Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 5 (2012), 259–270
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

On coupled generalised Banach and Kannan type contractions

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This paper is dedicated to Professor Ljubomir Ćirić

Communicated by Professor V. Berinde

Abstract

In this paper we have proved two theorems in which we have established the existence of coupled fixed point results in partially ordered complete metric spaces for generalised coupled Banach and Kannan type mappings. The generalisation has been accomplished by following the line of argument given by Geraghty [Proc. Amer. Math. Soc., 40 (1973), 604-608] . Here the mapping are assumed to satisfy certain contractive type inequalities. We have illustrated our result with two examples. First example is presented to show that our result is a proper generalizations of the corresponding results of Bhaskar et al [Nonlinear Anal. TMA, 65 (7) (2006), 1379-1393].

Keywords: Partially ordered set, Contractive-type mapping, Mixed monotone property, Coupled fixed point.
2010 MSC: Primary 54H25; Secondary 54H10.

1. Introduction and Preliminaries

Fixed point theory in recent has developed rapidly in partially ordered metric spaces; that is, metric spaces endowed with a partial ordering. References [1], [13], [25], [26], [27] are some examples of these works. Fixed point problems have also been considered in generalisation of metric spaces endowed with partial orderings as for example in partially order cone metric spaces [23], in partially ordered G-metric spaces [4] and in partially ordered probabilistic metric spaces [12]. In their paper Bhaskar and Lakshmikantham [16]

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Received 2011-6-14

established a coupled contraction mapping principle in partially ordered metric spaces for mappings having mixed monotone property. An application of their result to differential equations has also been given in the same work. After the publication of this work, several coupled fixed point results have appeared in the literature. The work of Bhaskar et. al; was further generalized to coupled coincidence point theorems in [7] and [24] under two separate sets of sufficient conditions. Several other coupled fixed and coincidence point results were proved in works like those noted in references [3], [6], [11], [17], [29].

Geraghty [15] introduced an extension of the Banach contraction mapping principle in which the contraction constant was replaced by a function having some specified properties. The method applied by Geraghty was utilized to obtain further new fixed point results works like [2] and [10].

Definition 1.1. [15] Let S is the class of functions $\beta : \Re^+ \to [0, 1)$ with

$$(i)\ \Re^+ = \{t \in \Re / t > 0\},$$

$$(ii)\ \beta(t_n) \to 1 \ \text{ implies } \ t_n \to 0. \tag{1.1}$$

With the help of the above class of functions Geraghty [15] had established a generalisation of the Banach contraction principle.

Theorem 1.2. *[15] Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping satisfying* $d(Tx, Ty) \le \beta(d(x, y)).d(x, y)$*, for $x, y \in X$,* *where $\beta \in S$. Then T has a unique fixed point $z \in X$ and $\{T^n(x)\}$ converges to z for each $x \in X$.*

A. Amini-Harandi and H. Emami, [2] has shown that the result which Geraghty had been proved in [15] is also valid in complete partially ordered metric spaces. The following is the result of A. Amini-Harandi and H. Emami.

Theorem 1.3. *[2] Let $(X, \le)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a nondecreasing mapping such that $d(Tx, Ty) \le \beta(d(x, y)).d(x, y)$, for $x, y \in X$ with $x \le y$,*
where $\beta \in S$. Assume that either T is continuous or X satisfies the following condition:
if $\{x_n\}$ is a non decreasing sequence in X such that $x_n \to x$, then $x_n \le x \ \ \forall n \in N$.

Besides, suppose that for each $x, y \in X$there exists $z \in X$ which is comparable to x and y. If there exists $x_0 \in X$ with $x_0 \le Tx_0$, then T has a unique fixed point.
The essential feature of this line of generalisation is that the contraction constant has been replaced by a function belonging to the class S. A further extension of the above result has been done in [10].

Kannan type mappings are a class of contractive mappings which are different from Banach contraction. Like Banach contraction they have unique fixed points in complete metric spaces. However, unlike the Banach condition, there exist discontinuous functions satisfying the definition of Kannan type mappings. Following their appearance in [20], [21], many persons created contractive conditions not requiring continuity of the mappings and established fixed points results of such mappings. Today , this line of research has a vast literature . Another reason for the importance of Kannan type mappings is that it characterizes completeness which the Banach contraction principle does not. It has been shown in [30], [32] that the necessary existence of fixed points for Kannan type mappings implies that the corresponding metric space is complete. The same is not true with the Banach contractions. There is an example of an incomplete metric space where every Banach contraction has a fixed point [14]. Kannan type mappings, its generalizations and extensions in various spaces have been considered in a large number of works some of which appear in [5], [8], [9], [18], [19], [22], [28], [31].

Definition 1.4. [20, 21] A mapping $T : X \to X$, where (X, d) is a metric space, is called a Kannan type mapping if there exists $0 < \lambda < 1$ such that, for all $x, y \in X$, the following inequality holds:

$$d(Tx, Ty) \leq \frac{\lambda}{2}[d(x, Tx) + d(y, Ty)]. \tag{1.2}$$

Let $(X, \preceq)$ be a partially ordered set and $F : X \longrightarrow X$. The mapping F is said to be non-decreasing if for all $x_1,\ x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1) \preceq F(x_2)$ and non-increasing if for all $x_1 \preceq x_2$ implies $F(x_1) \succeq F(x_2)$.

Definition 1.5. [16] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \longrightarrow X$. The mapping F is said to have the mixed monotone property if F is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument; that is, for any $x,\ y \in X$,

$$x_1,\ x_2 \in X,\ \ x_1 \preceq x_2 \Longrightarrow F(x_1,\ y) \preceq F(x_2,\ y)$$

and

$$y_1,\ y_2 \in X,\ \ y_1 \preceq y_2 \Longrightarrow F(x,\ y_1) \succeq F(x,\ y_2).$$

Definition 1.6. [16] An element $(x,\ y) \in X \times X$, is called a coupled fixed point of the mapping $F : X \times X \longrightarrow X$ if $F(x,\ y) = x$ and $F(y,\ x) = y$.

The following coupled contraction mapping theorem was established by Bhaskar et al [16].

Theorem 1.7. *[16] Let $(X,\ \preceq)$ be a partially ordered set and suppose there is a metric d on X such that $(X,\ d)$ is a complete metric space. Assume that X has the following property:*
1. if a non-decreasing sequence $\{x_n\} \longrightarrow x$, then $x_n \preceq x$, for all n,
2. if a non-increasing sequence $\{y_n\} \longrightarrow y$, then $y \preceq y_n$, for all n.
Let $F : X \times X \longrightarrow X$ be a mapping having the mixed monotone property on X. Assume that there exists a $k \in [0,\ 1)$ such that for $x,\ y, u, v \in X$ with $x \succeq u,\ y \preceq v$ the following inequality holds

$$d(F(x,\ y),\ F(u,\ v)) \leq \frac{k}{2}[d(x,\ u) + d(y,\ v)]. \tag{1.3}$$

If there exists $x_0,\ y_0 \in X$ such that $x_0 \preceq F(x_0,\ y_0)$ and $y_0 \succeq F(x_0,\ y_0)$, then there exist $x,\ y \in X$ such that $x = F(x,\ y)$ and $y = F(y,\ x)$.

In the present work, following the idea of Geraghty [15], we established two coupled coincidence point theorems to generalize Banach and Kannan type contractions in partially ordered metric spaces. Two illustrative examples are given. One of our theorems extends the work of Bhaskar et. al [16].

2. Main Results

Theorem 2.1. *Let $(X,\ \preceq)$ be a partially ordered set and suppose there is a metric d on X such that $(X,\ d)$ is a complete metric space. Let $F : X \times X \longrightarrow X$ be a mapping such that F has the mixed monotone property and satisfies*

$$d(F(x, y), F(u, v)) \leq \beta(\frac{d(x, u) + d(y, v)}{2})(\frac{d(x, u) + d(y, v)}{2}), \tag{2.1}$$

for all $x, y, u, v \in X$ with $x \preceq u$ and $y \succeq v$ and $\beta \in S$. Also suppose that
(a) F is continuous or
(b) X has the following properties:

$$i)\ \text{if a non-decreasing sequence } \{x_n\} \longrightarrow x,\ \text{ then } x_n \preceq x,\ \text{for all } n \geq 0, \tag{2.2}$$

$$ii)\ \text{if a non-increasing sequence } \{y_n\} \longrightarrow y,\ \text{ then } y_n \succeq y,\ \text{for all } n \geq 0. \tag{2.3}$$

If there are $x_0,\ y_0 \in X$ such that $x_0 \preceq F(x_0,\ y_0)$ and $y_0 \succeq F(y_0,\ x_0)$, then there exist $x,\ y \in X$ such that $x = F(x,\ y)$ and $y = F(y,\ x)$, that is, F has a coupled fixed point in X.

Proof. By the condition of the theorem there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. We define $x_1, y_1 \in X$ as $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. Then $x_1 \succeq x_0$ and $y_1 \preceq y_0$. I n the same way, using the mixed monotone property of F we define $x_2 = F(x_1, y_1)$ and $y_2 = F(y_1, x_1)$. Then $x_2 = F(x_1, y_1) \succeq F(x_0, y_1) \succeq F(x_0, y_0) = x_1$ and $y_2 = F(y_1, x_1) \preceq F(y_1, x_0) \preceq F(y_0, x_0) = y_1$. Continuing the above procedure we have two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$x_{n+1} = F(x_n, y_n) \text{ and } y_{n+1} = F(y_n, x_n) \text{ for all } n \geq 0. \tag{2.4}$$

Due to the mixed monotone property of F, we have,

$$x_0 \preceq F(x_0, y_0) = x_1 \preceq F(x_1, y_1) = x_2 \preceq ... \preceq x_n = F(x_{n-1}, y_{n-1}) \preceq x_{n+1} = F(x_n, y_n) \preceq ... \tag{2.5}$$

and

$$y_0 \succeq F(y_0, x_0) = y_1 \succeq F(y_1, x_1) = y_2 ... \succeq y_n = F(y_{n-1}, x_{n-1}) \succeq y_{n+1} = F(y_n, x_n) \succeq ... \ . \tag{2.6}$$

From (2.1), (2.4), (2.5) and (2.6), for all $n \geq 1$, it follows that

$$d(x_n, x_{n+1}) = d(F(x_{n-1}, y_{n-1}), F(x_n, y_n))$$
$$\leq \beta(\frac{d(x_{n-1}, x_n) + d(y_{n-1}, y_n)}{2})(\frac{d(x_{n-1}, x_n) + d(y_{n-1}, y_n)}{2})$$

and

$$d(y_{n+1}, y_n) = d(F(y_{n-1}, x_{n-1}), F(y_n, x_n))$$
$$\leq \beta(\frac{d(y_{n-1}, y_n) + d(x_{n-1}, x_n)}{2})(\frac{d(y_{n-1}, y_n) + d(x_{n-1}, x_n)}{2}).$$

Let, for all $n \geq 0$,

$$a_n = d(x_n, x_{n+1}),\ b_n = d(y_n, y_{n+1}) \text{ and } \delta_{n+1} = d(x_n, x_{n+1}) + d(y_n, y_{n+1}). \tag{2.7}$$

Then from the above two inequalities, for all $n \geq 1$,

$$\delta_{n+1} = d(x_{n+1}, x_n) + d(y_{n+1}, y_n) \leq \beta(\frac{d(y_{n-1}, y_n) + d(x_{n-1}, x_n)}{2})(d(y_{n-1}, y_n) + d(x_{n-1}, x_n))$$

$$\begin{aligned} &\leq \beta(\frac{d(y_{n-1}, y_n) + d(x_{n-1}, x_n)}{2})\ \delta_n \\ &\leq \delta_n. \end{aligned} \tag{2.8}$$

Therefore the sequence $\{\delta_n\}$ is a monotone decreasing sequence of non-negative real numbers. Hence there exists $\delta \geq 0$ such that $\lim_{n\to\infty} \delta_n = \delta$.

$$\text{Assume } \delta > 0. \tag{2.9}$$

Then from (2.8) we have

$$\frac{\delta_{n+1}}{\delta_n} \leq \beta(\frac{d(y_{n-1}, y_n) + d(x_n, x_{n-1})}{2}) < 1.$$

Letting $n \to \infty$ in the above inequality, and using (2.9), we get

$$\lim_{n\to\infty} \beta(\frac{d(y_{n-1}, y_n) + d(x_n, x_{n-1})}{2}) = 1.$$

By virtue of (1.1), this implies that

$$\lim_{n\to\infty} \{d(x_{n-1}, x_n) + d(y_{n-1}, y_n)\} = \lim_{n\to\infty} \delta_n = 0$$

which is a contraction with (2.9). Hence

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} d(x_{n-1}, x_n) = 0 \text{ and } \lim_{n\to\infty} b_n = \lim_{n\to\infty} d(y_{n-1}, y_n) = 0. \tag{2.10}$$

Next we show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. If possible, let at least one of $\{x_n\}$ and $\{y_n\}$ be not a Cauchy sequence. Then there exists $\epsilon > 0$ and sequences of natural numbers $\{m(k)\}$ and $\{l(k)\}$ for which

$$m(k) > l(k) \geq k,$$

and such that for all $k \geq 1$, either $d(x_{l(k)}, x_{m(k)}) \geq \varepsilon$ or $d(y_{l(k)}, y_{m(k)}) \geq \varepsilon$. Then for all $k \geq 1$,

$$d_k = d(x_{l(k)}, x_{m(k)}) + d(y_{l(k)}, y_{m(k)}) \geq \varepsilon. \tag{2.11}$$

Now corresponding to $l(k)$ we can choose $m(k)$ to be the smallest positive integer for which (2.11) holds. Then, for all $k \geq 1$,

$$d(x_{l(k)}, x_{m(k)-1}) + d(y_{l(k)}, y_{m(k)-1}) < \varepsilon. \tag{2.12}$$

Further from (2.7), (2.11) and (2.12), for all $k \geq 1$, we have

$$\begin{aligned}
\varepsilon &\leq d_k = d(x_{l(k)}, x_{m(k)}) + d(y_{l(k)}, y_{m(k)}) \\
&\leq d(x_{l(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}) + d(y_{l(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)}) \\
&= d(x_{l(k)}, x_{m(k)-1}) + d(y_{l(k)}, y_{m(k)-1}) + a_{m(k)-1} + b_{m(k)-1} < \varepsilon + a_{m(k)-1} + b_{m(k)-1}.
\end{aligned}$$

Taking the limit as $k \to \infty$, and using (2.10) we have

$$\lim_{k\to\infty} d_k = \varepsilon. \tag{2.13}$$

From (2.1), (2.4), (2.5), and (2.6), for all $k \geq 1$, we obtain

$$\begin{aligned}
&d(x_{l(k)+1}, x_{m(k)+1}) = d(F(x_{l(k)}, y_{l(k)}), F(x_{m(k)}, y_{m(k)})) \\
&\leq \beta(\frac{d(x_{l(k)}, x_{m(k)}) + d(y_{l(k)}, y_{m(k)})}{2})(\frac{d(x_{l(k)}, x_{m(k)}) + d(y_{l(k)}, y_{m(k)})}{2}).
\end{aligned} \tag{2.14}$$

Also from (2.1), (2.4), (2.5), and (2.6), for all $k \geq 0$, we have

$$\begin{aligned}
&d(y_{l(k)+1}, y_{m(k)+1}) = d(F(y_{l(k)}, x_{l(k)}), F(y_{m(k)}, x_{m(k)})) \\
&\leq \beta(\frac{d(y_{l(k)}, y_{m(k)}) + d(x_{l(k)}, x_{m(k)})}{2})(\frac{d(y_{l(k)}, y_{m(k)}) + d(x_{l(k)}, x_{m(k)})}{2}).
\end{aligned} \tag{2.15}$$

Again, for all $k \geq 1$, we have

$$\begin{aligned}
d_k &= d(x_{l(k)}, x_{m(k)}) + d(y_{l(k)}, y_{m(k)}) \leq d(x_{l(k)}, x_{l(k)+1}) + d(x_{l(k)+1}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{m(k)}) \\
&\quad + d(y_{l(k)}, y_{l(k)+1}) + d(y_{l(k)+1}, y_{m(k)+1}) + d(y_{m(k)+1}, y_{m(k)}) \\
&\leq d(x_{l(k)+1}, x_{m(k)+1}) + d(y_{l(k)+1}, y_{m(k)+1}) + d(x_{l(k)}, x_{l(k)+1}) + d(x_{m(k)+1}, x_{m(k)}) \\
&\quad + d(y_{l(k)}, y_{l(k)+1}) + d(y_{m(k)+1}, y_{m(k)})
\end{aligned}$$

and

$$\begin{aligned}
d(x_{l(k)+1}, x_{m(k)+1}) + d(y_{l(k)+1}, y_{m(k)+1}) &\leq d(x_{l(k)+1}, x_{l(k)}) + d(x_{l(k)}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)+1}) \\
&\quad + d(y_{l(k)+1}, y_{l(k)}) + d(y_{l(k)}, y_{m(k)}) + d(y_{m(k)}, y_{m(k)+1}) \\
&\leq d_k + d(x_{l(k)+1}, x_{l(k)}) + d(x_{m(k)+1}, x_{m(k)}) + d(y_{l(k)+1}, y_{l(k)}) \\
&\quad + d(y_{m(k)}, y_{m(k)+1}).
\end{aligned}$$

Taking the limit as $k \to \infty$ in the above two inequalities, using (2.10) and (2.13), we have

$$\lim_{k\to\infty}\{d(x_{l(k)+1}, x_{m(k)+1}) + d(y_{l(k)+1}, y_{m(k)+1})\} = \varepsilon. \tag{2.16}$$

Adding (2.14), (2.15), and using the notation of (2.11), we have,

$$d(x_{l(k)+1}, x_{m(k)+1}) + d(y_{l(k)+1}, y_{m(k)+1}) \le 2.\beta(\frac{d_k}{2})(\frac{d_k}{2}) \le d_k \text{ (Since } \beta \in S).$$

Letting $k \to \infty$ in the above inequality, using (2.13) and (2.16) we obtain $\varepsilon \le \lim_{k\to\infty}\beta(\frac{d_k}{2}).\varepsilon \le \varepsilon$, which implies $\lim_{k\to\infty} \beta(\frac{d_k}{2}) = 1$. Since $\beta \in S$, it follows that $\lim_{k\to\infty} d_k = \lim_{k\to\infty}\{d(x_{l(k)}, x_{m(k)}) + d(y_{l(k)}, y_{m(k)})\} = 0$, which, by the virtue of (2.13), contradicts the fact that $\varepsilon > 0$. Therefore, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and hence they are convergent in the complete metric space (X, d). Let

$$x_n \to x \text{ as } n \to \infty \tag{2.17}$$

and

$$y_n \to y \text{ as } n \to \infty. \tag{2.18}$$

Next we prove that $x = F(x, y)$ and $y = F(y, x)$.
Let condition (a) of the theorem 2.1 hold, that is, F is continuous. From (2.4), (2.17) and (2.18) we have respectively

$$x = \lim_{n\to\infty} x_{n+1} = F(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = F(x, y)$$

and

$$y = \lim_{n\to\infty} y_{n+1} = F(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n) = F(y, x).$$

Let condition (b) of the theorem 2.1 hold.
From (2.5), (2.6), (2.17) and (2.18) we have that $\{x_n\}$ is non-decreasing such that $x_n \to x$ and $\{y_n\}$ is non-increasing such that $y_n \to y$ as $n \to \infty$. Then by (2.2) and (2.3) we have, for all $n \ge 0$,

$$x_n \preceq x \text{ and } y_n \succeq y. \tag{2.19}$$

Then by (2.1), and using (2.19), for all $n \ge 0$, we have

$$d(F(x, y), x_{n+1}) = d(F(x, y), F(x_n, y_n))$$
$$\le \beta(\frac{d(x, x_n) + d(y, y_n)}{2})(\frac{d(x, x_n) + d(y, y_n)}{2}) \le (\frac{d(x, x_n) + d(y, y_n)}{2}) \text{ (Since } \beta \in S).$$

Taking $n \to \infty$ in the above inequality, and using (2.17), we have $d(F(x, y), x) = 0$, that is, $x = F(x, y)$.
Similarly, we have $y = F(y, x)$.
Thus we have proved that F has a coupled fixed point in X. This completes the proof of the theorem. □

Our next theorem is a Kannan type coupled fixed point result.

Theorem 2.2. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \longrightarrow X$ be a mapping such that F has the mixed monotone property and satisfies*

$$d(F(x, y), F(u, v)) \le \beta(M(x, y, u, v)).(M(x, y, u, v)), \tag{2.20}$$

where, $M(x, y, u, v) = \dfrac{d(x, F(x, y)) + d(y, F(y, x)) + d(u, F(u, v)) + d(v, F(v, u))}{4}$

for all $x, y, u, v \in X$ with $x \preceq u$ and $y \succeq v$ and $\beta \in S$,. Also suppose that
(a) F is continuous or
(b) X has the properties noted in (2.2) and (2.3).
If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point in X.

Proof. Following the proof of the theorem (2.1) we have two sequences $\{x_n\}$ and $\{y_n\}$ given by (2.4) and satisfying (2.5) and (2.6). Then $\{x_n\}$ is an increasing sequence and $\{y_n\}$ is a decreasing sequence in X. Now by (2.1), (2.4), (2.5) and (2.6) and the fact that $\beta \in S$, we have for all $n \geq 1$,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\leq \beta(M(x_n, y_n, x_{n-1}, y_{n-1}))(M(x_n, y_n, x_{n-1}, y_{n-1})) \end{aligned}$$

and

$$\begin{aligned} d(y_{n+1}, y_n) &= d(F(y_n, x_n), F(y_{n-1}, x_{n-1})) \\ &\leq \beta(M(y_n, x_n, y_{n-1}, x_{n-1}))(M(y_n, x_n, y_{n-1}, x_{n-1})) \end{aligned}$$

Let, for all $n \geq 1$,

$$a_n = d(x_{n+1}, x_n), \quad b_n = d(y_{n+1}, y_n) \text{ and } \delta_{n+1} = d(x_{n+1}, x_n) + d(y_{n+1}, y_n). \tag{2.21}$$

Then adding the above two inequalities and using (2.21), for all $n \geq 1$, we have

$$\delta_{n+1} = d(x_{n+1}, x_n) + d(y_{n+1}, y_n) \leq 2.\beta(M(x_n, y_n, x_{n-1}, y_{n-1}))(M(x_n, y_n, x_{n-1}, y_{n-1}))$$

$$\begin{aligned} \text{where, } M(x_n, y_n, x_{n-1}, y_{n-1}) &= \frac{d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(x_{n-1}, x_n) + d(y_{n-1}, y_n)}{4} \\ &= \frac{\delta_n + \delta_{n+1}}{4} \quad \text{(by (2.21))}. \end{aligned}$$

Therefore, for all $n \geq 1$, it follows that

$$\begin{aligned} \delta_{n+1} &\leq \beta(M(x_n, y_n, x_{n-1}, y_{n-1}))\ (\frac{\delta_n + \delta_{n+1}}{2}) \qquad (2.22) \\ &\leq \frac{\delta_n + \delta_{n+1}}{2}, \quad \text{since } \beta \in S, \\ \text{that is, } \delta_{n+1} &\leq \delta_n. \end{aligned}$$

Therefore the sequence $\{\delta_n\}$ is a monotone decreasing sequence of non-negative real numbers. Hence there exists $\delta \geq 0$ such that $\lim_{n\to\infty} \delta_n = \delta$.

$$\text{Assume } \delta > 0. \tag{2.23}$$

. Then by (2.22) we have

$$\frac{\delta_{n+1}}{(\frac{\delta_n + \delta_{n+1}}{2})} \leq \beta(M(x_n, y_n, x_{n-1}, y_{n-1})) < 1.$$

Letting $n \to \infty$ in the above inequality, and using (2.23), we get

$$\lim_{n\to\infty} \beta(M(x_n, y_n, x_{n-1}, y_{n-1})) = 1, \tag{2.24}$$

Due to (1.1) the above limit in (2.24) implies that

$$\lim_{n\to\infty} \{d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(x_{n-1}, x_n) + d(y_{n-1}, y_n)\} = \lim_{n\to\infty} \{\delta_{n+1} + \delta_n\} = 0.$$

But this contradicts (2.23). Hence

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} d(x_{n+1}, x_n) = 0 \text{ and } \lim_{n\to\infty} b_n = \lim_{n\to\infty} d(y_{n+1}, y_n) = 0. \tag{2.25}$$

Next we show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. If possible, let at least one of $\{x_n\}$ and $\{y_n\}$ be not a Cauchy sequence. Then there exists $\epsilon > 0$ and sequences of natural numbers $\{m(k)\}$ and $\{l(k)\}$ for which

$$m(k) > l(k) \geq k,$$

and such that for all $k \geq 1$, either $d(x_{l(k)}, x_{m(k)}) \geq \varepsilon$ or $d(y_{l(k)}, y_{m(k)}) \geq \varepsilon$. Then for all $k \geq 1$,

$$d_k = d(x_{l(k)}, x_{m(k)}) + d(y_{l(k)}, y_{m(k)}) \geq \varepsilon. \tag{2.26}$$

Now corresponding to $l(k)$ we can choose $m(k)$ to be the smallest positive integer for which (2.26) holds. Then,

$$d(x_{l(k)}, x_{m(k)-1}) + d(y_{l(k)}, y_{m(k)-1}) < \varepsilon. \tag{2.27}$$

Further from (2.21), (2.26) and (2.27), for all $k \geq 0$, we have

$$\begin{aligned}
\varepsilon &\leq d_k = d(x_{l(k)}, x_{m(k)}) + d(y_{l(k)}, y_{m(k)}) \\
&\leq d(x_{l(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}) + d(y_{l(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)}) \\
&= d(x_{l(k)}, x_{m(k)-1}) + d(y_{l(k)}, y_{m(k)-1}) + a_{m(k)-1} + b_{m(k)-1} < \varepsilon + a_{m(k)-1} + b_{m(k)-1}.
\end{aligned}$$

Taking the limit as $k \to \infty$, and using (2.25) we have

$$\lim_{k\to\infty} d_k = \varepsilon. \tag{2.28}$$

From (2.4), (2.5), (2.6) and (2.20), for all $k \geq 0$, we obtain

$$\begin{aligned}
d(x_{l(k)}, x_{m(k)}) &= d(F(x_{l(k)-1}, y_{l(k)-1}), F(x_{m(k)-1}, y_{m(k)-1})) \\
&\leq \beta(M(x_{l(k)-1}, y_{l(k)-1}, x_{m(k)-1}, y_{m(k)-1}))(M(x_{l(k)-1}, y_{l(k)-1}, x_{m(k)-1}, y_{m(k)-1})).
\end{aligned} \tag{2.29}$$

Also by (2.4), (2.5), (2.6) and (2.20), for all $k \geq 0$, we have

$$\begin{aligned}
d(y_{l(k)}, y_{m(k)}) &= d(F(y_{l(k)-1}, x_{l(k)-1}), F(y_{m(k)-1}, x_{m(k)-1})) \\
&\leq \beta(M(y_{l(k)-1}, x_{l(k)-1}, y_{m(k)-1}, x_{m(k)-1}))(M(y_{l(k)-1}, x_{l(k)-1}, y_{m(k)-1}, x_{m(k)-1})).
\end{aligned} \tag{2.30}$$

Adding (2.29) and (2.30), we have,

$$\begin{aligned}
d_k &= d(x_{l(k)}, x_{m(k)}) + d(y_{l(k)}, y_{m(k)}) \\
&\leq 2.\beta(M(x_{l(k)-1}, y_{l(k)-1}, x_{m(k)-1}, y_{m(k)-1}))(M(x_{l(k)-1}, y_{l(k)-1}, x_{m(k)-1}, y_{m(k)-1})).
\end{aligned} \tag{2.31}$$

Further, by (2.25),

$$\begin{aligned}
&\lim_{k\to\infty} M(x_{l(k)-1}, y_{l(k)-1}, x_{m(k)-1}, y_{m(k)-1}) \\
&= \lim_{k\to\infty} \frac{d(x_{l(k)-1}, x_{l(k)}) + d(y_{l(k)-1}, y_{l(k)}) + d(x_{m(k)-1}, x_{m(k)}) + d(y_{m(k)-1}, y_{m(k)})}{4} = 0.
\end{aligned} \tag{2.32}$$

Taking $k \to \infty$ in (2.31), and using (2.28) and (2.32), we obtain $\varepsilon = 0$, which is a contradiction. Therefore, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and hence they are convergent in the complete metric space (X, d). Let

$$x_n \to x \text{ as } n \to \infty \tag{2.33}$$

and

$$y_n \to y \text{ as } n \to \infty. \tag{2.34}$$

Next we prove that $x = F(x, y)$ and $y = F(y, x)$.

Let condition (a) of the theorem 2.2 hold, that is, F is continuous. From (2.4), (2.30) and (2.31) we have respectively

$$x = \lim_{n\to\infty} x_{n+1} = F(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = F(x, y),$$

and

$$y = \lim_{n\to\infty} y_{n+1} = F(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n) = F(y, x).$$

Let condition (b) of the theorem 2.2 hold.
Using (2.5), (2.6), (2.33) and (2.34) we have that $\{x_n\}$ is non-decreasing such that $x_n \to x$ and $\{y_n\}$ is non-increasing such that $y_n \to y$ as $n \to \infty$. Then by (2.2) and (2.3) we have, for all $n \geq 0$,

$$x_n \preceq x \quad \text{and } y_n \succeq y. \tag{2.35}$$

By the virtue of (2.35), and since $\beta \in S$, from (2.20), for all $n \geq 0$, we have

$$\begin{aligned} d(F(x,y), x_{n+1}) &= d(F(x,y), F(x_n, y_n)) \\ &\leq \beta(\frac{d(x,F(x,y)) + d(y,F(y,x)) + d(x,x_{n+1}) + d(y,y_{n+1})}{4}) \\ &\quad (\frac{d(x,F(x,y)) + d(y,F(y,x)) + d(x,x_{n+1}) + d(y,y_{n+1})}{4}) \\ &< (\frac{d(x,F(x,y)) + d(y,F(y,x)) + d(x,x_{n+1}) + d(y,y_{n+1})}{4}). \end{aligned} \tag{2.36}$$

Similarly, due to (2.35) and the fact that $\beta \in S$, from (2.20), for all $n \geq 0$, we have

$$\begin{aligned} d(F(y,x), y_{n+1}) &= d(F(y,x), F(y_n, x_n)) \\ &\leq \beta(\frac{d(x,F(x,y)) + d(y,F(y,x)) + d(x,x_{n+1}) + d(y,y_{n+1})}{4}) \\ &\quad (\frac{d(x,F(x,y)) + d(y,F(y,x)) + d(x,x_{n+1}) + d(y,y_{n+1})}{4}) \\ &< (\frac{d(x,F(x,y)) + d(y,F(y,x)) + d(x,x_{n+1}) + d(y,y_{n+1})}{4}). \end{aligned} \tag{2.37}$$

Adding (2.36) and (2.37), for all $n \geq 0$, we obtain

$$d(F(x,y), x_{n+1}) + d(F(y,x), y_{n+1}) < 2.(\frac{d(x,F(x,y)) + d(y,F(y,x)) + d(x,x_{n+1}) + d(y,y_{n+1})}{4})$$

Taking $n \to \infty$, and using (2.25), (2.33) and (2.34), we have

$$d(F(x,y), x) + d(F(y,x), y) \leq (\frac{d(F(x,y), x) + d(F(y,x), y)}{2}),$$

which implies that $d(F(x,y), x) + d(F(y,x), y) = 0$, that is, $x = F(x,y)$ and $y = F(y,x)$.
Thus (x,y) is a coupled fixed point of F in X. This completes the proof of the theorem 2.2. □

Remark 2.3. Since inequality (2.20) bears the same idea of Kannan's inquality described in (1.2), we recognised (2.20) as a coupled Kannan type inequality.

3. Example

In this section we have two examples which illustrated the results of theorems 2.1 and 2.2 respectively.

Example 3.1. Let $X = [0,1]$. Then $(X, \preceq)$ is a partially ordered set with $x \preceq y$ whenever $x \geq y$. Let
$d(x,y) = |x - y|$ for $x, y \in [0,1]$.

Then (X,d) is a complete metric space.

Let $F : X \times X \to X$ be defined as

$$F(x,y) = \begin{cases} \frac{1}{2}[(x-y) - \frac{1}{2}(x-y)^2], & \text{if } \quad x, y \in [0,1], \ x \geq y, \\ 0, & \text{if } \quad x < y. \end{cases}$$

Let $\beta : [0,\infty) \to [0,1)$ be defined as

$$\beta(t) = \begin{cases} 1-t, & \text{if } t \leq 1, \\ \alpha < 1, & \text{if } t > 1. \end{cases}$$

For the two points, $x_0 = 0$ and $y_0 = c > 0$ in X we have $x_0 = 0 = F(x_0, y_0)$ and $y_0 = c > \frac{c}{2}[1 - \frac{c}{2}] = F(y_0, x_0)$.

Let $x \succeq u$ and $y \preceq v$ (equivalently $u \geq x$ and $y \geq v$). The inequality (2.1) is trivially satisfied except in the following two cases.

Case-1 $x \geq y$ and $u \geq v$ $(x \preceq y$ and $u \preceq v)$.
Then $d(F(x,y), F(u,v)) = d(\frac{1}{2}[(x-y) - \frac{1}{2}(x-y)^2], \frac{1}{2}[(u-v) - \frac{1}{2}(u-v)^2])$
$= \frac{1}{2}(u-v) - \frac{1}{4}(u-v)^2 - \frac{1}{2}(x-y) + \frac{1}{4}(x-y)^2$

$= \frac{1}{2}[(u-x) + (y-v)] - \frac{1}{4}[\{(u-v) + (x-y)\}\{(u-v) - (x-y)\}]$

$\leq \frac{1}{2}[(u-x) + (y-v)] - \frac{1}{4}[\{(u-v) - (x-y)\}^2]$

$= \frac{1}{2}[(u-x) + (y-v)] - \frac{1}{4}[\{(u-x) + (y-v)\}^2]$.

Therefore,
$d(F(x,y), F(u,v))) \leq \frac{1}{2}[(u-x) + (y-v)] - \frac{1}{4}[(u-x) + (y-v)]^2$
$= \left[\frac{(u-x) + (y-v)}{2}\right] - \left[\frac{(u-x) + (y-v)}{2}\right]^2$
$= \left(1 - \left[\frac{(u-x) + (y-v)}{2}\right]\right)\left[\frac{(u-x) + (y-v)}{2}\right]$
$= \beta(\frac{d(x,u) + d(y,v)}{2})(\frac{d(x,u) + d(y,v)}{2})$.

Case-2 $x < y$ and $u \geq v$ $(y \prec x$ and $u \preceq v)$.
We have, $d(F(x,y), F(u,v)) = d(0, \frac{1}{2}(u-v) - \frac{1}{4}(u-v)^2) = \frac{1}{2}(u-v) - \frac{1}{4}(u-v)^2$
$= \frac{1}{2}(u-v+x-x) - \frac{1}{4}(u-v+x-x)^2$

$< \frac{1}{2}(u-x-v+y) - \frac{1}{4}(u-v+y-x)^2$ (since $y > x$)

$= \frac{1}{2}[(u-x) + (y-v)] - \frac{1}{4}[(u-x) + (y-v)]^2$.
Then,
$d(F(x,y), F(u,v)) < \frac{1}{2}[(u-x) + (y-v)] - \frac{1}{4}[(u-x) + (y-v)]^2$
$= \left[\frac{(u-x) + (y-v)}{2}\right] - \left[\frac{(u-x) + (y-v)}{2}\right]^2 = \left(1 - \left[\frac{(u-x) + (y-v)}{2}\right]\right)\left[\frac{(u-x) + (y-v)}{2}\right]$
$= \beta(\frac{d(x,u) + d(y,v)}{2})(\frac{d(x,u) + d(y,v)}{2})$.

Thus in both of the above two cases inequality (2.1) is satisfied. Also the functions F and β satisfy all the conditions required by them in theorem 2.1. Then, by an application of theorem2.1, F has a coupled fixed point. Here $(0,0)$ is a coupled fixed point of F in X.

Example 3.2. Let $X = [0,1]$. Then X with the usual order "$\leq$" be a partially ordered set. Let d be the usual metric on X. Then (X,d) is a complete metric space.
We define $F : X \times X \to X$ as

$$F(x,y) = \begin{cases} \frac{1}{16}, & \text{if } x \geq \frac{1}{2} \text{ and } y < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $$\beta(t) = \begin{cases} 1 - \dfrac{t}{8}, & \text{if } t \le 1, \\ \alpha < 1, & \text{if } \ t > 1. \end{cases}$$

Then all the properties required by F and β in the theorem 2.2 are satisfied.
The inequality (2.20) is trivially satisfied except in the following two cases.

Case-1: $x \in [\frac{1}{2}, 1]$, $y \in [0, \frac{1}{2})$. and $u, v \in [\frac{1}{2}, 1]$

$d(F(x, y), F(u, v)) = \frac{1}{16}$

Now $$\frac{d(x, F(x, y)) + d(y, F(y, x)) + d(u, F(u, v)) + d(v, F(v, u))}{4} = \frac{x - \frac{1}{16} + y + u + v}{4}$$

Then the inequality (2.20) becomes $\frac{1}{16} \le (\frac{x - \frac{1}{16} + y + u + v}{4}) - \frac{(x - \frac{1}{16} + y + u + v)^2}{32}$.

Since $\frac{3}{2} \le x + y + u + v \le \frac{7}{2}$, the right hand side of the above inequality is bounded below by 0.35, Hence (2.20) is satisfied.

Case-2: $x \in [\frac{1}{2}, 1]$, $y \in [0, \frac{1}{2})$ and $u, v \in [0, \frac{1}{2}]$

$d(F(x, y), F(u, v)) = \frac{1}{16}$

Therefore, $\frac{1}{16} \le (\frac{x - \frac{1}{16} + y + u + v}{4}) - \frac{(x - \frac{1}{16} + y + u + v)^2}{32}$.

Since $\frac{1}{2} \le x + y + u + v \le \frac{5}{2}$, therefore the value of the right hand side is greater than $\frac{1}{16}$, which shows that (2.20) is satisfied.

This shows that theorem 2.2 is applicable to this example Here $(0, 0)$ is a coupled fixed point of F in X.

Remark: If, in particular, we consider the function $\beta(t) = k$, $0 < k < 1$, then the inequality (2.1) reduces to inequality (1.3), thus the result of Bhaskar et al; is a special case of theorem 2.1.
Then result of T. Gnana Bhaskar and V. Lakshmikantham in [16] is not applicable to example 3.1. Further inequality (1.3) is not satisfied for the choice of $x = 0, y = 0, u = c$ and $v = 0$. This shows that the result of theorem 2.1 is an actual improvement over the corresponding results of Bhaskar et. al [16].

Acknowledgements

The work is partially supported by Council of Scientific and Industrial Research, India(Project No. 25(0168)/ 09/EMR-II). The first author gratefully acknowledges the support.

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