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Application of fixed point theorems to best simultaneous approximation in ordered semi-convex structure

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`This paper is dedicated to Professor Ljubomir Ćirić`

`Communicated by Professor V. Berinde`

Abstract

In this paper, we establish some common fixed point results for uniformly C_q-commuting asymptotically S-nonexpansive maps in a Banach space with semi-convex structure. We also extend the main results of Ćirić [Lj. B. Ćirić, Publ. Inst. Math., 49 (1991), 174-178] and [Lj. B. Ćirić, Arch. Math. (BRNO), 29 (1993), 145-152] to semi-convex structure and obtain common fixed point results for Banach operator pair. The existence of invariant best simultaneous approximation in ordered semi-convex structure is also established.

Keywords: Common fixed point, uniformly C_q-commuting, asymptotically S-nonexpansive map, Banach operator pair, best simultaneous approximation
2010 MSC: 47H10, 54H25.

1. Introduction

In best approximation theory, it is pertinent, viable, meaningful and potentially productive to know whether some useful properties of the function being approximated is inherited by the approximating function. In this perspective, Meinardus [28] observed the general principle that could be applied, while doing

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so the author has employed a fixed point theorem as a tool to establish it. The result of Meinardus was further generalized by Habiniak [15], Smoluk [38] and Subrahmanyam [39].

On the other hand, Al-Thagafi [2], Singh [36, 37], Hussain et al. [17, 19, 20], Hussain and Rhoades [18], Jungck and Hussain [23], O'Regan and Hussain [30], Pathak et al. [31] and many others have used fixed point theorems in approximation theory, to prove existence of best approximation. Various types of applications of fixed point theorems may be seen in Klee [27], Meinardus [28] and Pathak and Hussain [32]. Some applications of the fixed point theorems to best simultaneous approximation are given by Sahney and Singh [35]. For the detail survey of the subject we refer the reader to Cheney [6].

The class of asymptotically nonexpansive mappings was introduced by Goeble and Kirk [13] and further studied by various authors (see [3, 26] and references therein). Recently, Beg et al. [3], have proved common fixed point results for uniformly R-subweakly commuting pair $\{S, T\}$. In this paper, we introduce a more general class of uniformly C_q-commuting selfmaps and study common fixed point results for uniformly C_q-commuting asymptotically S-nonexpansive maps in a Banach space with semi-convex structure. We also extend the main results of Ćirić [8, 9] to semi-convex structure. Recently, Chen and Li [5] introduced the class of Banach operator pairs, as a new class of noncommuting maps and it has been further studied by Hussain [16] and Pathak and Hussain [32]. We also obtain common fixed point and approximation results for Banach operator pair (T, S) satisfying Ćirić type contractive condition.

2. Preliminaries and Definitions

Let $X, \|\cdot\|$ be a normed space, M a subset of of X. We shall use $\mathbb{N}$ to denote the set of positive integers, $cl(M)$ to denote the closure of a set M and $wcl(M)$ to denote the weak closure of a set M. Let $I : M \to M$ be a mapping. A mapping $T : M \to M$ is called an I-contraction if there exists $0 \le k < 1$ such that $\|Tx - Ty\| \le k\|Ix - Iy\|$ for any $x, y \in M$. If $k = 1$, then T is called I-nonexpansive. The map T is called asymptotically I-nonexpansive if there exists a sequence $\{k_n\}$ of real numbers with $k_n \ge 1$ and $lim_n k_n = 1$ such that $\|T^n x - T^n y\| \le k_n\|Ix - Iy\|$ for all $x, y \in M$ and $n = 1, 2, 3, \ldots$. The map T is called uniformly asymptotically regular on M [3, 12], if for each $\eta > 0$, there exists $N(\eta) = N$ such that $\|T^n x - T^{n+1}x\| < \eta$ for all $n \ge N$ and all $x \in M$. The set of fixed points of T(resp. I) is denoted by $F(T)$(resp. $F(I)$). A point $x \in M$ is a coincidence point (common fixed point) of I and T if $Ix = Tx$ ($x = Ix = Tx$). The set of coincidence points of I and T is denoted by $C(I, T)$. The pair $\{I, T\}$ is called (1) commuting if $TIx = ITx$ for all $x \in M$, (2) R-weakly commuting if for all $x \in M$, there exists $R > 0$ such that $\|ITx - TIx\| \le R\|Ix - Tx\|$. If $R = 1$, then the maps are called weakly commuting; (3) compatible if $\lim_n \|TIx_n - ITx_n\| = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n Ix_n = t$ for some t in M; (4) weakly compatible if they commute at their coincidence points, i.e.,if $ITx = TIx$ whenever $Ix = Tx$. The set M is called q-starshaped with $q \in M$, if the segment $[q, x] = \{(1 - k)q + kx : 0 \le k \le 1\}$ joining q to x is contained in M for all $x \in M$. Suppose that M is q-starshaped with $q \in F(I)$ and is both T- and I-invariant. Then T and I are called (5) C_q-commuting [**?** 18] if $ITx = TIx$ for all $x \in C_q(I, T)$, where $C_q(I, T) = \cup\{C(I, T_k) : 0 \le k \le 1\}$ where $T_k = (1 - k)q + kT$; (6) R-subweakly commuting on M if for all $x \in M$, there exists a real number $R > 0$ such that $\|ITx - TIx\| \le Rdist(Ix, [q, Tx])$; (7) uniformly R-subweakly commuting on $M \setminus \{q\}$ (see [3]) if there exists a real number $R > 0$ such that $\|IT^n x - T^n Ix\| \le Rdist(Ix, [q, T^n x])$, for all $x \in M \setminus \{q\}$ and $n \in \mathbb{N}$.
The ordered pair (T, I) of two self maps of a metric space (X, d) is called a Banach operator pair, if the set $F(I)$ is T-invariant, namely $T(F(I)) \subseteq F(I)$. Obviously commuting pair (T, I) is a Banach operator pair but not conversely in general, see [5]. If (T, I) is a Banach operator pair then (I, T) need not be a Banach operator pair (cf. Example 1 [5]).

Now we give the notion of convex structure introduced by Gudder [14](see also, Petrusel [33]). Let X be a set and $F : [0, 1] \times X \times X \to X$ a mapping. Then the pair (X, F) forms a *convex prestructure*. Let

(X, F) be a convex prestructure. If F satisfies the following conditions:

(i) $F(\lambda, x, F(\mu, y, z)) = F(\lambda + (1-\lambda)\mu, \ F(\lambda(\lambda + (1-\lambda)\mu)^{-1}, \ x, \ y), z)$ for every $\lambda, \mu \in (0,1)$ with $\lambda + (1-\lambda)\mu \neq 0$ and $x, y, z \in X$.
(ii) $F(\lambda, x, x) = x$ for any $x \in X$ and $\lambda \in (0,1)$,

then (X, F) forms a *semi-convex structure.* If (X, F) is a semi-convex structure, then

$(SC1)$ $F(1, x, y) = x$ for any $x, y \in X$.

A semi-convex structure is said to be *regular* if

$(SC2)$ $\lambda \leq \mu \Rightarrow F(\lambda, x, y) \leq F(\mu, x, y)$ where $\lambda, \mu \in (0,1)$.

A semi-convex structure (X, F) is said to form a *convex structure* if F also satisfies the conditions

(iii) $F(\lambda, x, y) = F(1-\lambda, y, x)$ for every $\lambda \in (0,1)$ and $x, y \in X$.
(iv) if $F(\lambda, x, y) = F(\lambda, x, z)$ for some $\lambda \neq 1, x \in X$ then $y = z$.

Let (X, F) be a convex structure. A subset Y of X is called (a) *F-starshaped* if there exist $p \in Y$ so that for any $x \in Y$ and $\lambda \in (0,1), F(\lambda, x, p) \in Y$. (b) *F-convex* if for any x, y in Y and $\lambda \in (0,1), F(\lambda, x, y) \in Y$. For $F(\lambda, x, y) = \lambda x + (1-\lambda)y$, we obtain the known notion of starshaped convexity from linear spaces. Petrusel [33] noted with an example that a set can be a F-semi convex structure without being a convex structure. Let (X, F) be a semi-convex structure. A subset Y of X is called F *semi-starshaped* if there exists $p \in Y$ so that for any $x \in Y$ and $\lambda \in (0,1)$, $F(\lambda, x, p) \in Y$. A Banach space X with semi-convex structure F is said to satisfy condition (P_1) at $p \in K$ (where K is semi-starshaped and p is star centre) if F is continuous relative to the following argument : for any $x, y \in X, \lambda \in (0,1)$

$$\| (F(\lambda, x, p) - F(\lambda, y, p) \| \leq \lambda \| x - y \| .$$

3. Common Fixed Point Results

We begin with the definition of uniformly C_q-commuting mappings.

Definition 1. *Let M be a q-starshaped subset of a normed space X. Let $I, T : M \to M$ be maps with $q \in F(I)$. Then I and T are said to be uniformly C_q-commuting on M if $IT^n x = T^n I x$ for all $x \in C_q(I, T^n)$ and $n \in \mathbb{N}$.*

It is clear from Definition 1 that uniformly C_q-commuting maps on M are C_q-commuting but not conversely in general as the following example shows.

Example 1. Let $X = R$ with usual norm and $M = [1, \infty)$. Let $Tx = 2x - 1$ and $Ix = x^2$, for all $x \in M$. Let $q = 1$. Then M is q-starshaped with $Iq = q$, $C_q(I, T) = \{1\}$ and $C_q(I, T^2) = [1, 3]$. Note that I and T are C_q-commuting maps but not uniformly C_q-commuting because $IT^2 x \neq T^2 I x$ for all $x \in (1, 3] \subset C_q(I, T^2)$.

Uniformly R-subweakly commuting maps are uniformly C_q-commuting but the converse does not hold in general, to see this we consider the following example.

Example 2. Let $X = R$ with usual norm and $M = [0, \infty)$. Let $Ix = \frac{x}{2}$ if $0 \leq x < 1$ and $Ix = x$ if $x \geq 1$, and $Tx = \frac{1}{2}$ if $0 \leq x < 1$ and $Tx = x^2$ if $x \geq 1$. Then M is 1-starshaped with $I1 = 1$ and $C_q(I,T) = [1, \infty]$ and $C_q(I,T^n) \subseteq [1, \infty]$ for each $n > 1$. Clearly, I and T are uniformly C_q-commuting but not R-weakly commuting for all $R > 0$. Thus I and T are neither R-subweakly commuting nor uniformly R-subweakly commuting maps.

We can extend these concepts on F-starshaped set in the convex structure (X, F)(see [17, 18]).

Definition 2. *Let $(X, F, \leq)$ be a ordered semi-convex structure and, T be a self-map on a nonempty subset M of X. We define, $Y_p^{T^n x} = \{F(\lambda, T^n x, p) : 0 \leq \lambda \leq 1\}$.*

The following result improves and extends Lemma 3.3 [3].

Lemma 3. *Let $(X, F, \leq)$ be a ordered semi-convex structure and S and T be self-maps on a nonempty subset M of X. Suppose that M is F-starshaped with respect to an element p in $F(S)$, S satisfies $F(\lambda, Sx, p) = S(F(\lambda, x, p))$ and $S(M) = M$. Assume that T and S are uniformly C_p-commuting and satisfy for each $n \geq 1$*

$$\|T^n x - T^n y\| \leq k_n max \left\{ \begin{array}{c} \|Sx - Sy\|, dist(Sx, Y_p^{T^n x}), dist(Sy, Y_p^{T^n y}), \\ dist(Sx, Y_p^{T^n y}), dist(Sy, Y_p^{T^n x})\} \end{array} \right\} \tag{1}$$

for all $x, y \in M$, where $\{k_n\}$ is a sequence of real numbers with $k_n \geq 1$ and $lim_n k_n = 1$. For each $n \geq 1$, define a mapping T_n on M by

$$T_n x = F(\mu_n, T^n x, p)$$

where $\mu_n = \frac{\lambda_n}{k_n}$ and $\{\lambda_n\}$ is a sequence of numbers in $(0, 1)$ such that $lim_n \lambda_n = 1$. Then for each $n \geq 1$, T_n and S have exactly one common fixed point x_n in M such that

$$Sx_n = x_n = F(\mu_n, T^n x_n, p)$$

provided one of the following conditions hold;
(i) M is closed and for each n, $clT_n(M)$ is complete,
(ii) M is weakly closed and for each n, $wclT_n(M)$ is complete.

Proof. By definition,

$$T_n x = F(\mu_n, T^n x, p).$$

As S and T are uniformly C_p-commuting and $F(\lambda, Sx, p) = S(F(\lambda, x, p))$ with $Sp = p$, then for each $x \in C(S, T_n) \subseteq C_p(S, T^n)$

$$\begin{aligned} T_n Sx &= F(\mu_n, T^n Sx, p) \\ &= F(\mu_n, ST^n x, p) \\ &= S(F(\mu_n, T^n x, p)) \\ &= ST_n x. \end{aligned}$$

Hence S and T_n are weakly compatible for all n. Also by (1),

$$\begin{aligned} \|T_n x - T_n y\| &= \mu_n \|T^n x - T^n y\| \\ &\leq \lambda_n \max\{\|Sx - Sy\|, dist(Sx, Y_p^{T^n x}), dist(Sy, Y_p^{T^n y}), \\ &\quad dist(Sx, Y_p^{T^n y}), dist(Sy, Y_p^{T^n x})\} \\ &\leq \lambda_n \max\{\|Sx - Sy\|, \|Sx - T_n x\|, \|Sy - T_n y\|, \\ &\quad \|Sx - T_n y\|, \|Sy - T_n x\|\}, \end{aligned}$$

for each $x, y \in M$.
(i) As M is closed, therefore, for each n, $clT_n(M) \subset M = S(M)$. By Theorem 2.1[23], for each $n \geq 1$, there exists $x_n \in M$ such that $x_n = Sx_n = T_n x_n$. Thus for each $n \geq 1$, $M \cap F(T_n) \cap F(I) \neq \emptyset$.
(ii) As $wclT_n(M) \subset M = S(M)$, for each n, by Theorem 2.1[23], the conclusion follows.

The following result extends the recent results due to Hussain and Rhoades [18] and Theorem 3.4 of Beg et al. [3] to uniformly C_p-commuting asymptotically S-nonexpansive maps defined on nonstarshaped domain.

Theorem 4. *Let $(X, F, \leq)$ be a ordered semi-convex structure with F regular and, S and T be self-maps on a nonempty subset M of X. Suppose that M is F-starshaped with respect to an element p in $F(S)$, S satisfies $F(\lambda, Sx, p) = S(F(\lambda, x, p))$ and $S(M) = M$. Assume that T and S are uniformly C_p-commuting maps, T is uniformly asymptotically regular and asymptotically S-nonexpansive map. Then $F(T) \cap F(S) \neq \emptyset$, provided one of the following conditions holds;*
(i) M is closed, T is continuous and $clT(M)$ is compact,
(ii) X is complete, M is weakly closed, S is weakly continuous, $wclT(M)$ is weakly compact and $I - T$ is demiclosed at 0.

Proof. (i) Notice that compactness of $clT(M)$ implies that $clT_n(M)$ is compact and hence complete. From Lemma 3, for each $n \geq 1$, there exists $x_n \in M$ such that $x_n = Sx_n = T_n x_n = F(\mu_n, T^n x_n, p)$. Hence $x_n \in C_p(S, T^n)$.
Therefore

$$\begin{aligned} x_n - T^{n+1}x_n &= T_n x_n - T^{n+1}x_n \\ &= F(\mu_n, T^n x_n, p) - T^{n+1}x_n \\ &\leq F(\limsup_{n\to\infty} \mu_n, T^n x_n, p) - T^{n+1}x_n \\ &\leq F(1, T^n x_n, p) - T^{n+1}x_n \\ &\leq T^n x_n - T^{n+1}x_n. \end{aligned}$$

Applying the same argument as above, we also have

$$x_n - T^n x_n \leq 0.$$

Since T is uniformly asymptotically regular on M it follows that
$T^n x_n - T^{n+1}x_n \to 0$ as $n \to \infty$.
Therefore $x_n - T^{n+1}x_n \to 0$ as $n \to \infty$.
Now

$$\begin{aligned} \| x_n - Tx_n \| &\leq \| x_n - T^{n+1}x_n \| + \| T^{n+1}x_n - Tx_n \| \\ &\leq \| x_n - T^{n+1} \| + k_1 \| S(T^n x_n) - Sx_n \| \qquad \text{for some } \ k_1 \geq 1 \\ &= \| x_n - T^{n+1}x_n \| + k_1 \| T^n x_n - x_n \| \end{aligned}$$

Since S commutes with T^n on $C_p(S, T^n)$ and $x_n \in C_p(S, T^n)$, $x_n = Sx_n$, therefore $x_n - Tx_n \to 0$ as $n \to \infty$ Since $clT(M)$ is compact, there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m \to x_0$ as $m \to \infty$. By the continuity of T , we have $T(x_0) = x_0$. Since $T(M) \subset S(M)$, it follows that $x_0 = T(x_0) = Sy$, for some $y \in M$. Taking the limit as $m \to \infty$, we get $Tx_0 = Ty$. Thus, $Tx_0 = Sy = Ty = x_0$. Since S and T are uniformly C_q- commuting on M and $y \in C(S, T)$, therefore

$$\| Tx_0 - Sx_0 \| = \| TSy - STy \| = 0.$$

Hence we have $y \in F(T) \cap F(S)$. Thus $F(T) \cap F(S) \neq \emptyset$.

(ii) The weak compactness of $wclT(M)$ implies that $wclT_n(M)$ is weakly compact and hence complete due to completeness of X (see [23]). From Lemma 3, for each $n \geq 1$, there exists $x_n \in M$ such that $x_n = Sx_n = F(\mu_n, T^n x_n, p)$. The analysis in (i), implies that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. The weak compactness of $wclT(M)$ implies that there is a subsequence $\{x_m\}$ of $\{x_n\}$ converging weakly to $y \in M$ as $m \to \infty$. As S is weakly continuous, so $Sy = y$. Also we have, $Sx_m - Tx_m = x_m - Tx_m \to 0$ as $m \to \infty$. If $S - T$ is demiclosed at 0, then $Sy = Ty$. Thus $F(T) \cap F(S) \neq \emptyset$. This completes the proof.

Remark 1. *Notice that the conditions of the continuity and linearity of S are not needed in Theorem 3.4 of Beg et al. [3]. The result is also true for affine mapping S.*

Now we introduce the concept of upper semi-convex structure in a Banach space as follows:

Definition 5. *(i) Let $(X, \|\cdot\|)$ be a Banach space with semi-convex structure F. A continuous map $F : [\frac{1}{2}, 1] \times X \times X \to X$ is said to be an upper semi-convex structure on X if for all x, y in X, λ in $[\frac{1}{2}, 1]$,*

$$\|u - F(\lambda, x, F(\lambda, y, y))\| \leq \lambda\|u - x\| + (1 - \lambda)\|u - y\|$$

for all u in X.

(ii) Let $(X, \| . \|)$ be a Banach space with upper semi-convex structure F. Then the triplet $(X, F, \|\cdot\|)$ is called an upper semi-convex Banach space (or, in brief, USCBS).

(iii) Let $(X, F, \|\cdot\|)$ be an upper semi-convex Banach space, K a subset of X and let '$\leq$' be an order relation defined on X by

$$x \leq y \qquad \text{iff} \quad y - x \in K.$$

Then the triplet $(X, F, \|\cdot\|)$ is said to be an ordered USCBS induced by $(K, \leq)$.

The following result extends main theorems in [8, 9, 11, 22].

Theorem 6. *Let M be a nonempty, subset of an ordered USCBS $(X, F, \|\cdot\|)$ induced by $(M, \leq)$, and $T, S : M \to M$ be weakly compatible pair satisfying the following condition:*

$$\| Tx - Ty \|^p \leq a \| Sx - Sy \|^p + (1 - a)\, max\{\| Tx - Sx \|^p, \| Ty - Sy \|^p\} \tag{2}$$

for all $x, y \in M$, where $0 < a < 1/2^{p-1}$ and $p \geq 1$. If $cl(T(M)) \cup F\Big([\frac{1}{2}, 1], T(M) \times T(M)\Big) \subseteq S(M)$, where F is a upper semi-convex structure on M and $cl(T(M))$ is complete, then T and S have a unique common fixed point in M; i.e., $M \cap F(T) \cap F(S)$ is singleton.

Proof. Let x be an arbitrary point of M. Choose points x_1, x_2, x_3 in M and some $\lambda \in [\frac{1}{2}, 1]$ such that

$$Sx_1 = Tx, Sx_2 = Tx_1, Sx_3 = F(\lambda, Tx_1, Tx_2).$$

This choice is possible because $Tx, Tx_1, Tx_2, F(\lambda, Tx_1, Tx_2)$ are in $S(M)$.
By (2), we have

$$\begin{aligned} \|Sx_1 - Sx_2\|^p &= \|Tx - Tx_1\|^p \\ &\leq a\,\|Sx - Sx_1\|^p + (1 - a)\, max\{\|Sx - Tx\|^p, \|Sx_1 - Tx_1\|^p\} \\ &= a\,\|Sx - Sx_1\|^2 + (1 - a)\, max\{\|Sx - Sx_1\|^2, \|Sx_1 - Sx_2\|^2\}. \end{aligned}$$

Hence we have

$$\|Sx_1 - Sx_2\| \leq \|Sx - Sx_1\|. \tag{3}$$

Form (2) and (3),

$$\|Sx_2 - Tx_2\|^p = \|Tx_1 - Tx_2\|^p$$

$$\le a\,\|Sx_1 - Sx_2\|^p + (1-a)\,max\{\|Sx_1 - Tx_1\|^p, \|Sx_2 - Tx_2\|^p\}$$
$$\le a\,\|Sx - Sx_1\|^p + (1-a)\,max\{\|Sx - Sx_1\|^p, \|Sx_2 - Tx_2\|^p\}$$

which implies

$$\|Sx_2 - Tx_2\| \le \|Sx - Sx_1\| \tag{4}$$

As $f(x) = x^p$ is increasing for $x \ge 0$, we have from (2),

$$\begin{aligned}
\|Sx_1 - Tx_2\|^p &= \|Tx - Tx_2\|^p \\
&\le a\,\|Sx - Sx_2\|^p + (1-a)\,max\{\|Sx - Tx\|^p, \|Sx_2 - Tx_2\|^p\} \\
&\le a\,[\|Sx - Sx_1\| + \|Sx_1 - Sx_2\|]^p + (1-a)\,max\{\|Sx - Sx_1\|^p, \|Sx_2 - Tx_2\|^p\}.
\end{aligned}$$

Hence, using (3) and (4), we have

$$\|Sx_1 - Tx_2\|^p \le (2^p a + 1 - a)\,\|Sx - Sx_1\|^p. \tag{5}$$

Now using Definition (5) and convexity of $f(x) = x^p (p \ge 1)$, we have

$$\begin{aligned}
\|Sx_1 - Sx_3\|^p &= \|Sx_1 - F(\lambda, Tx_1, Tx_2)\|^p \\
&= \|Sx_1 - F(\lambda, Tx_1, F(\lambda, Tx_2, Tx_2))\|^p \\
&\le [\lambda\,\|Sx_1 - Tx_1\| + (1-\lambda)\,\|Sx_1 - Tx_2\|]^p \\
&\le \lambda\,\|Sx_1 - Sx_2\|^p + (1-\lambda)\,\|Sx_1 - Tx_2\|^p.
\end{aligned}$$

Hence, from (1) and (3), we obtain

$$\|Sx_1 - Sx_3\|^p \le [1 + (1-\lambda)2^p a\{1 - 2^{-p}\}]\|Sx - Sx_1\|^p. \tag{6}$$

Further,

$$\begin{aligned}
\|Sx_2 - Sx_3\|^p &= \|Sx_2 - F(\lambda, Tx_1, Tx_2)\|^p \\
&= \|Sx_2 - F(\lambda, Tx_1, F(\lambda, Tx_2, Tx_2))\|^p \\
&\le [\lambda\,\|Sx_2 - Sx_2\| + (1-\lambda)\,\|Sx_2 - Tx_2\|]^p
\end{aligned}$$

hence by (2) we get

$$\|Sx_2 - Sx_3\| \le (1-\lambda)\,\|Sx - Sx_1\|. \tag{7}$$

Now we choose $x_4 \in M$ such that $Sx_4 = Tx_3$. Then from (2), (3) and (4) we have

$$\begin{aligned}
\|Sx_3 - Sx_4\|^p &= \|Tx_3 - F(\lambda, Tx_1, Tx_2)\|^p \\
&= \|Tx_3 - F(\lambda, Tx_1, F(\lambda, Tx_2, Tx_2))\|^p \\
&\le [\lambda\,\|Tx_1 - Tx_3\| + (1-\lambda)\,\|Tx_2 - Tx_3\|]^p \\
&\le \lambda\,[a\,[\|Sx_1 - Sx_3\|^p + (1-a)\,max\{\|Sx_1 - Sx_2\|^p, \|Sx_3 - Sx_4\|^p\}] \\
&\quad +(1-\lambda)\,[a\,[\|Sx_2 - Sx_3\|^p + (1-a)\,max\{\|Sx_2 - Tx_2\|^p, \|Sx_3 - Sx_4\|^p\}] \\
&\le a\,[\lambda\,\|Sx_1 - Sx_3\|^p + (1-\lambda)\,\|Sx_2 - Sx_3\|^p] + (1-a) \\
&\quad max\{\|Sx - Sx_1\|^p, \|Sx_3 - Sx_4\|^p\}.
\end{aligned}$$

Hence, using (6) and (7), we have

$$\|Sx_3 - Sx_4\|^p \le \mu^p\,max\{\|Sx - Sx_1\|^p, \|Sx_3 - Sx_4\|^p\},$$

where $\mu^p = \Big(a\,\lambda[1 + (1-\lambda)2^p a\{1 - 2^{-p}\} + (1-\lambda)^p] + (1-a)\Big)$. Since $p \ge 1$, $0 < a < \left(\frac{1}{2}\right)^{p-1}$ and $\lambda \in [\frac{1}{2}, 1]$, we obtain $\mu^p < 1$. To see this, we observe that

$$\mu^p = \Big(a\,\lambda[1 + (1-\lambda)2^p a\{1 - 2^{-p}\} + (1-\lambda)^p] + (1-a)\Big)$$

$$< \Big(a\,\lambda[1+2(1-\lambda)\{1-2^{-p}\}+(1-\lambda)^p]+(1-a)\Big), \text{ as } a<\Big(\frac{1}{2}\Big)^{p-1}$$
$$\le \Big(a\cdot 2^{-1}[1+2\cdot 2^{-1}\{1-2^{-p}\}+2^{-p}]+(1-a)\Big)=1, \text{ as } 1-\lambda\le\frac{1}{2}.$$

Therefore,

$$\|Sx_3-Sx_4\|\le\mu\,\|Sx-Sx_1\| \qquad (0<k<1). \tag{8}$$

Now we shall consider the sequence $\{Sx_n\}_{n=0}^{\infty}$ which possess the properties (3), (4), (7) and (8); i.e., the sequence $\{Sx_n\}_{n=0}^{\infty}$ is defined as follows:

$$Sx_{3k+1}=Tx_{3k}; Sx_{3k+2}=Tx_{3k+1}; Sx_{3(k+1)}=F(\lambda,Tx_{3k+1},Tx_{3k+2}),\ k=0,1,2\cdots$$

By induction it can easily be shown that from (8), (3) and (7) we have

$$\|Sx_{3k}-Sx_{3k+1}\|\le\mu\,\|Sx_{3(k-1)}-Sx_{3(k-1)+1}\|\le\cdots\le\mu^k\,\|Sx-Sx_1\|,$$
$$\|Sx_{3k+1}-Sx_{3k+2}\|\le\|Sx_{3k}-Sx_{3k+1}\|\le\mu^k\,\|Sx-Sx_1\|,$$
$$\|Sx_{3k+2}-Sx_{3(k+1)}\|\le(1-\lambda)\,\|Sx_{3k}-Sx_{3k+1}\|\le(1-\lambda)\,\mu^k\,\|Sx-Sx_1\|. \tag{9}$$

Hence for $m>n>N$, we have

$$\|Sx_m-Sx_n\|\le\sum_{i=N}^{\infty}\|Sx_i-Sx_{i+1}\|\le\Big((3-\lambda)\mu^{[N/3]}/(1-\mu)\Big)\|Sx-Sx_1\|,$$

where $[N/3]$ means the greatest integer not exceeding $N/3$. Take $x_0=x$, then it follows from the above inequality that the sequence $\{Sx_n\}_{n=0}^{\infty}$ is a Cauchy sequence in M, hence convergent. So, let $\underset{n\to\infty}{lim}Sx_n=u$. As $Tx_{3k}=Sx_{3k+1}, Tx_{3k+1}=Sx_{3k+2}$, from (4) and (9) we have

$$\|Tx_{3k+2}-Sx_{3k+2}\|\le\|Sx_{3k}-Sx_{3k+1}\|\le\mu^p\|Sx-Sx_1\|.$$

Therefore,

$$\lim_n Sx_n=\lim_n Tx_n=u\in cl(T(M))\subseteq S(M),$$

which implies that there exists some $y\in M$ such that $u=Sy$. For each $n\ge1$,

$$\begin{aligned}\|u-Ty\| &\le \|u-Tx_n\|+\|Tx_n-Ty\|\\ &\le [\|u-Tx_n\|+a^{\frac{1}{p}}\|Sx_n-Sy\|+(1-a)^{\frac{1}{p}}\max\{\|Tx_n-Sx_n\|,\|Ty-Sy\|\}].\end{aligned}$$

Taking the limit as $n\to\infty$ yields

$$\|u-Ty\|\le(1-a)^{\frac{1}{p}}\|u-Ty\|,$$

which implies that $Sy=u=Ty$. Since S and T are weakly compatible, $T^2y=TSy=STy$. Using (2),

$$\|TTy-Ty\|^p \le a\|STy-Sy\|^p+(1-a)\max\{\|TTy-STy\|^p,\|Ty-Sy\|^p\},$$

which implies that $TTy=Ty$. Since $TTy=STy$, $Ty=u$ is a common fixed point of T and S. Condition (2) ensures that u is the unique common fixed point of T and S; i.e., $M\cap F(T)\cap F(S)$ is singleton.

Theorem 7. *Let $(X,F,\|\cdot\|)$ be an ordered USCBS induced by $(M,\le)$, where F is a upper semi-convex structure on M and let $T,S:M\to M$ be C_p-commuting mappings. Let M be F-starshaped with respect to an element $p\in F(S)$ and S satisfies $F(\lambda,Sx,p)=S(F(\lambda,x,p))$ for each $x\in M$. If $M=S(M)$, and for all $x,y\in M$, and all $k\in(0,1)$,*

$$\|\,Tx-Ty\,\|\le\|\,Sx-Sy\,\|+\frac{1-k}{k}max\{dist(Sx,Y_p^{Tx}),dist(Sy,Y_p^{Ty})\}, \tag{10}$$

then $M \cap F(S) \cap F(T) \neq \emptyset$, provided one of the following conditions holds;

(i) T is continuous and $cl(T(M))$ is compact;
(ii) S is weakly continuous, $wcl(T(M))$ is weakly compact and either $S-T$ is demiclosed at 0 or X satisfies Opial's condition.

Proof. Define $T_n : M \to M$ by

$$T_n x = F(k_n, Tx, p)$$

for some $p \in F(S)$ and all $x \in M$ and a fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1. As S and T are C_p-commuting and $F(\lambda, Sx, p) = S(F(\lambda, x, p))$ with $Sp = p$, then for each $x \in C_p(S,T)$

$$\begin{aligned} T_n Sx &= F(k_n, TSx, p) \\ &= F(k_n, STx, p) \\ &= S(F(k_n, Tx, p)) \\ &= ST_n x. \end{aligned}$$

Thus $ST_n x = T_n Sx$ for each $x \in C(S, T_n) \subset C_p(S,T)$. Hence S and T_n are weakly compatible for all n. Also

$$\begin{aligned} \| T_n x - T_n y \| &= k_n \| Tx - Ty \| \\ &\leq k_n \{ \| Sx - Sy \| + \frac{1-k_n}{k_n} \max\{ \| Sx - T_n x \|, \| Sy - T_n y \| \} \} \\ &= k_n \| Sx - Sy \| + (1-k_n) max\{ \| Sx - T_n x \|, \| Sy - T_n y \| \} \end{aligned}$$

for each $x, y \in M$ and $0 < k_n < 1$.
(i) By Theorem 6, for each $n \geq 1$, there exist an $x_n \in M$ such that $x_n = Sx_n = T_n x_n$. The compactness of $cl(T(M))$ implies that there exists a subsequence Tx_m such that $Tx_m \to z$ as $m \to \infty$. Also

$$\lim x_m = \lim T_m(x_m) = \lim F(k_m, T(x_m), p) = F(1, z, p) = z.$$

As $z \in cl(T(M)) \subset S(M)$, $z = Su$ for some $u \in M$ and hence $Su = z = Tz$. Further, for each m,

$$\begin{aligned} \|Tx_m - Tu\| &\leq \|Sx_m - Su\| + \frac{1-k_m}{k_m} \max\{\|Sx_m - T_m x_m\|, \|Su - T_m u\|\}\} \\ &= \|x_m - z\| + \frac{1-k_m}{k_m} \max\{\|Sx_m - T_m x_m\|, \|Su - T_m u\|\}\}, \end{aligned}$$

which, on letting $m \to \infty$, implies that $Su = z = Tz = Tu$. Since S and T are also weakly compatible, we have $Sz = STu = TSu = Tz = z$. This shows that $M \cap F(S) \cap F(T) \neq \emptyset$.
(ii) Proof is similar to the proof of Theorem 2.4 [19], here we use Theorem (6) instead of Theorem 2.1 [19]
Theorem (7) extends Theorem 2.2 in [2] and Theorems 2.3 and 2.4 in [19].

Lemma 8. *Let M be a nonempty subset of an ordered USCBS $(X, F, \|\cdot\|)$ induced by $(M, \leq)$, and $T,\ S : M \to M$ be a pair of maps satisfying inequality (2), $F(S)$ is nonempty and F is an upper semi-convex structure on $F(S)$. Suppose that $cl(T(M))$ is complete and $clT(F(S)) \subseteq F(S)$, then T and S have a unique common fixed point in M.*

Proof. By our assumptions, $T(F(S)) \subseteq F(S)$ and $F(S)$ is nonempty, and has an upper semi-convex structure. The completeness of $cl(T(M))$ implies that $cl(T(F(S)))$ is complete. Further for all $x,\ y \in F(S)$, we have by inequality 2,

$$\begin{aligned} \|Tx - Ty\| &\leq a \| Sx - Sy \|^p + (1-a) \max\{ \| Tx - Sx \|^p, \| Ty - Sy \|^p \} \\ &= a \| x - y \|^p + (1-a) \max\{ \| Tx - x \|^p, \| Ty - y \|^p \} \end{aligned}$$

By Theorem 6, T has a unique fixed point y in $F(S)$ and consequently $M \cap F(T) \cap F(S)$ is singleton.

Corollary 9. *Let M be a nonempty subset of an ordered USCBS $(X, F, \|\cdot\|)$ induced by $(M, \leq)$, and $T,\ S: M \to M$ be a pair of maps satisfying inequality (2), $F(S)$ is nonempty and closed and F is an upper semi-convex structure on $F(S)$. Suppose that $cl(T(M))$ is complete, (T, S) is a Banach operator pair, then T and S have a unique common fixed point in M.*

The following result extends and improves Theorem 3.3 of [5] and Theorems 2.2 and 2.4 in [16].

Theorem 10. *Let $(X, F, \|\cdot\|)$ be an ordered USCBS induced by $(M, \leq)$ and let $T, S : M \to M$ be pair of self-mappings. Assume that $F(S)$ is F-starshaped with respect to an element $p \in F(S)$, where F is an upper semi-convex structure on $F(S)$, $clT(F(S)) \subseteq F(S)$ [resp. $wclT(F(S)) \subseteq F(S)$], $cl(T(M))$ is compact [resp. $wcl(T(M))$ is weakly compact and either $id - T$ is demiclosed at 0 or X satisfies Opial's condition] and (T, S) satisfies (10), for all $x, y \in M$, and all $k \in (0, 1)$, then $M \cap F(S) \cap F(T) \neq \emptyset$.*

Proof. Define $T_n : F(S) \to F(S)$ as in Theorem 7. As $F(S)$ is F-starshaped with respect to an element p in $F(S)$, for each $x \in F(S)$ $T_n x = F(k_n, Tx, p) \in F(S)$, since $Tx \in F(S)$ and $F(S)$ is F-starshaped with respect to $p \in F(S)$. Thus $clT_n(F(S)) \subseteq F(S)$ for each n. Also

$$\begin{aligned}
\| T_n x - T_n y \| &= k_n \| Tx - Ty \| \\
&\leq k_n\{\| Sx - Sy \| + \frac{1-k_n}{k_n} \max\{\| Sx - T_n x \|, \| Sy - T_n y \|\}\} \\
&= k_n \| Sx - Sy \| + (1-k_n) \max\{\| Sx - T_n x \|, \| Sy - T_n y \|\}
\end{aligned}$$

for each $x, y \in F(S)$ and $0 < k_n < 1$.
If $cl(T(M))$ is compact, for each $n \in \mathbb{N}$, $cl(T_n(M))$ is compact and hence complete. By Lemma 8, for each $n \geq 1$, there exist an $x_n \in M$ such that $x_n = Sx_n = T_n x_n$. The compactness of $cl(T(M))$ implies that there exist a subsequence Tx_{n_i} such that $Tx_{n_i} \to z \in cl(T(F(S))) \subseteq F(S)$ as $i \to \infty$. Since T is continuous, so

$$z = \lim Tx_{n_i} = \lim T(T_{n_i}(x_{n_i})) = \lim T(F(k_{n_i}, T(x_{n_i}), p)) = T(F(1, z, p)) = T(z).$$

This shows that $M \cap F(S) \cap F(T) \neq \emptyset$.
Similarly we obtain the proof of second part.

Corollary 11. *Let $(X, F, \|\cdot\|)$ be an ordered USCBS induced by $(M, \leq)$ and let $T, S : M \to M$ be pair of self-mappings. Assume that (T, S) is a Banach operator pair on M and F-starshaped with respect to an element $p \in F(S)$, where F is an upper semi-convex structure on $F(S)$. If , $F(S)$ is closed [resp. weakly closed], $cl(T(M))$ is compact [resp. $wcl(T(M))$ is weakly compact and either $id - T$ is demiclosed at 0 or X satisfies Opial's condition] and (T, S) satisfies (9), for all $x, y \in M$, and all $k \in (0, 1)$, then $M \cap F(S) \cap F(T) \neq \emptyset$.*

We now furnish a non-trivial example to validate Theorem (6).

Example 3. Let $X = \mathbb{R}$ be equipped with usual norm $\|\cdot\| = |\cdot|$. Let $F : [\frac{1}{2}, 1] \times X \times X \to X$ be defined by $F(\lambda, x, y) = \lambda x + (1-\lambda)y$ for all x, y in M and λ in $[\frac{1}{2}, 1]$. Clearly, F is an *upper semi-convex structure* on X. Take $M = [-1, 1]$. Let $T, S : M \to M$ be a pair of self-mappings on M such that $Tx = \frac{1}{3}|x|$ and $Sx = -x$. Obviously, T and S are weakly compatible pair of mappings. Also $cl(T(M)) \cup F\Big([\frac{1}{2}, 1], T(M), T(M)\Big) \subseteq S(M)$ and $q = 0$ is the starcenter. For all $x, y \in M$, $p \geq 1$ and $0 < a = \frac{1}{3^p} < \frac{1}{2^{p-1}}$, we have

$$\|Tx - Ty\|^p = |Tx - Ty|^p = \frac{1}{3^p}|\,|x| - |y|\,|^p \leq \frac{1}{3^p}|x - y|^p = \frac{1}{3^p}| - x + y|^p = \frac{1}{3^p}\|Sx - Sy\|^p$$

$$\leq \frac{1}{3^p}\|Sx - Sy\|^p + (1 - \frac{1}{3^p}) \max\{\|Tx - Sx\|^p, \|Ty - Sy\|^p\}.$$

Thus, all the conditions of Theorem 6 are satisfied. Clearly, 0 is the unique fixed point of S and T in M i.e., $M \cap F(T) \cap F(S)$ is singleton.

4. Best Simultaneous Approximation Results

Let M be a subset of a Banach space $(X, \| . \|)$. The set $P_M(u) = \{x \in M : \| x - u \| = dist(u, M)\}$ is called the set of best approximants to $u \in X$ out of M, where $dist(u, M) = inf\{\| y - u \| : y \in M\}$. Suppose A , G, are bounded subsets of X, then we write

$$r_G(A) = inf_{g \in G} sup_{a \in A} \| a - g \|$$

$$cent_G(A) = \{g_0 \in G : sup_{a \in A} \| a - g_0 \| = r_G(A)\}.$$

The number $r_G(A)$ is called the *Chebyshev radius* of A w.r.t G and an element $y_0 \in cent_G(A)$ is called a *best simultaneous approximation* of A w.r.t G. If $A = \{u\}$, then $r_G(A) = d(u, G)$ and $cent_G(A)$ is the set of all best approximations, $P_G(u)$, of u out of G. We also refer the reader to Cheney [6], Klee [27] and Milman [29] for further details.

Sahab et al. [34], Jungck and Sessa [24] and Al-Thagafi [2] generalized main result of Singh [37] to nonexpansive mapping T with respect to continuous mapping S in the context of best approximation in normed linear space. In this section, as an application of our common fixed point results, we prove the corresponding results in semi-convex structure in the context of best simultaneous approximation for more general pair of mappings.
In the following result we extend corresponding results in [2, 3, 18, 24] to asymptotically S-nonexpansive maps defined on F-starshaped domain.

Theorem 12. *Let $(X, F, \leq)$ be an ordered semi-convex structure with F regular and, G and A are nonempty subsets of X such that $cent_G(A)$, set of best simultaneous approximations of elements in A by G, is nonempty. Let T and S are self mapping on $cent_G(A)$. Suppose that $cent_G(A)$ is F-starshaped with respect to an element p in $F(S)$, $F(\lambda, Sx, p) = S(F(\lambda, x, p))$ for all $x \in cent_G(A)$ and $S(cent_G(A)) = cent_G(A)$. Assume that T and S are uniformly C_p-commuting, T is uniformly asymptotically regular and asymptotically $S-$nonexpansive. Then $F(T) \cap F(S) \cap cent_G(A) \neq \emptyset$, provided one of the following conditions holds:*
(i) $cent_G(A)$ is closed and $clT(cent_G(A))$ is compact.
(ii)X is complete, $cent_G(A)$ is weakly closed, S is weakly continuous, $wclT(cent_G(A)$ is weakly compact and $I - T$ is demiclosed at 0.

Proof. In both of the cases (i) -(ii), Lemma 4 implies that, for each $n \geq 1$, there exists $x_n \in cent_G(A)$ such that $x_n = Sx_n = F(\mu_n, T^n x_n, p)$. The result now follows from Theorem 4.

Corollary 13. *([40], Theorem 2.3). Let K be a nonempty subset of a normed space X and $y_1, y_2 \in X$. Suppose that T and S are self-mappings of K such that T is asymptotically $S-$nonexpansive. Suppose that the set $F(S)$ is nonempty. Let the set D, of best simultaneous K-approximates to y_1 and y_2, is nonempty compact and starshaped with respect to an element p in $F(S)$ and D is invariant under T. Assume further that T and S are commuting, T is uniformly asymptotically regular on D, S is affine with $S(D) = D$. Then D contains a $T-$ and $S-$invariant points.*

Another extension of Theorem 2.3 due to Vijayraju [40] is presented below;

Theorem 14. *Let K be a nonempty subset of a normed space X and $y_1, y_2 \in X$. Suppose that T and S are self-mappings of K. Assume that the set D, of best simultaneous K-approximants to y_1 and y_2, is nonempty and invariant under T and S, (T, S) is a Banach operator pair on D, $D_0 := F(S) \cap D$ is closed and F-starshaped with respect to an element $p \in D_0$, where F is an upper semi-convex structure on D_0. If $cl(T(D_0))$ is compact and (T, S) satisfies (10), for all $x, y \in D_0$, and all $k \in (0, 1)$, then D contains a $T-$ and $S-$invariant point.*

Proof. Proof is similar to that of Theorem 12 instead of applying Theorem 4 we apply Corollary 11 to obtain the conclusion.

Remark 2. *As an application of Theorems 7 and corolary 9, best simultaneous approximation results similar to Theorem 12 can be established which extend the recent results of Akbar and A. R. Khan [1], Al-Thagafi [2], Chen and Li [5], Habiniak [15], Hussain, O'Regan and Agarwal [17], Hussain and Rhoades [18], Hussain, Rhoades and Jungck [19], Jungck and Sessa [24], Khan et al. [25], Sahab, Khan and Sessa [34], Sahney and Singh [35], Singh [36, 37], Smoluk [38], Subrahmanyam [39] and Vijayraju [40] to ordered semi-convex structure* $(X, F, \leq)$.

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