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# Center and pseudo-isochronous conditions in a quasi analytic system

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**Abstract**

The center conditions and pseudo-isochronous center conditions at origin or infinity in a class of non-analytic polynomial differential system are classified in this paper. By proper transforms, the quasi analytic system can be changed into an analytic system, and then the first 77 singular values and periodic constants are computed by Mathematics. Finally, we investigate the center conditions and pseudo-isochronous center conditions at infinity for the system. Especially, this system was investigated when $\lambda = 1$ in [Y. Wu, W. Huang, H. Dai, Qual. Theory Dyn. Syst., **10** (2011), 123–138]. 

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## 1. Introduction

It is well known that the study of limit cycles bifurcation from infinity and center problem is an important part of the so called weakened $16-th$ Hilbert problem. As far as limit cycles bifurcated from infinity are concerned, there have been many results [2, 3, 6, 11, 14, 29] for special continuous systems. Bifurcation of a periodic orbit from infinity has been also studied for polynomial planar vector fields, see for instance Sotomayor and Paterlini [26], Blows and Rousseau [2], and Gunez, Saez and Szanto [4]. Other papers about bifurcation of periodic orbits from infinity are due to Keith and Rand [7], Malaguti [22], A. K. Alomari[1] and Sabatini [25], where they study the Rayleigh, Vanderpol and Lienard systems. But for general system

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it is still a hard work to solve its center problem. A special system with a singular point at infinity

$$\begin{aligned}\frac{dx}{dt} &= (\delta x - y)(x^2+y^2)^n + \sum_{k=0}^{2n} X_k(x,y),\\ \frac{dy}{dt} &= (x+\delta y)(x^2+y^2)^n + \sum_{k=0}^{2n} Y_k(x,y),\end{aligned} \tag{1.1}$$

where

$$X_k(x,y) = \sum_{\alpha+\beta=k} A_{\alpha\beta}x^\alpha y^\beta, \quad Y_k(x,y) = \sum_{\alpha+\beta=k} B_{\alpha\beta}x^\alpha y^\beta, \tag{1.2}$$

was discussed by Liu *et al.* in [8], where $X_k(x,y)$ and $Y_k(x,y)$ are homogeneous polynomials of order $k$. System (1.1) can be changed into

$$\begin{aligned}\frac{d\xi}{d\tau} &= \frac{-\delta}{2n+1}\xi - \eta + \sum_{k=1}^{2n+1}\left[\left(-\frac{1}{2n+1}\xi^2+\eta^2\right)X_{2n+1-k}(\xi,\eta) - \frac{2n+2}{2n+1}\xi\eta Y_{2n+1-k}(\xi,\eta)\right]\\ &\qquad \times (\xi^2+\eta^2)^{(n+1)(k-1)},\\ \frac{d\eta}{d\tau} &= \xi - \frac{\delta}{2n+1}\eta + \sum_{k=1}^{2n+1}\left[\left(\xi^2-\frac{1}{2n+1}\eta^2\right)Y_{2n+1-k}(\xi,\eta) - \frac{2n+2}{2n+1}\xi\eta X_{2n+1-k}(\xi,\eta)\right]\\ &\qquad \times (\xi^2+\eta^2)^{(n+1)(k-1)},\end{aligned} \tag{1.3}$$

by transformations

$$x = \frac{\xi}{(\xi^2+\eta^2)^{n+1}}, \quad y = \frac{\eta}{(\xi^2+\eta^2)^{n+1}}, \quad dt = (\xi^2+\eta^2)^{n(2n+1)}d\tau. \tag{1.4}$$

In [18, 19, 20, 21], Llibre studied the following systems

$$\begin{aligned}\dot z &= (\lambda+i)z + (z\bar z)^{\frac{d-5}{2}}(Az^{4+j}\bar z^{1-j} + Bz^3\bar z^2 + Cz^{2-j}\bar z^{3+j} + D\bar z^5), \ d=2m+1\ge 5;\\ \dot z &= iz + (z\bar z)^{\frac{d-4}{2}}(Az^3\bar z + Bz^2\bar z^2 + C\bar z^4), \ d=2m\ge 4;\\ \dot z &= (\lambda+i)z + (z\bar z)^{\frac{d-3}{2}}(Az^3 + Bz^2\bar z + Cz\bar z^2 + D\bar z^3), \ d=2m+1\ge 3;\\ \dot z &= (\lambda+i)z + (z\bar z)^{\frac{d-2}{2}}(Az^2 + Bz\bar z + C\bar z^2), \ d=2m\ge 2.\end{aligned}$$

They obtained the conditions of centers and isochronous centers, but the $d$ is restricted in order to make the system to be polynomial system. $d$ was restricted strictly in order to make those systems to be analytic system. In fact, quasi-analytic systems have been widely used in modeling many practical problems in science and engineering recently. For example, an axis-symmetric quasi-analytical model was presented in order to simulate the behavior of a RFEC system during its operation [23]. A quasi-analytical model to predict and analyze signals on layered samples measured by infrared scattering type scanning near-field optical microscopy was modeled in [5]. A simple quasi-analytical model was developed to study the response of ice-sheets to climate change in [24]. It is also noted that a type of quasi-analytic systems, described by

$$\begin{aligned}\dot x &= \delta x - y + \sum_{k=2}^{\infty}(x^2+y^2)^{\frac{(k-1)(\lambda-1)}{2}}X_k(x,y),\\ \dot y &= x + \delta y + \sum_{k=2}^{\infty}(x^2+y^2)^{\frac{(k-1)(\lambda-1)}{2}}Y_k(x,y),\end{aligned}$$

where

$$X_k(x,y) = \sum_{\alpha+\beta=k} A_{\alpha\beta}x^\alpha y^\beta, \quad Y_k(x,y) = \sum_{\alpha+\beta=k} B_{\alpha\beta}x^\alpha y^\beta,$$

has been studied by Liu *et al.* [10, 17]. As special cases, quasi-analytic quadratic systems have been studied in [16] and cubic systems in [28]. As far as center conditions at origin are concerned, there are very few results for the case of non-analytic systems, several special systems have been studied, see [10, 17].

The problems of center conditions and pseudo-isochronous center conditions for quasi-analytic system are poorly-understood. In this paper, we investigate center conditions and pseudo-isochronous center conditions for a class of quasi-analytic septic system

$$\begin{aligned}\frac{dx}{dt} &= (\delta x - \beta y) + X_5(x,y)(x^2+y^2)^{\lambda-2} - y(x^2+y^2)^{2\lambda},\\ \frac{dy}{dt} &= (\beta x + \delta y) + Y_5(x,y)(x^2+y^2)^{\lambda-2} + x(x^2+y^2)^{2\lambda},\end{aligned} \tag{1.5}$$

where

$$X_5(x,y) = \sum_{k+j=5} A_{kj}x^k y^j, \quad Y_5(x,y) = \sum_{k+j=5} B_{kj}x^k y^j,$$

$$\begin{array}{ll} A_{50} = \beta_{03}+\beta_{12}+\beta_{21}+\beta_{30}, & A_{41} = -5\alpha_{03}-3\alpha_{12}-\alpha_{21}+\alpha_{30},\\ A_{32} = -2(5\beta_{03}+\beta_{12}-\beta_{21}-\beta_{30}), & A_{23} = 2(5\alpha_{03}-\alpha_{12}-\alpha_{21}+\alpha_{30}),\\ A_{14} = 5\beta_{03}-3\beta_{12}+\beta_{21}+\beta_{30}, & A_{05} = -\alpha_{03}+\alpha_{12}-\alpha_{21}+\alpha_{30},\\ B_{50} = \alpha_{03}+\alpha_{12}+\alpha_{21}+\alpha_{30}, & B_{41} = 5\beta_{03}+3\beta_{12}+\beta_{21}-\beta_{30},\\ B_{32} = -2\left(5\alpha_{03}+\alpha_{12}-\alpha_{21}-\alpha_{30}\right), & B_{23} = -2(5\beta_{03}-\beta_{12}-\beta_{21}+\beta_{30}),\\ B_{14} = 5\alpha_{03}-3\alpha_{12}+\alpha_{21}+\alpha_{30}, & B_{05} = \beta_{03}-\beta_{12}+\beta_{21}-\beta_{30},\\ \lambda\in\mathbb{R}\ . & \end{array} \tag{1.6}$$

The paper will be organized as follows. Some preliminary results are given in Section 2. In Section 3, system (1.5) is reduced to analytic system by some proper transformations. Furthermore, we compute the singular point quantities and derive the center conditions of the origin for the transformed system. In Section 4, we compute the period constants and discuss isochronous and pseudo-isochronous center conditions.

## 2. Some preliminary results

Complex center and isochronous center for the following system

$$\begin{aligned}\frac{dz}{dT} &= z + \sum_{k=2}^{\infty} Z_k(z,w) = Z(z,w),\\ \frac{dw}{dT} &= -w - \sum_{k=2}^{\infty} W_k(z,w) = -W(z,w),\end{aligned} \tag{2.1}$$

were defined in [9, 12, 13, 15]. The following theorems could be used to compute focal values and periodic constants.

**Theorem 2.1** ([9])**.** *For system* (2.1)*, we can derive successively the terms of the following formal series:*

$$M(z,w) = \sum_{\alpha+\beta=0}^{\infty} c_{\alpha\beta}z^\alpha w^\beta,$$

*such that*

$$\frac{\partial(MZ)}{\partial z} - \frac{\partial(MW)}{\partial w} = \sum_{m=1}^{\infty}(m+1)\mu_m(zw)^m,$$

*where* $c_{00} = 1, \forall c_{kk} \in \mathbb{R}, k = 1, 2, \cdots$ *, and for any integer* $m$*,* $\mu_m$ *is determined by the following recursive formulae:*

$$c_{00} = 1; \qquad \text{when } (\alpha = \beta > 0) \text{ or } \alpha < 0, \text{ or } \beta < 0, c_{\alpha\beta} = 0;$$

else;

$$c_{\alpha\beta} = \frac{1}{\beta - \alpha} \sum_{k+j=3}^{\alpha+\beta+2} [(\alpha + 1)a_{k,j-1} - (\beta + 1)b_{j,k-1}]c_{\alpha-k+1,\beta-j+1},$$

$$\mu_m = \sum_{k+j=3}^{2m+2} (a_{k,j-1} - b_{j,k-1})c_{m-k+1,m-j+1}.$$

**Theorem 2.2** ([13])**.** *For system* (2.1)*, we can derive uniquely the following formal series:*

$$f(z, w) = z + \sum_{k+j=2}^{\infty} c'_{kj} z^k w^j, g(z, w) = w + \sum_{k+j=2}^{\infty} d'_{kj} w^k z^j,$$

*where* $c'_{k+1,k} = d'_{k+1,k} = 0, k = 1, 2, \cdots$ *, such that*

$$\frac{df}{dT} = f(z, w) + \sum_{j=1}^{\infty} p'_j z^{j+1} w^j, \frac{dg}{dT} = -g(z, w) - \sum_{j=1}^{\infty} q'_j w^{j+1} z^j,$$

*and when* $k - j - 1 \neq 0, c'_{kj}$ *and* $d'_{kj}$ *are determined by the following recursive formulae:*

$$c'_{kj} = \frac{1}{j + 1 - k} \sum_{\alpha+\beta=3}^{k+j+1} [(k - \alpha + 1)a_{\alpha,\beta-1} - (j - \beta + 1)b_{\beta,\alpha-1}]c'_{k-\alpha+1,j-\beta+1},$$

$$d'_{kj} = \frac{1}{j + 1 - k} \sum_{\alpha+\beta=3}^{k+j+1} [(k - \alpha + 1)b_{\alpha,\beta-1} - (j - \beta + 1)a_{\beta,\alpha-1}]d'_{k-\alpha+1,j-\beta+1},$$

*and for any positive integer* $j, p'_j$ *and* $q'_j$ *are determined by the following recursive formulae:*

$$p'_j = \sum_{\alpha+\beta=3}^{2j+2} [(j - \alpha + 2)a_{\alpha,\beta-1} - (j - \beta + 1)b_{\beta,\alpha-1}]c'_{j-\alpha+2,j-\beta+1},$$

$$q'_j = \sum_{\alpha+\beta=3}^{2j+2} [(j - \alpha + 2)b_{\alpha,\beta-1} - (j - \beta + 1)a_{\beta,\alpha-1}]d'_{j-\alpha+2,j-\beta+1}.$$

*In the above expression, we have let* $c'_{10} = d'_{10} = 1, c'_{01} = d'_{01} = 0$*, and if* $\alpha < 0$ *or* $\beta < 0$*, let* $a_{\alpha\beta} = b_{\alpha\beta} = c'_{\alpha\beta} = d'_{\alpha\beta} = 0$.

**Theorem 2.3** ([15])**.** (The extended symmetric principle) *Let* $g$ *denote an elementary Lie invariant of system* (2.1)*. If for all* $g$ *the symmetric condition* $g = g^*$ *is satisfied, then the origin of system* (2.1) *is a complex center. Namely, all singular point quantities of the origin are zero.*

## 3. Singular point quantities and center conditions

System (1.5) with $\delta = 0$ could be changed into its concomitant complex system

$$\begin{aligned} \frac{du}{dT} &= u(uv)^{\lambda} + (uv)^{2(\lambda-1)}(a_{03}u^5 + a_{12}vu^4 + a_{21}u^3v^2 + a_{30}u^2v^3) - \beta u(uv)^{3\lambda}, \\ \frac{dv}{dT} &= -v(uv)^{\lambda} - (uv)^{2(\lambda-1)}(b_{03}v^5 + b_{12}v^4u + b_{21}u^2v^3 + b_{30}u^3v^2) + \beta v(uv)^{3\lambda)}, \end{aligned} \tag{3.1}$$

by using transformation

$$u = x + iy, \quad v = x - iy, \quad T = it, \quad i = \sqrt{-1}, \tag{3.2}$$

where

$$\begin{array}{llll} a_{30} = \alpha_{30} + i\beta_{30}, & a_{21} = \alpha_{21} + i\beta_{21}, & a_{12} = \alpha_{12} + i\beta_{12}, & a_{03} = \alpha_{03} + i\beta_{03}, \\ b_{30} = \alpha_{30} - i\beta_{30}, & b_{21} = \alpha_{21} - i\beta_{21}, & b_{12} = \alpha_{12} - i\beta_{12}, & b_{03} = \alpha_{03} - i\beta_{03}. \end{array} \tag{3.3}$$

Then, by means of transformations

$$\xi = u^{\frac{\lambda+1}{2}} v^{\frac{\lambda-1}{2}}, \ \eta = v^{\frac{\lambda+1}{2}} u^{\frac{\lambda-1}{2}},$$

and $(1.4)_{n=3}$, system (3.1) is reduced to

$$\begin{aligned} l\frac{dz}{d\tau} &= z + (\frac{1}{2}b_{03} + \frac{1}{14}b_{03}\lambda)w^9z^6 + \frac{1}{2}(a_{30} + b_{12})w^8z^7 + \frac{\lambda}{14}(b_{12} - a_{30})w^8z^7 \\ &+ \frac{1}{2}(a_{21} + b_{21})w^7z^8 + \frac{\lambda}{14}(b_{21} - a_{21})w^7z^8 + \frac{1}{2}(a_{12} + b_{30})w^6z^9 \\ &+ \frac{\lambda}{14}(a_{12} + b_{30})w^6z^9 + \frac{1}{14}(7a_{03} - a_{03}\lambda)w^5z^{10} + \beta w^{14}z^{15}, \\ \frac{dw}{d\tau} &= -w - (\frac{1}{2}a_{03} + \frac{1}{14}a_{03}\lambda)z^9w^6 - \frac{1}{2}(b_{30} + a_{12})z^8w^7 - \frac{\lambda}{14}(a_{12} - b_{30})z^8w^7 \\ &- \frac{1}{2}(b_{21} + a_{21})z^7w^8 - \frac{\lambda}{14}(a_{21} - b_{21})z^7w^8 - \frac{1}{2}(b_{12} + a_{30})z^6w^9 \\ &- \frac{\lambda}{14}(b_{12} + a_{30})z^6w^9 - \frac{1}{14}(7b_{03} - b_{03}\lambda)z^5w^{10} - \beta z^{14}w^{15}. \end{aligned} \tag{3.4}$$

The singular point quantities at the origin of system (3.4) can be computed by using the recursive formulae of Theorem 2.1 and simplify them with the constructive theorem of singular point quantities, we get the following theorem.

**Theorem 3.1.** *The first 77 singular point quantities at the origin of system* (3.4) *are as follows:*

$$\begin{aligned} \mu_7 &= -\frac{1}{7}(a_{21} - b_{21})\lambda, \\ \mu_{14} &= \frac{1}{7}(a_{30}a_{12} - b_{30}b_{12})\lambda, \end{aligned}$$

*Case 1* $a_{12}b_{12} \neq 0$, *then there exist* $k$ *to make* $a_{30} = kb_{12}, b_{30} = ka_{12}$,

$$\begin{aligned} \mu_{21} &= \frac{\lambda}{56}(a_{03}b_{12}^2 - b_{03}a_{12}^2)(-1 + 3k)(-2 - 2k - \lambda + k\lambda), \\ \mu_{28} &= -\frac{\lambda^2}{14(\lambda - 2)}b_{21}(a_{03}b_{12}^2 - b_{03}a_{12}^2)(3k - 1), \\ \mu_{35} &= -\frac{\lambda^2}{336(\lambda - 2)^3}(3k - 1)(a_{03}b_{12}^2 - b_{03}a_{12}^2)(-32a_{03}b_{03} - 28a_{03}b_{03}\lambda + 32a_{12}b_{12}\lambda^2 \\ &\quad +5a_{03}b_{03}\lambda^3 + a_{03}b_{03}\lambda^4 + 192\beta + 192\lambda\beta + 48\lambda^2\beta), \\ \mu_{42} &= 0, \\ \mu_{49} &= \frac{\lambda^2}{26880(\lambda - 2)^5}(3k - 1)(a_{03}b_{12}^2 - b_{03}a_{12}^2)(1024a_{03}^2b_{03}^2 + 1920a_{03}^2b_{03}^2\lambda \\ &\quad -13824a_{12}a_{03}b_{12}b_{03}\lambda^2 + 224a_{03}^2b_{03}^2\lambda^2 - 11392a_{12}a_{03}b_{12}b_{03}\lambda^3 - 1584a_{03}^2b_{03}^2\lambda^3 \end{aligned}$$

$$
\begin{aligned}
&+12800a_{12}^2b_{12}^2\lambda^4+2432a_{12}a_{03}b_{12}b_{03}\lambda^4-1056a_{03}^2b_{03}^2\lambda^4+4064a_{12}a_{03}b_{12}b_{03}\lambda^5\\
&+864a_{12}a_{03}b_{12}b_{03}\lambda^6+206a_{03}^2b_{03}^2\lambda^6+69a_{03}^2b_{03}^2\lambda^7+7a_{03}^2b_{03}^2\lambda^8),\\
\mu_{56} \;=\;& -\frac{7\lambda^2}{576(\lambda-2)^5}(3k-1)(a_{03}b_{12}^2-b_{03}a_{12}^2)(a_{03}b_{12}^2+b_{03}a_{12}^2)(\lambda-1)\\
&\times(-32a_{03}b_{03}-24a_{03}b_{03}\lambda+32a_{12}b_{12}\lambda^2+4a_{03}b_{03}\lambda^2+6a_{03}b_{03}\lambda^3+a_{03}b_{03}\lambda^4),
\end{aligned}
$$

*If* $a_{12}b_{12}=-\frac{a_{03}b_{03}(-2+\lambda)(2+\lambda)^2(4+\lambda)}{32\lambda^2}$,

$$\mu_{63}=-\tfrac{24131}{3674160}(3k-1)a_{03}^3b_{03}^3(a_{03}b_{12}^2-b_{03}a_{12}^2)\lambda^2(\lambda-1).$$

*If* $a_{03}b_{12}^2+b_{03}a_{12}^2=0$, *then there exist* $m$ *to make* $a_{03}=ma_{12}^2, b_{03}=-mb_{12}^2$,

$$
\begin{aligned}
\mu_{63} \;=\;& \frac{\lambda^2}{6773760000(\lambda-2)^3}(3k-1)(\lambda-1)a_{12}b_{12}a_{03}^2b_{03}^2(a_{03}b_{12}^2-b_{03}a_{12}^2)\\
&\times(-23257088a_{12}b_{12}m^2+6577280a_{12}b_{12}m^2\lambda-23257088\lambda^2+26650064a_{12}b_{12}m^2\lambda^2\\
&+75164416\lambda^3-2764244a_{12}b_{12}m^2\lambda^3+8304896\lambda^4-10922212a_{12}b_{12}m^2\lambda^4-18884608\lambda^5\\
&-916685a_{12}b_{12}m^2\lambda^5+1341691a_{12}b_{12}m^2\lambda^6+255854a_{12}b_{12}m^2\lambda^7),\\
\mu_{70} \;=\;& 0,\\
\mu_{77} \;=\;& -\frac{8\lambda^{11}}{6806835}(3k-1)(a_{03}b_{12}^2-b_{03}a_{12}^2)(\lambda-1).
\end{aligned}
$$

*Case 2* $a_{12}=b_{12}=0$,

$$
\begin{aligned}
\mu_{21} &= \frac{3\lambda}{56}(a_{03}a_{30}^2-b_{03}b_{30}^2)(\lambda-2),\\
\mu_{28} &= \frac{3}{14}(a_{03}a_{30}^2-b_{03}b_{30}^2)b_{21},\\
\mu_{35} &= -\frac{1}{56}(a_{03}a_{30}^2-b_{03}b_{30}^2)(4a_{30}b_{30}-3a_{03}b_{03}+24\beta),\\
\mu_{42} &= 0,\\
\mu_{49} &= -\frac{1}{560}(a_{03}a_{30}^2-b_{03}b_{30}^2)(-11a_{03}^2b_{03}^2-19a_{03}b_{03}a_{30}b_{30}+50a_{30}^2b_{30}^2),\\
\mu_{56} &= \frac{1}{96}(a_{03}a_{30}^2-b_{03}b_{30}^2)(a_{03}a_{30}^2+b_{03}b_{30}^2)(a_{03}b_{03}-a_{30}b_{30}),\\
\mu_{63} &= -\frac{1}{560}a_{30}^2b_{30}^2(a_{03}a_{30}^2-b_{03}b_{30}^2)(-50-19a_{30}b_{30}m^2+11a_{30}^2b_{30}^2m^4),
\end{aligned}
$$

*where* $\mu_k=0, k\neq 7i, i\leq 11, i\in N$. *In the above expression of* $\mu_k$, *we have already let* $\mu_1=\cdots=\mu_{k-1}=0$, $k=2,3,\cdots,77$.

Theorem 3.1 implies that

**Theorem 3.2.** *The first 77 singular point quantities of system* (3.4) *are zero if and only if one of the following conditions holds,*

$$a_{21}=b_{21},\quad a_{12}=b_{12}=0,\quad a_{30}^2a_{03}=b_{03}b_{30}^2; \tag{3.5}$$

$$a_{21}=b_{21},\quad a_{30}=\frac{1}{3}b_{12},\quad b_{30}=\frac{1}{3}a_{12},\quad a_{12}b_{12}\neq 0; \tag{3.6}$$

$$a_{21}=b_{21},\quad a_{30}a_{12}=b_{30}b_{12},\quad a_{12}^2b_{03}=b_{12}^2a_{03},\quad a_{12}b_{12}\neq 0. \tag{3.7}$$

$$\lambda = 1, \quad \beta = 0, \quad a_{21} = b_{21} = 0, \quad a_{30} = -3b_{12}, \quad b_{30} = -3a_{12}, \quad a_{03}b_{03} = 4a_{12}b_{12}, \quad a_{12}b_{12} \neq 0. \tag{3.8}$$

*Correspondingly, the conditions in Theorem 3.2 are the center conditions of the origin.*

In order to prove this Theorem, according to the technique used in [17], we can find out all the elementary Lie invariants of system (3.4) firstly which are given in following Lemma.

**Lemma 3.3.** *All the elementary Lie invariants of system* (3.4) *are as follows:*

$$\begin{gathered} \beta, a_{21}, b_{21}, a_{30}b_{30}, a_{12}b_{12}, a_{03}b_{03}, a_{30}a_{12}, b_{30}b_{12}, \\ a_{30}^2a_{03}, a_{30}b_{12}a_{03}, b_{12}^2a_{03}, b_{30}^2b_{03}, b_{30}a_{12}b_{03}, a_{12}^2b_{03}. \end{gathered} \tag{3.9}$$

*Proof.* When condition (3.5) or (3.7) holds, system (3.4) satisfies the conditions of Theorem 2.3. If condition (3.6) holds, system (3.4) has the first integral

$$\begin{cases} zwe^{3-3a_{03}z^9w^5-4a_{12}z^8w^6-6b_{21}z^7w^7-4b_{12}z^6w^8-3b_{03}z^5w^9-3z^{14}w^{14}\beta}, & \lambda = 2, \\ (zw)^{-\frac{7(\lambda-2)}{\lambda}} f_1, & \lambda \neq 2, \end{cases}$$

where

$$\begin{aligned} f_1 = & -24 - 12\lambda(-12a_{03} + 3a_{03}\lambda^2)z^9w^5 + (4a_{12}\lambda^2 - 16a_{12})z^8w^6 + (6b_{21}\lambda^2 - 24b_{21})z^7w^7 \\ & + (4b_{12}\lambda^2 - 16b_{12})z^6w^8 + (3b_{03}\lambda^2 - 12b_{03})z^5w^9 - (24\beta - 12\lambda\beta)z^{10}w^{10}. \end{aligned}$$

When condition (3.8) is satisfied, system (3.4) can be rewritten as

$$\begin{aligned} \frac{dz}{dT} &= \frac{1}{14}(14 + 6a_{03}z^9w^5 - 18a_{12}z^8w^6 - 10b_{12}z^6w^8 + 8b_{03}z^5w^9), \\ \frac{dw}{dT} &= -\frac{1}{14}(14 + 6b_{03}z^5w^9 - 18b_{12}w^8z^6 - 10a_{12}w^6z^8 + 8a_{03}w^5z^9). \end{aligned} \tag{3.10}$$

By means of transformation

$$z = \frac{u}{(uv)^{\frac{3}{7}}}, \quad w = \frac{w}{(uv)^{\frac{3}{7}}},$$

system (3.10) is transformed into

$$\begin{aligned} \frac{du}{dT} &= u + b_{03}v^3 + b_{12}v^2u - 3a_{12}u^3 = U(u,v), \\ \frac{dv}{dT} &= -(v + a_{03}u^3 + a_{12}u^2v - 3b_{12}v^3) = -V(u,v), \end{aligned} \tag{3.11}$$

which has a integrating factor $f_2^{-\frac{5}{6}}$, where

$$\begin{aligned} f_2 = 1 & - 6(b_{12}u^2 + a_{12}v^2) \\ & + 3(3b_{12}^2u^4 - 2a_{12}b_{03}u^3v + 2a_{12}b_{12}u^2v^2 - 2b_{12}a_{03}v^3u + 3a_{12}^4v^4) \\ & + \frac{1}{2}(2a_{12}u - a_{03}v)(2b_{12}v - b_{03}u)(b_{03}u^4 - 2b_{12}u^3v - 2a_{12}v^3u + a_{03}v^4) \end{aligned} \tag{3.12}$$

and

$$\frac{df_2}{dt} = -12(b_{12}u^2 - a_{12}v^2)f_2 = \frac{6}{5}(\frac{\partial U}{\partial u} - \frac{\partial V}{\partial v})f_2. \tag{3.13}$$

□

## 4. Period constants and isochronous center conditions

In this section, we devote to discuss the isochronous center conditions for system (3.4). First of all, we compute period constants according to Theorem 2.2 from the center conditions given in Section 4. Then the sufficiency are proved by different means. The complex isochronous center conditions are given in following Theorem.

**Theorem 4.1.** *The origin of system* (3.4) *is a complex isochronous center if and only if one of the following conditions holds*

$$\beta = a_{21} = b_{21} = a_{12} = b_{12} = a_{30} = b_{30} = 0,\ \lambda = -2; \tag{4.1}$$

$$\beta = a_{21} = b_{21} = a_{12} = b_{12} = a_{03} = b_{03} = 0,\ \lambda = 1,\ a_{30}b_{30} \neq 0; \tag{4.2}$$

$$\lambda = -2,\ \beta = a_{21} = b_{21} = a_{03} = b_{03} = 0,\ a_{30} = \tfrac{1}{3}b_{12},\ b_{30} = \tfrac{1}{3}a_{12}. \tag{4.3}$$

$$\beta = a_{21} = b_{21} = a_{03} = b_{03} = 0,\ a_{30} = -b_{12},\ b_{30} = -a_{12}; \tag{4.4}$$

$$\beta = a_{21} = b_{21} = a_{03} = b_{03} = 0,\ a_{30} = \tfrac{1+\lambda}{\lambda-1}b_{12},\ b_{30} = \tfrac{1+\lambda}{\lambda-1}a_{12}. \tag{4.5}$$

*Proof.* When condition (4.1) is satisfied, system (3.4) could be rewritten as

$$\begin{aligned} \frac{dz}{dT} &= \frac{1}{14}z(14 + 9a_{03}z^9w^5 + 5b_{03}z^5w^9), \\ \frac{dw}{dT} &= -\frac{1}{14}w(14 + 5a_{03}z^9w^5 + 9b_{03}z^5w^9), \end{aligned} \tag{4.6}$$

there exists a transformation

$$u = \frac{z(1 + b_{03}z^5w^9)^{\frac{5}{56}}}{(1 + a_{03}z^9w^5)^{\frac{9}{56}}},\quad v = \frac{w(1 + a_{03}z^9w^5)^{\frac{5}{56}}}{(1 + b_{03}z^5w^9)^{\frac{9}{56}}},$$

such that system (4.6) is reduced to a linear system.

When condition (4.2) holds, system (3.4) becomes

$$\begin{aligned} \tfrac{dz}{dT} &= \tfrac{1}{14}z(14 + 8b_{30}z^8w^6 + 6a_{30}z^6w^8), \\ \tfrac{dw}{dT} &= -\tfrac{1}{14}w(14 + 6b_{30}z^8w^6 + 8a_{30}z^6w^8). \end{aligned} \tag{4.7}$$

By means of a transformation

$$u = \frac{z(1 + a_{30}z^6w^8)^{\frac{3}{14}}}{(1 + b_{30}z^8w^6)^{\frac{2}{7}}},\quad v = \frac{w(1 + b_{30}z^8w^6)^{\frac{3}{14}}}{(1 + a_{30}z^6w^8)^{\frac{2}{7}}}, \tag{4.8}$$

system (4.7) is reduced to a linear system.

When condition (4.3) is satisfied, system (3.4) is rewritten as

$$\begin{aligned} \frac{dz}{dT} &= \frac{1}{14}z(14 + 8a_{12}z^8w^6 + \frac{32}{3}b_{12}z^6w^8), \\ \frac{dw}{dT} &= -\frac{1}{14}w(14 + 8b_{12}z^6w^8 + \frac{32}{3}a_{12}z^8w^6), \end{aligned} \tag{4.9}$$

it also could be reduced to a linear system by a transformation

$$u = \frac{z(3 + 4b_{12}z^6w^8)^{\frac{3}{14}}}{(3 + 4a_{12}z^8w^6)^{\frac{2}{7}}},\quad v = \frac{w(3 + 4a_{12}z^8w^6)^{\frac{3}{14}}}{(3 + 4b_{12}z^6w^8)^{\frac{2}{7}}}.$$

When condition (4.4) is fulfilled, system (3.4) becomes

$$\begin{aligned}\frac{dz}{dT} &= \frac{1}{14}z(14-2a_{12}\lambda z^8w^6+2b_{12}\lambda z^6w^8),\\ \frac{dw}{dT} &= -\frac{1}{14}w(14-2b_{12}\lambda z^6w^8+2b_{12}\lambda z^8w^6),\end{aligned}\tag{4.10}$$

we have for system (4.10) that

$$\frac{d\theta}{dt}=\frac{1}{2}\left(\frac{1}{z}\frac{dz}{dT}-\frac{1}{w}\frac{dw}{dT}\right)=1.\tag{4.11}$$

When condition (4.5) is satisfied, system (3.4) could be rewritten as

$$\begin{aligned}\tfrac{dz}{dT} &= \tfrac{1}{7(\lambda-1)}z(7(\lambda-1)+8a_{12}\lambda z^8w^6+6b_{12}\lambda z^6w^8),\\ \tfrac{dw}{dT} &= -\tfrac{1}{7(\lambda-1)}w(5(\lambda-1)+8b_{12}\lambda z^6w^8+6b_{12}\lambda z^8w^6),\end{aligned}\tag{4.12}$$

there exists a transformation

$$u=\frac{z(-1+\lambda+2b_{12}\lambda z^6w^8)^{\frac{3}{14}}}{(-1+\lambda+2a_{12}\lambda z^8w^6)^{\frac{2}{7}}},\quad v=\frac{w(-1+\lambda+2a_{12}\lambda z^8w^6)^{\frac{3}{14}}}{(-1+\lambda+2b_{12}\lambda z^6w^8)^{\frac{2}{7}}},$$

such that system (4.12) is reduced to a linear system. □

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