



Existence of solutions for fractional impulsive neutral functional infinite delay integrodifferential equations with nonlocal conditions

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This paper is dedicated to Professor Ljubomir Ćirić

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Abstract

This paper is mainly concerned with the existence of solutions for fractional impulsive neutral functional integrodifferential equations with nonlocal initial conditions and infinite delay. The results are obtained by the fixed point theorem. ©2012. All rights reserved.

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1. Introduction

Fractional Calculus deals with the generalization of integrals and derivatives of noninteger order. Fractional calculus involves a wide area of applications by bringing into a broader paradigm concepts of physics, mathematics and engineering [11, 13]. Infact fractional differential equation is considered as an alternative model to nonlinear differential equations [8]. In [2, 12], the authors have proved the existence of solutions of abstract fractional differential equations by using fixed point techniques. In consequence, the subject of fractional differential equations is gaining much importance and attention. For details, see [14, 15, 16] and the

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references therein. Subsequently several authors have discussed the problem for different types of nonlinear differential and integro differential equations including functional differential equations in Banach spaces.

The theory of impulsive differential equations has undergone rapid development over the years and played a very important role in modern applied mathematical models of real processes arising in phenomena studied in physics, population dynamics, chemical technology and economics. In [1, 7], Benchohra et al. established sufficient conditions for the existence of solutions for a class of initial value problems for impulsive fractional differential equations involving the Caputo fractional derivative of order $0 < q \leq 1$ and $1 < q \leq 2$. In [10], Mophou proved the existence and uniqueness results of a mild solution to impulsive fractional semilinear differential equations. Anguraj and Karthikeyan [3] proved Existence for impulsive neutral integrodifferential inclusions with nonlocal initial conditions via fractional operators. Benchohra and Seba [6] studied the existence of fractional impulsive differential equations in Banach spaces while Balachandran and Kiruthika [5] discussed the existence of nonlocal Cauchy problem for semilinear fractional evolution equations. Balachandran and Trujillo [4] investigated the nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces.

In this paper, we consider the following fractional impulsive neutral integrodifferential systems with infinite delay

$$\begin{cases} D_t^q(x(t) - u(t, x_t)) = A(t, x)(x(t) - u(t, x_t)) + f(t, x_t, \int_0^t h(t, s, x_s) ds), \\ \qquad \qquad \qquad t \in J = [0, b], \quad t \neq t_k \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)), \qquad \qquad t = t_k, \quad k = 1, 2, \dots, n \\ x(0) + g(x) = \phi, \quad \phi \in B_\vartheta \end{cases} \tag{1.1}$$

where $0 < q < 1$ and the state $x(\cdot)$ belongs to Banach space X endowed with the norm $\|\cdot\|$. Operator A generates a strongly continuous bounded linear operator on a Banach space X . D_t^q is the Caputo fractional derivative. u and f are two given continuous functions, $I_k : X \rightarrow X$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ with $x(t_k^+) = \lim_{h \rightarrow 0^+} x(t_k + h)$, $x(t_k^-) = \lim_{h \rightarrow 0^-} x(t_k + h)$, $k = 1, 2, 3, \dots, n$, $0 = t_0 < t_1 < t_2 < \dots < t_n < t_{n+1} = b$. Let $x_t(\cdot)$ denote $x_t(\theta) = x(t + \theta)$, $\theta \in (-\infty, 0]$.

The rest of this paper is organized as follows. In Section 2, some preliminaries are presented. In Section 3, we study the existence and the uniqueness of solutions for the impulsive fractional system 1.1. In Section 4, an example.

2. Preliminaries

In this section, we shall introduce some basic definitions, notations, lemmas and proposition which are used throughout this paper.

Assume that $\vartheta : (-\infty, 0] \rightarrow (0, +\infty)$ is a continuous function satisfy $\ell = \int_{-\infty}^0 \vartheta(t) dt < +\infty$. The Banach space $(B_\vartheta, \|\cdot\|_{B_\vartheta})$ induced by the function ϑ is defined as follows

$$B_\vartheta = \left\{ \varphi : (-\infty, 0] \rightarrow X : \text{for any } c > 0, \varphi(\theta) \text{ is a bounded and measurable function on } [-c, 0] \text{ and } \int_{-\infty}^0 \vartheta(t) \sup_{t \leq \theta \leq 0} |\varphi(\theta)| dt < +\infty \right.$$

endowed with the norm $\|\varphi\|_{B_\vartheta} = \int_{-\infty}^0 \vartheta(s) \sup_{s \leq \theta \leq 0} |\varphi(\theta)| ds$.

Let us define the space

$$B_{\vartheta'} = \left\{ \varphi : (-\infty, b] \rightarrow X : \varphi_k \in C(J_k, X), \quad k = 0, 1, 2, \dots, n \text{ and there exist } \varphi(t_k^-) \text{ and } \varphi(t_k^+) \text{ with } \varphi(t_k) = \varphi(t_k^-), \varphi_0 = \varphi(0) + g(\varphi) = \phi \in B_\vartheta \right.$$

where φ_k is the restriction of φ to J_k , $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \dots, n$. Denote by $\|\cdot\|_{B_{\vartheta'}}$, a seminorm in the space $B_{\vartheta'}$, which is defined by

$$\|\varphi\|_{B_{\vartheta'}} = \|\varphi\|_{B_\vartheta} + \max \|\varphi_k\|_{J_k}, \quad k = 1, 2, \dots, n \text{ where } \|\varphi_k\|_{J_k} = \sup_{s \in J_k} \|\varphi_k(s)\|.$$

Definition 2.1. A function $x : (-\infty, b] \rightarrow X$ is said to be a solution of system 1.1 if $x(0)+g(x) = \phi \in B_{\mathcal{D}}$, the impulsive condition $\Delta x|_{t=t_k} = I_k(x(t_k^-))$, $k = 1, 2, \dots, n$ is verified, the restriction of $x(\cdot)$ to the interval $J_k(k = 0, 1, 2, \dots, n)$ is continuous and the following integral equation holds for $t \in J$,

$$x(t) = \begin{cases} [\phi(0) - g(x) - u(0, \phi)] + u(t, x_t) \\ + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s, x)x(s)ds \\ + \frac{1}{\Gamma(q)} \int_{t_k}^t (t - s)^{q-1} A(s, x)x(s)ds \\ - \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s, x)u(s, x_s)ds \\ - \frac{1}{\Gamma(q)} \int_{t_k}^t (t - s)^{q-1} A(s, x)u(s, x_s)ds \\ + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, x_s, \int_0^s h(s, \tau, x_\tau)d\tau)ds \\ + \frac{1}{\Gamma(q)} \int_{t_k}^t (t - s)^{q-1} f(s, x_s, \int_0^s h(s, \tau, x_\tau)d\tau)ds \\ + \sum_{0 < t_k < t} I_k(x(t_k^-)) \end{cases} \tag{2.1}$$

Definition 2.2. The Riemann - Liouville fractional integral operator of order $q \geq 0$ of function $f \in L^1(\mathbb{R}^+)$ is defined as

$$I_{0+}^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s)ds, \quad t > 0$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.3. The Caputo fractional derivative of order $q \geq 0, n - 1 < q < n$, is defined as

$$D_{0+}^q f(t) = \frac{1}{\Gamma(n - q)} \int_0^t (t - s)^{(n-q-1)} f^{(n)}(s)ds, \quad t > 0$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$.

If $0 < q < 1$, then

$$D_{0+}^q f(t) = \frac{1}{\Gamma(1 - q)} \int_0^t (t - s)^{(-q)} f^{(1)}(s)ds$$

where $f^{(1)}(s) = Df(s) = \frac{df(s)}{ds}$ and f is an abstract function with values in X .

We shall state some properties of the operators I_{0+}^α and D_{0+}^α .

Lemma 2.4. . For $\alpha, \beta > 0$ and f as a suitable function, we have

(i) $I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t)$

(ii) $I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t)$

(iii) $I^\alpha (f(t) + g(t)) = I^\alpha f(t) + I^\alpha g(t)$

(iv) $I^\alpha {}^c D^\alpha f(t) = f(t) - f(0), 0 < \alpha < 1$

(v) ${}^c D^\alpha I^\alpha f(t) = f(t)$

(vi) ${}^c D^\alpha f(t) = I^{(1-\alpha)} Df(t) = I^{(1-\alpha)} f'(t), 0 < \alpha < 1, D = \frac{d}{dt}$

(vii) ${}^c D^\alpha {}^c D^\beta f(t) \neq {}^c D^{(\alpha+\beta)} f(t)$

$$(viii) \quad {}^c D^\alpha {}^c D^\beta f(t) \neq {}^c D^\beta {}^c D^\alpha f(t)$$

Zhang and Xiyue Huang [18] proved the existence and uniqueness of mild solutions for impulsive fractional equations with nonlocal conditions and infinite delay, in which A is a infinitesimal generator of strongly continuous semigroup. But in [4], Balachandran and Trujillo observed that both the R - L and the Caputo fractional differential operators do not possess neither semigroup nor commutative properties, which are inherent to the derivatives on integer order.

Theorem 2.5. ([17])

Let B be a convex, bounded and closed subset of a Banach space X and $N : B \rightarrow B$ be a condensing map. Then N has a fixed point in B .

Lemma 2.6. ([9])

Assume that $x \in B_{\vartheta'}$ then, for $t \in J$, $x_t \in B_{\vartheta}$. Moreover

$$\ell \|x(t)\| \leq \|x_t\|_{B_{\vartheta}} \leq \|\phi\|_{B_{\vartheta}} + \ell \sup_{s \in [0,t]} \|x(s)\|.$$

3. Main results

For $\phi \in B_{\vartheta}$, we define $\hat{\phi}$ by

$$\hat{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0] \\ \phi(0), & t \in J \end{cases}$$

then $\hat{\phi} \in B_{\vartheta'}$.

Let $x(t) = y(t) + \hat{\phi}(t)$, $-\infty < t < b$.

It is evident that y satisfies $y_0 = 0$, $t \in (-\infty, 0]$, and

$$\begin{aligned} x(t) = & [-g(y + \hat{\phi}) - u(0, \phi)] + u(t, y_t + \hat{\phi}_t) \\ & + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s, y + \hat{\phi})(y + \hat{\phi})(s) ds \\ & + \frac{1}{\Gamma(q)} \int_{t_k}^t (t - s)^{q-1} A(s, y + \hat{\phi})(y + \hat{\phi})(s) ds \\ & - \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s, y + \hat{\phi})u(s, y_s + \hat{\phi}_s) ds \\ & - \frac{1}{\Gamma(q)} \int_{t_k}^t (t - s)^{q-1} A(s, y + \hat{\phi})u(s, y_s + \hat{\phi}_s) ds \\ & + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, y_s + \hat{\phi}_s, \int_0^s h(s, \tau, y_\tau + \hat{\phi}_\tau) d\tau) ds \\ & + \frac{1}{\Gamma(q)} \int_{t_k}^t (t - s)^{q-1} f(s, y_s + \hat{\phi}_s, \int_0^s h(s, \tau, y_\tau + \hat{\phi}_\tau) d\tau) ds \\ & + \sum_{0 < t_k < t} I_k(y(t_k^-) + \hat{\phi}(t_k^-)), \end{aligned} \quad t \in J$$

if and only if x satisfies $x(t) = \phi(t)$, $t \in (-\infty, 0]$ and x satisfies equation 2.1.

For brevity let us take

$$H(x_s) = \int_0^s h(s, \tau, x_\tau) d\tau.$$

We assume the following conditions to prove the existence of solution of the equation 1.1.

- (H₁) $A : J \times B_{\vartheta} \rightarrow B(X)$ is a continuous bounded linear operator and there exists a constant $M > 0$, such that $\|A(t, x) - A(t, y)\| \leq M\|x - y\|_{B_{\vartheta}}$, for all $x, y \in B_{\vartheta}$.
- (H₂) The function $u : J \times B_{\vartheta} \rightarrow X$, and there exist two positive constants λ_1 and λ_2 such that the function satisfies the Lipschitz condition $\|u(s, x_t) - u(s, y_t)\| \leq \lambda_1(\|x_t - y_t\|_{B_{\vartheta'}})$, $\lambda_2 = \sup_{t \in J} \|u(t, 0)\|$.
- (H₃) $f : J \times B_{\vartheta} \times X \rightarrow X$, and there exist two positive constants K_1, K_2 such that $\|f(t, \phi_1, y_1) - f(t, \phi_2, y_2)\| \leq K_1(\|\phi_1 - \phi_2\|_{B_{\vartheta}} + \|y_1 - y_2\|)$, $K_2 = \sup_{t \in J} \|f(t, 0, 0)\|$.
- (H₄) $h : \Delta \times B_{\vartheta} \rightarrow X$, where $\Delta = \{(t, s) : 0 \leq s \leq b\}$, equipped with positive constants P_1, P_2 satisfying $\|h(t, s, \phi_1) - h(t, s, \phi_2)\| \leq P_1(\|\phi_1 - \phi_2\|_{B_{\vartheta'}})$, $P_2 = \sup_{(t,s)} \|h(t, s, 0)\|$.
- (H₅) $I_k : X \rightarrow X$ are continuous, and there exists a constant $\mu > 0$ such that $\|I_k(x) - I_k(y)\| \leq \mu\|x - y\|$, $k = 1, 2, 3, \dots, n$.
- (H₆) $g : B_{\vartheta'} \rightarrow X$ is continuous and there exist some positive constant δ_1, δ_2 such that $\|g(x) - g(y)\| \leq \delta_1\|x - y\|_{B_{\vartheta'}}$ and $\|g(x)\| \leq \delta_1\|x\|_{B_{\vartheta'}} + \delta_2$.
- (H₇) $[\delta_1 L + \delta_2 + \lambda_1(L_1 + \|\phi\|_{B_{\vartheta}}) + 2\lambda_2 + T + \mu n(r + \ell^{-1}\|\phi\|_{B_{\vartheta}} + |\phi(0)|)] \leq r$
 where $T = \frac{b^q(n+1)}{\Gamma(q+1)} [(ML + K)(L + \lambda_1 L_1 + \lambda_2) + K_1 L_1 + K_1 b(P_1 L_1 + P_2) + K_2]$
 and $L = r + \|\phi\|_{B_{\vartheta}} + |\phi(0)|$, $L_1 = \ell(r + |\phi(0)|) + \|\phi\|_{B_{\vartheta}}$.

Theorem 3.1. *Suppose that conditions (H₁) – (H₇) are satisfied with $\delta_1 + \lambda_1 \ell < 1$ then system 1.1 has a solution.*

Proof 1. Define $\Theta : B_{\vartheta'} \rightarrow B_{\vartheta'}$ by

$$\begin{aligned}
 \Theta y(t) &= 0, & t \in (-\infty, 0] \\
 \Theta y(t) &= [-g(y + \hat{\phi}) - u(0, \phi)] + u(t, y_t + \hat{\phi}_t) \\
 &+ \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s, y + \hat{\phi})(y + \hat{\phi})(s) ds \\
 &+ \frac{1}{\Gamma(q)} \int_{t_k}^t (t - s)^{q-1} A(s, y + \hat{\phi})(y + \hat{\phi})(s) ds \\
 &- \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s, y + \hat{\phi})u(s, y_s + \hat{\phi}_s) ds \\
 &- \frac{1}{\Gamma(q)} \int_{t_k}^t (t - s)^{q-1} A(s, y + \hat{\phi})u(s, y_s + \hat{\phi}_s) ds \\
 &+ \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, y_s + \hat{\phi}_s, H(y_s + \hat{\phi}_s)) ds \\
 &+ \frac{1}{\Gamma(q)} \int_{t_k}^t (t - s)^{q-1} f(s, y_s + \hat{\phi}_s, H(y_s + \hat{\phi}_s)) ds \\
 &+ \sum_{0 < t_k < t} I_k(y(t_k^-) + \hat{\phi}(t_k^-)), & t \in J.
 \end{aligned}$$

Clearly, y is a fixed point of Θ then $y + \hat{\phi}$ is a solution of the system 1.1. We shall show that Θ satisfies the hypotheses of Theorem 2.5.

Define the Banach space $(B_{\vartheta''}, \|\cdot\|_{B_{\vartheta''}})$ induced by $B_{\vartheta'}$,

$B_{\vartheta''} = \{y \in B_{\vartheta'} : y_0 = 0 \in B_{\vartheta'}\}$ with norm $\|y\|_{B_{\vartheta''}} = \sup\{|y(s)| : s \in [0, b]\}$.

Set $B_r = \{y \in B_{\vartheta''} : \|y\|_{B_{\vartheta''}} \leq r\}$ for some $r > 0$ then B_r , for each r , is bounded, closed convex subset of X . For any $y \in B_r$, by Lemma 2.6, we have

$$\|y_t + \hat{\phi}_t\|_{B_{\vartheta}} \leq \|\phi\|_{B_{\vartheta}} + \ell[r + |\phi(0)|],$$

$$\|y + \hat{\phi}\|_{B_{\vartheta'}} \leq r + \|\phi\|_{B_{\vartheta}} + |\phi(0)|,$$

$$\sup_{t \in J} |y(t) + \hat{\phi}(t)| \leq r + \ell^{-1} \|\phi\|_{B_{\vartheta}} + |\phi(0)|.$$

Now we proceed in Two steps.

Step I: We claim that there exists a positive integer $r \in N$ such that $\Theta(B_r) \subset B_r$.

There exists a positive number r such that B_r is clearly a closed bounded convex set in $B_{\vartheta'}$.

For each positive integer r , there exist $y_r \in B_r$ and $t(r) \in (-\infty, b]$ such that

$$\begin{aligned} \|\Theta(y^r)(t(r))\| &\leq \|[-g(y^r + \hat{\phi}) - u(0, \phi)]\| + \|u(t(r), y_{t(r)} + \hat{\phi}_{t(r)})\| \\ &+ \frac{1}{\Gamma(q)} \sum_{0 < t_k < t(r)} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|A(s, y^r + \hat{\phi})(y^r + \hat{\phi})(s)\| ds \\ &+ \frac{1}{\Gamma(q)} \int_{t_k}^{t(r)} (t(r) - s)^{q-1} \|A(s, y^r + \hat{\phi})(y^r + \hat{\phi})(s)\| ds \\ &- \frac{1}{\Gamma(q)} \sum_{0 < t_k < t(r)} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|A(s, y^r + \hat{\phi})u(s, y_s^r + \hat{\phi}_s)\| ds \\ &- \frac{1}{\Gamma(q)} \int_{t_k}^{t(r)} (t(r) - s)^{q-1} \|A(s, y^r + \hat{\phi})u(s, y_s^r + \hat{\phi}_s)\| ds \\ &+ \frac{1}{\Gamma(q)} \sum_{0 < t_k < t(r)} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|f(s, y_s^r + \hat{\phi}_s, H(y_s^r + \hat{\phi}_s))\| ds \\ &+ \frac{1}{\Gamma(q)} \int_{t_k}^{t(r)} (t(r) - s)^{q-1} \|f(s, y_s^r + \hat{\phi}_s, H(y_s^r + \hat{\phi}_s))\| ds \\ &+ \sum_{0 < t_k < t(r)} \|I_k(y^r(t_k^-) + \hat{\phi}(t_k^-))\| \\ &\leq \delta_1 L + \delta_2 + \lambda_1(L_1 + \|\phi\|_{B_{\vartheta}}) + 2\lambda_2 \\ &+ \frac{b^q(n+1)}{\Gamma(q+1)} [(ML + K)(L + \lambda_1 L_1 + \lambda_2) + K_1 L_1 \\ &+ K_1 b(P_1 L_1 + P_2) + K_2] + \mu n(r + \ell^{-1} \|\phi\|_{B_{\vartheta}} + |\phi(0)|) \\ &\leq r \end{aligned}$$

where $L = r + \|\phi\|_{B_{\vartheta}} + |\phi(0)|$, $L_1 = \ell(r + |\phi(0)|) + \|\phi\|_{B_{\vartheta}}$.

Using (H_7) , for some positive integer r , $\Theta(B_r) \subset B_r$.

Step II: Now we claim that the operator $\Theta = \Theta_1 + \Theta_2$ is condensing, that is Θ_1 is a contraction and Θ_2 is compact.

The operators Θ_1 and Θ_2 are defined on B_r respectively by,

$$(\Theta_1 y)(t) = \begin{cases} 0, & t \in (-\infty, 0] \\ [-g(y + \hat{\phi}) - u(0, \phi)] + u(t, y_t + \hat{\phi}_t), & t \in J. \end{cases}$$

$$(\Theta_2 y)(t) = \begin{cases} 0, & t \in (-\infty, 0] \\ \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s, y + \hat{\phi})(y + \hat{\phi})(s) ds \\ \quad + \frac{1}{\Gamma(q)} \int_{t_k}^t (t - s)^{q-1} A(s, y + \hat{\phi})(y + \hat{\phi})(s) ds \\ \quad - \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s, y + \hat{\phi})u(s, y_s + \hat{\phi}_s) ds \\ \quad - \frac{1}{\Gamma(q)} \int_{t_k}^t (t - s)^{q-1} A(s, y + \hat{\phi})u(s, y_s + \hat{\phi}_s) ds \\ \quad + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, y_s + \hat{\phi}_s, H(y_s + \hat{\phi}_s)) ds \\ \quad + \frac{1}{\Gamma(q)} \int_{t_k}^t (t - s)^{q-1} f(s, y_s + \hat{\phi}_s, H(y_s + \hat{\phi}_s)) ds \\ \quad + \sum_{0 < t_k < t} I_k(y(t_k^-) + \hat{\phi}(t_k^-)), & t \in J. \end{cases}$$

We take $y_1, y_2 \in B_r$ arbitrarily.

By (H_2) and (H_6) , we have

$$\|(\Theta_1 y_1)(t) - (\Theta_1 y_2)(t)\| \leq \delta_1 \|y_1 - y_2\|_{B_{\vartheta'}} + \lambda_1 \ell \|y_1 - y_2\|_{B_{\vartheta'}}$$

since $\|y_t\|_{B_{\vartheta}} \leq \ell \|y\|_{B_{\vartheta'}}$

$$\begin{aligned} \|(\Theta_1 y_1)(t) - (\Theta_1 y_2)(t)\| &\leq (\delta_1 + \lambda_1 \ell) \|y_1 - y_2\|_{B_{\vartheta'}} \\ &\leq \|y_1 - y_2\|_{B_{\vartheta'}} \end{aligned}$$

since $\delta_1 + \lambda_1 \ell < 1$. Therefore Θ_1 is a contraction.

Next, we prove that Θ_2 is continuous on B_r .

Let $\{y_m\}_{k=0}^\infty \subseteq B_r$, with $y_m \rightarrow y$ in B_r . By $(H1)$, $(H3)$, $(H4)$ and $(H5)$ we have

$$\begin{aligned} \|(\Theta_2 y)(t) - (\Theta_2 y_m)(t)\| &\leq \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|A(s, y + \hat{\phi})(y + \hat{\phi})(s) \\ &\quad - A(s, y_m + \hat{\phi})(y_m + \hat{\phi})(s)\| ds \\ &\quad + \frac{1}{\Gamma(q)} \int_{t_k}^t (t - s)^{q-1} \|A(s, y + \hat{\phi})(y + \hat{\phi})(s) \\ &\quad - A(s, y_m + \hat{\phi})(y_m + \hat{\phi})(s)\| ds \\ &\quad + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|A(s, y + \hat{\phi})u(s, y_s + \hat{\phi}_s) \\ &\quad - A(s, y_m + \hat{\phi})u(s, y_{ms} + \hat{\phi}_s)\| ds \\ &\quad + \frac{1}{\Gamma(q)} \int_{t_k}^t (t - s)^{q-1} \|A(s, y + \hat{\phi})u(s, y_s + \hat{\phi}_s) \\ &\quad - A(s, y_m + \hat{\phi})u(s, y_{ms} + \hat{\phi}_s)\| ds \\ &\quad + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|f(s, y_s + \hat{\phi}_s, H(y_s + \hat{\phi}_s)) \\ &\quad - f(s, y_{ms} + \hat{\phi}_s, H(y_{ms} + \hat{\phi}_s))\| ds \\ &\quad + \frac{1}{\Gamma(q)} \int_{t_k}^t (t - s)^{q-1} \|f(s, y_s + \hat{\phi}_s, H(y_s + \hat{\phi}_s)) \\ &\quad - f(s, y_{ms} + \hat{\phi}_s, H(y_{ms} + \hat{\phi}_s))\| ds \\ &\quad + \sum_{0 < t_k < t} \|I_k(y(t_k^-) + \hat{\phi}(t_k^-)) - I_k(y_m(t_k^-) + \hat{\phi}(t_k^-))\| \end{aligned}$$

$$\begin{aligned} &\leq \frac{(n+1)b^q}{\Gamma(q+1)} [(M[r + \|\phi\|_{B_{\vartheta}} + |\phi(0)|] + K)(1 + \ell\lambda_1) \\ &\quad + M(r + |\phi(0)| + \|\phi\|_{B_{\vartheta}} + \lambda_1[\ell(r + |\phi(0)|) \\ &\quad + \|\phi\|_{B_{\vartheta}}] + \lambda_2) + K_1\ell(1 + \frac{P_1b}{(q+1)})] \|y - y_m\|_{B_{\vartheta'}} \\ &\quad + \sum_{0 < t_k < t} \|I_k(y(t_k^-) + \hat{\phi}(t_k^-)) - I_k(y_m(t_k^-) + \hat{\phi}(t_k^-))\|. \end{aligned}$$

$$\|(\Theta_2 y)(t) - (\Theta_2 y_m)(t)\| \rightarrow 0 \text{ as } m \rightarrow \infty$$

Thus, Θ_2 is continuous.

Next, we prove that $\{\Theta_2 y : y \in B_r\}$ is a family of equicontinuous functions.

Let $0 < t_1 < t_2 \leq b$. Then

$$\|(\Theta_2 y)(t_2) - (\Theta_2 y)(t_1)\| \leq I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7$$

where

$$\begin{aligned} I_1 &= \frac{1}{\Gamma(q)} \left[\left\| \sum_{0 < t_k < t_2} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s, y + \hat{\phi})(y + \hat{\phi})(s) ds \right. \right. \\ &\quad \left. \left. - \sum_{0 < t_k < t_1} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s, y + \hat{\phi})(y + \hat{\phi})(s) ds \right\| \right], \\ I_2 &= \frac{1}{\Gamma(q)} \left[\left\| \int_{t_k}^{t_2} (t_2 - s)^{q-1} A(s, y + \hat{\phi})(y + \hat{\phi})(s) ds - \int_{t_k}^{t_1} (t_1 - s)^{q-1} A(s, y + \hat{\phi})(y + \hat{\phi})(s) ds \right\| \right], \\ I_3 &= \frac{1}{\Gamma(q)} \left[\left\| \sum_{0 < t_k < t_2} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s, y + \hat{\phi})u(s, y_s + \hat{\phi}_s) ds \right. \right. \\ &\quad \left. \left. - \sum_{0 < t_k < t_1} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s, y + \hat{\phi})u(s, y_s + \hat{\phi}_s) ds \right\| \right], \\ I_4 &= \frac{1}{\Gamma(q)} \left[\left\| \int_{t_k}^{t_2} (t_2 - s)^{q-1} A(s, y + \hat{\phi})u(s, y_s + \hat{\phi}_s) ds - \int_{t_k}^{t_1} (t_1 - s)^{q-1} A(s, y + \hat{\phi})u(s, y_s + \hat{\phi}_s) ds \right\| \right], \\ I_5 &= \frac{1}{\Gamma(q)} \left[\left\| \sum_{0 < t_k < t_2} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, y_s + \hat{\phi}_s, H(y_s + \hat{\phi}_s)) ds \right. \right. \\ &\quad \left. \left. - \sum_{0 < t_k < t_1} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, y_s + \hat{\phi}_s, H(y_s + \hat{\phi}_s)) ds \right\| \right], \\ I_6 &= \frac{1}{\Gamma(q)} \left[\left\| \int_{t_k}^{t_2} (t_2 - s)^{q-1} f(s, y_s + \hat{\phi}_s, H(y_s + \hat{\phi}_s)) ds - \int_{t_k}^{t_1} (t_1 - s)^{q-1} f(s, y_s + \hat{\phi}_s, H(y_s + \hat{\phi}_s)) ds \right\| \right], \\ I_7 &= \left\| \sum_{0 < t_k < t_2} I_k(y(t_k^-) + \hat{\phi}(t_k^-)) - \sum_{0 < t_k < t_1} I_k(y(t_k^-) + \hat{\phi}(t_k^-)) \right\|. \end{aligned}$$

Clearly $I_1, I_2, I_3, I_4, I_5, I_6, I_7$ tends to 0 when $t_1 \rightarrow t_2$.

Therefore, we conclude that $\lim_{t_1 \rightarrow t_2} I_i = 0, i = 1, 2, 3, 4, 5, 6, 7$. Hence $\Theta_2 B_r$ is equicontinuous.

Next, we claim that $\Theta_2 B_r$ is precompact.

Let $0 < t \leq b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < b$. For $y \in B_r$, we define

$$\begin{aligned}
 (\Theta_2^\epsilon y)(t) &= \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s, y + \hat{\phi})(y + \hat{\phi})(s) ds \\
 &+ \frac{1}{\Gamma(q)} \int_{t_k}^{t-\epsilon} (t - s)^{q-1} A(s, y + \hat{\phi})(y + \hat{\phi})(s) ds \\
 &- \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s, y + \hat{\phi})u(s, y_s + \hat{\phi}_s) ds \\
 &- \frac{1}{\Gamma(q)} \int_{t_k}^{t-\epsilon} (t - s)^{q-1} A(s, y + \hat{\phi})u(s, y_s + \hat{\phi}_s) ds \\
 &+ \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, y_s + \hat{\phi}_s, H(y_s + \hat{\phi}_s)) ds \\
 &+ \frac{1}{\Gamma(q)} \int_{t_k}^{t-\epsilon} (t - s)^{q-1} f(s, y_s + \hat{\phi}_s, H(y_s + \hat{\phi}_s)) ds \\
 &+ \sum_{0 < t_k < t} I_k(y(t_k^-) + \hat{\phi}(t_k^-))
 \end{aligned}$$

since the operator $A(t,x)$ is compact for $t > 0$, for every ϵ sufficiently small, $0 < \epsilon < b$,

$$\begin{aligned}
 \|(\Theta_2 y)(t) - (\Theta_2^\epsilon y)(t)\| &= \left\| \frac{1}{\Gamma(q)} \int_{t-\epsilon}^t (t - s)^{q-1} A(s, y + \hat{\phi})(y + \hat{\phi})(s) ds \right. \\
 &- \frac{1}{\Gamma(q)} \int_{t-\epsilon}^t (t - s)^{q-1} A(s, y + \hat{\phi})u(s, y_s + \hat{\phi}_s) ds \\
 &+ \frac{1}{\Gamma(q)} \int_{t-\epsilon}^t (t - s)^{q-1} f(s, y_s + \hat{\phi}_s, H(y_s + \hat{\phi}_s)) ds \left. \right\| \\
 &\leq \frac{1}{\Gamma(q+1)} [(ML + K)(\ell^{-1}L_1 + \lambda_1 L_1 + \lambda_2) \\
 &+ (K_1 L_1 + K_1 b(P_1 L_1 + P_2) + K_2)] \epsilon^q.
 \end{aligned}$$

Therefore, letting $\epsilon \rightarrow 0$, we see that there are relatively compact sets arbitrarily close to the set $\{(\Theta_2 y)(t) : y \in B_r\}$. Hence $\{(\Theta_2 y)(t) : y \in B_r\}$ is relatively compact in $B_{\mathcal{Y}}$.

As a consequence of the above steps and the Arzela-Ascoli Theorem, we can conclude that Θ_2 is a compact operator. These arguments enable us to conclude that $\Theta = \Theta_1 + \Theta_2$ is a condensing map on B_r and Theorem 2.5 gives the conclusion that the system 1.1 has a solution.

4. An Example

Consider the following fractional integrodifferential equation with impulsive condition of the form

$$\begin{cases}
 D_t^q(x(t) - \frac{e^{-t}x}{(9+e^t)(1+x)}) = \frac{1}{4} \sin x(t)[x(t) - \frac{e^{-t}x}{(9+e^t)(1+x)}] + \frac{1}{(t+2)^2} \frac{|x|}{(1+|x|)} + \frac{1}{4} \int_0^s e^{-\frac{x_s}{3}} ds \\
 \Delta x|_{t=\frac{1}{2}} = \frac{|x(\frac{1}{2})^-|}{3+|x(\frac{1}{2})^-|} \\
 x(0) + \frac{x}{5+x} = x_0
 \end{cases} \tag{4.1}$$

where $0 < q \leq 1$ Take $J = [0, 1]$, $b = 1$

Let $A(t, x) = \frac{1}{4} \sin x(t)$, $H(x_s) = \int_0^s e^{-\frac{x_s}{3}} ds$,

$$f(t, x, H(x_s)) = \frac{1}{(t+2)^2} \frac{|x|}{(1+|x|)} + \frac{1}{4} \int_0^s e^{-\frac{x_s}{3}} ds,$$

$$u(t, x) = \frac{e^{-tx}}{(9+e^t)(1+x)}, \text{ where } t \in J, x \in X = R.$$

If $x, y \in X$ and $t \in J$ then we have

$$\|A(t, x) - A(t, y)\| \leq \frac{1}{4} \|x - y\|, \|H(x_s) - H(y_s)\| \leq \frac{1}{3} \|x - y\|,$$

$$\|f(t, x, H(x_s)) - f(t, y, H(y_s))\| \leq \frac{1}{4} (\|x - y\| + \|H(x_s) - H(y_s)\|),$$

$$\|I_k(x) - I_k(y)\| \leq \frac{1}{3} \|x - y\|, \|u(t, x_t) - u(t, y_t)\| \leq \frac{1}{10} \|x_t - y_t\|, \|g(x) - g(y)\| \leq \frac{1}{5} \|x - y\|.$$

here $P_1 = \frac{1}{3}$, $K_1 = \frac{1}{4}$, $\mu = \frac{1}{3}$, $\lambda_1 = \frac{1}{10}$, $\delta_1 = \frac{1}{5}$.

Let $\vartheta(t) = e^t$, therefore $\ell = \int_{-\infty}^0 \vartheta(t) dt = \int_{-\infty}^0 e^t dt = 1 < +\infty$

hence $\delta_1 + \lambda_1 \ell = \frac{3}{10} < 1$.

For some $q \in [0, 1]$, all the hypotheses of the Theorem 2.5 are satisfied. Hence the problem 4.1 has a solution.

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