)/\|x_n - z\|]$.

Next we have to prove $\liminf_{n\to\infty} \|x_n - z\| = 0$.

If not, let $\liminf_{n\to\infty}\|x_n - z\| \neq 0$.

By the above consideration and the condition imposed on 'ϕ' we can consider for some positive integer n_k, $\liminf_{n\to\infty} \phi(\|x_n - z\|)/\|x_n - z\| > \rho$ where ρ is a positive number, for $n \geq n_k$. Let G=max$\{n_l, n_k\}$

Now for all $n \geq G$,
$\|x_{n+1} - z\|^2 \leq [1 - 2\alpha_n(1-\beta_n)\rho]\|x_n - z\|^2 + \alpha_n^2 M_0[1 - \phi(\|x_n - z\|)/\|x_n - z\|]$
$\leq [1 - 2\alpha_n(1-d)\rho]\|x_n - z\|^2 + \alpha_n^2 M_0[1 - \phi(\|x_n - z\|)/\|x_n - z\|]$.

Using the lemma (1.5) we can say that $\liminf_{n\to\infty} \|x_n - z\|$=0 which is a contradiction to our initial assumption. Therefore we are left with only alternative $\liminf_{n\to\infty} \|x_n - z\|$=0. This proves the theorem. □

Note: As an example, if $C = [0, 1]$, then in the viscosity iteration (2.1) - (2.3) in Theorem (2.1) we can choose the weak contraction f as $fx = x - \frac{1}{2}x^2$ for all $x \in C$.

Theorem 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, A be an α-inverse strongly monotone mapping from C into H , S be a nonexpansive mapping of C into itself and f be a weakly contractive mapping of C into itself. Let F be a bifunction from $C\times C \to R$ satisfying (C1)-(C4). Suppose that $F(S) \cap GEP(F) \neq \phi$. The sequences $\{z_n\} \subset C$ and $\{x_n\} \subset C$ are constructed as in Theorem (2.1) Assume that,*

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $\lim_{n\to\infty} (r_{n+1} - r_n) = 0$,
(iv) $0 < a \leq r_n \leq b < 2\alpha$,
(v) $0 < c \leq \beta_n \leq d < 1$.

If $\{x_n\}$ is convergent then it converges to a point in $F(S) \cap GEP(F)$.

Proof. Since a convergent sequence is bounded , proceeding exactly as in Theorem (2.1) we obtain,

$$\|x_{n+1} - z\|^2 \leq [1 - 2\alpha_n(1-\beta_n) + 2\alpha_n(1-\beta_n)\{1 - \phi(\|x_n - z\|)/\|x_n - z\|\}]\|x_n - z\|^2$$
$$+\alpha_n^2 M_0[2 - \phi(\|x_n - z\|)/\|x_n - z\|]. \tag{2.21}$$

Therefore by (2.21) we can say that $\{x_n\}$ is convergent and converges to z. □

Theorem 2.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, A be an α-inverse strongly monotone mapping from C into H , S be a nonexpansive mapping of C into itself and f be a contractive mapping of C into itself. Let F be a bifunction from $C\times C\to R$ satisfying (C1)-(C4). Suppose that $F(S)\cap GEP(F) \neq \phi$.The sequences $\{z_n\} \subset C$ and $\{x_n\} \subset C$ are constructed as in Theorem (2.1) Assume that,*

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $\lim_{n\to\infty} (r_{n+1} - r_n)=0$,
(iv) $0< a \leq r_n \leq b < 2\alpha$,
(v) $0< c \leq \beta_n \leq d < 1$.

Then $\{x_n\}$ is convergent to a point in $F(S)\cap GEP(F)$.

Proof. The proof is identical with that of theorem (2.1) except that the boundedness of the sequence $\{x_n\}$ now follows from the given condition of the theorem.
By (2.3) we have for any $z \in F(S)\cap GEP(F)$,

$$\begin{aligned}\|y_n - z\| &= \|\alpha_n(f(x_n) - z) + (1-\alpha_n)(z_n - z)\| \\ &\leq \alpha_n\|f(x_n) - f(z) + f(z) - z\| + (1-\alpha_n)\|z_n - z\| \\ &\leq \alpha_n\|f(x_n) - f(z)\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)\|x_n - z\| \\ &\leq \alpha_n\theta\|x_n - z\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)\|x_n - z\|\end{aligned}$$

where θ is the contraction coefficient.

$$\begin{aligned}&= (1 - \alpha_n(1-\theta))\|x_n - z\| + \alpha_n\|f(z) - z\| \\ &\leq \max\{\|x_n - z\|, (1/(1-\theta))\|f(z) - z\|\}.\end{aligned} \tag{2.22}$$

Therefore by (2.2) we have,

$$\begin{aligned}\|x_{n+1} - z\| &= \|\beta_n(x_n - z) + (1-\beta_n)(Sy_n - z)\| \\ &\leq \beta_n\|x_n - z\| + (1-\beta_n)\|y_n - z)\| \\ &\leq \max\{\|x_n - z\|, (1/(1-\theta))\|f(z) - z\|\}. \text{ (by (2.22))}\end{aligned}$$

Applying the same process we obtain
$\|x_{n+1} - z\| \leq \max\{\|x_1 - z\|, (1/(1-\theta))\|f(z) - z\|\}$.
This implies that $\{x_n\}$ is bounded in H.
Now proceeding exactly as in theorem (2.1) the theorem can be proved. □

Remark 1. In this paper we have generalized the viscosity approximation scheme by replacing the contraction mapping conventionally present in the definition of the viscosity approximation with a weak contraction. Its effect on an application of the scheme to a generalized equilibrium problem has been studied here. From the results obtained in this paper we have the following observation. We have strong convergence result $\|x_n - z\| \to 0$ when we use a contraction as in theorem (2.3) while in the theorem (2.1), where we have used the iteration scheme with a weak contraction, we can only conclude that the limit infimum of $\{\|x_n - z\|\}$ is zero. In the above sense the viscosity iteration is weakened with the weak contraction replacing the contraction.

References

[1] Y. I. Alber and S. Guerre-Delabriere, *Principles of weakly contractive maps in Hilbert spaces, New Results in Operator Theory*, Advances and Application (eds. I. Gohberg, Y. Lyubich, Birkhauser, Basel, 722, 98(1997). 1

[2] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math Student, 63 (1994), 123-145. 1

[3] L. C.Ceng and J. C. Yao, *A hybrid iterative scheme for mixed equilibrium problems and fixed point problems*, J. Comput. Appl. Math., 214 (2008), 186-201. 1

[4] L. C. Ceng, A. Petrusel and J. C. Yao, *Iterative approaches to solving equilibrium problems and fixed point problems of infinitely many nonexpansive mappings*, J. Optim. Theory. Appl., 143 (2009), 37-58. 1

[5] L. C. Ceng, S. Al-Homidan, Q. H. Ansari and J. C. Yao, *An iterative scheme for equilibrium problems and fixed point problems of strict pseudo-contraction mappings*, J. Comput. Appl. Math., 223 (2009), 967-974. 1

[6] C. E. Chidume, H. Zegeye and S. J. Aneke, *Approximation of fixed points of weakly contractive non self maps in Banach spaces*, J. Math. Anal. Appl., 270 (2002), 189-199. 1

[7] B. S. Choudhury and N. Metiya, *Fixed points of weak contractions in cone metric spaces*, Nonlinear Analysis TMA, 72 (2010), 1589-1593. 1

[8] V. Colao, G. Marino and H. K. Xu, *An iterative method for finding common solutions of equilibrium problem and fixed point problems*, J. Math. Anal. Appl., 344 (2008), 340-352. 1

[9] P. L. Combettes and S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal., 6(1) (2005), 117-136.
1, 1.3

[10] P. N. Dutta and B. S. Choudhury, *A generalization of contraction principle in metric spaces*, Fixed point Theory Appl., Article ID: 406368(2008)8 pages. 1

[11] C. Jaiboon and P. Kumam, *Strong convergence theorems for solving equilibrium problems and fixed point problems of ξ- strict pseudo contraction mappings by two hybrid projection methods*, J. Comput. Appl. Math., 234 (2010), 722-732. 1

[12] P. Kumam, N. Petrot and R. Wangkeeree, *A hybrid iterative scheme for equilibrium problems and fixed point problems of asymptotically k-strict pseudo contractions*, J. Comput. Appl. Math., 233 (2010), 2013-2026. 1

[13] L. S. Liu, *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl., 194 (1995), 114-125. 1.5

[14] A. Moudafi, *Viscosity approximation methods for fixed point problems*, J. Math. Anal. Appl., 241 (2000), 46-55. 1

[15] O. Popescu, *Fixed points for (ϕ, ψ)-weak contractions*, Appl. Math. Lett., 24 (2000), 1-4. 1

[16] X. Qin, Y. J. Cho and S. M. Kang, *Viscosity approximation methods for generalized equilibrium problems and fixed point problems with application*, Nonlinear Analysis TMA., 72 (2010), 99-112.

[17] S. Radenovi and Z. Kadelburg, *Generalized weak contractions in partially ordered metric spaces, Comp Math Appl*, 60 (2010), 1776-1783. 1, 1.6, 1.7

[18] B. E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Analysis TMA., 47 (2001), 2683-2693. 1

[19] Y. Song, *Coincidence points for noncommuting f-weakly contractive mappings*, Int. J. Comput. Appl. Math., 2 (2007), 51-57. 1, 1.1

[20] T. Suzuki, *Strong convergence of Krasnoselskii and Mann's Type sequences for one parameter nonexpansive semigroups without Bochner integrals*, J. Math. Anal. Appl., 305 (2005), 227-239. 1

[21] A. Tada and W. Takahashi, *Strong convergence theorem for an equilibrium problem and a nonexpansive mapping*, J. Optim. Theory. Appl., 133 (2007), 359-370. 1.2

[22] S. Takahashi and W. Takahashi, *Viscosity approximation methods for equilibrium problem and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl., 331 (2007), 506-515. 1

[23] S. Takahashi and W. Takahashi, *Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space*, Nonlinear Analysis TMA., 69 (2008), 1025-1033. 1

[24] Z. Wang and Y. Su, *An iterative scheme for equilibrium problems and fixed point problems of asymptotically k-strict pseudo contractive mappings*, Commun. Korean. Math. Soc., 25 (2007), 69-82. 1, 1.4

[25] Y. Yao, M. A. Noor and Y. C. Liou, *On iterative methods for equilibrium problems*, Nonlinear Analysis TMA., 70 (2009), 497-509. 1

[26] Q. Zhang, Y. Song, *Fixed point theory for ϕ-weak contractions*, Appl. Math. Lett., 22 (2009), 75-78. 1