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Algorithms for the variational inequalities and fixed point problems

Yaqiang Liu[a], Zhangsong Yao[b,*], Yeong-Cheng Liou[c], Li-Jun Zhu[d]

[a]*School of Management, Tianjin Polytechnic University, Tianjin 300387, China.*

[b]*School of Information Engineering, Nanjing Xiaozhuang University, Nanjing 211171, China.*

[c]*Department of Information Management, Cheng Shiu University, Kaohsiung 833, Taiwan and Center for General Education, Kaohsiung Medical University, Kaohsiung 807, Taiwan.*

[d]*School of Mathematics and Information Science, Beifang University of Nationalities, Yinchuan 750021, China.*

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Abstract

A system of variational inequality and fixed point problems is considered. Two algorithms have been constructed. Our algorithms can find the minimum norm solution of this system of variational inequality and fixed point problems.

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1. Introduction

Variational inequality problems were initially studied by Stampacchia [17] in 1964. Variational inequalities have applications in diverse disciplines such as physical, optimal control, optimization, mathematical programming, mechanics and finance, see [12], [16], [17], [29] and the references therein. The main purpose of this paper is devoted to find the minimum norm solution of some system of variational inequality and fixed point problems.

Let $\mathbb{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let $\mathbb{C}$ be a nonempty closed convex subset of $\mathbb{H}$.

*Corresponding author
Email addresses: yani3115791@126.com (Yaqiang Liu), yaozhsong@163.com (Zhangsong Yao), simplex_liou@hotmail.com (Yeong-Cheng Liou), zhulijun1995@sohu.com (Li-Jun Zhu)

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Definition 1.1. A mapping $\mathbb{F} : \mathbb{C} \to \mathbb{H}$ is called ζ-inverse strongly monotone if there exists a real number $\zeta > 0$ such that

$$\langle \mathbb{F}x - \mathbb{F}y, x - y\rangle \geq \zeta\|\mathbb{F}x - \mathbb{F}y\|^2,\ \forall x, y \in \mathbb{C}.$$

Definition 1.2. A mapping $\mathbb{R} : \mathbb{C} \to \mathbb{H}$ is called κ-contraction, if there exists a constant $\kappa \in [0, 1)$ such that $\|\mathbb{R}(x) - \mathbb{R}(y)\| \leq \kappa\|x - y\|$ for all $x, y \in \mathbb{C}$.

Definition 1.3. A mapping $N : \mathbb{C} \to \mathbb{C}$ is said to be nonexpansive if

$$\|Nx - Ny\| \leq \|x - y\|, \forall x, y \in \mathbb{C}.$$

We use $Fix(N)$ to denote the set of fixed points of N.

Definition 1.4. We call $Proj_{\mathbb{C}} : \mathbb{H} \to \mathbb{C}$ is the metric projection if $Proj_{\mathbb{C}} : \mathbb{H} \to \mathbb{C}$ assigns to each point $x \in \mathbb{C}$ the unique point $Proj_{\mathbb{C}}x \in \mathbb{C}$ satisfying the property

$$\|x - Proj_{\mathbb{C}}x\| = \inf_{y\in\mathbb{C}} \|x - y\| =: d(x, \mathbb{C}).$$

Let $\mathbb{F} : \mathbb{C} \to \mathbb{H}$ be a nonlinear mapping. Recall that the classical variational inequality is to find $x^* \in \mathbb{C}$ such that

$$\langle \mathbb{F}x^*, x - x^*\rangle \geq 0 \quad \text{for all } x \in \mathbb{C}. \tag{1.1}$$

The set of solutions of the variational inequality (1.1) is denoted by $VI(\mathbb{F}, \mathbb{C})$. The variational inequality problem has been extensively studied in the literature. Related works, please see, e.g. [1]-[11], [13], [15], [19]-[28], [30]-[34] and the references therein. For finding an element of $Fix(N) \cap VI(\mathbb{F}, \mathbb{C})$, Takahashi and Toyoda [19] introduced the following iterative scheme:

$$x^{n+1} = \zeta_n x^n + (1 - \zeta_n)NProj_{\mathbb{C}}(x^n - \eta_n\mathbb{F}x^n), n \geq 0, \tag{1.2}$$

where $Proj_{\mathbb{C}}$ is the metric projection of $\mathbb{H}$ onto $\mathbb{C}$, $\{\zeta_n\}$ is a sequence in $(0, 1)$, and $\{\eta_n\}$ is a sequence in $(0, 2\zeta)$. Takahashi and Toyoda showed that the sequence $\{x^n\}$ converges weakly to some $z \in Fix(N) \cap VI(\mathbb{F}, \mathbb{C})$. Consequently, Nadezhkina and Takahashi [11] and Zeng and Yao [34] proposed some so-called extragradient methods for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem.

Recently, Ceng, Wang and Yao [3] considered a general system of variational inequality of finding $x^* \in \mathbb{C}$ such that

$$\begin{cases} \langle \eta\mathbb{F}y^* + x^* - y^*, x - x^*\rangle \geq 0, \forall x \in \mathbb{C}, \\ \langle \xi\mathbb{G}x^* + y^* - x^*, x - y^*\rangle \geq 0, \forall x \in \mathbb{C}, \end{cases} \tag{SVI}$$

where $\mathbb{F}, \mathbb{G} : \mathbb{C} \to \mathbb{H}$ are two nonlinear mappings, $y^* = Proj_{\mathbb{C}}(x^* - \xi\mathbb{G}x^*)$, $\eta > 0$ and $\xi > 0$ are two constants. The solutions set of SVI is denoted by Ω.

If take $\mathbb{F} = \mathbb{G}$, then SVI reduces to finding $x^* \in \mathbb{C}$ such that

$$\begin{cases} \langle \eta\mathbb{F}y^* + x^* - y^*, x - x^*\rangle \geq 0, \forall x \in \mathbb{C}, \\ \langle \xi\mathbb{F}x^* + y^* - x^*, x - y^*\rangle \geq 0, \forall x \in \mathbb{C}, \end{cases}$$

which is introduced by Verma [20] (see also Verma [21]). Further, if we add up the requirement that $x^* = y^*$, then SVI reduces to the classical variational inequality problem (1.1). For finding an element of $Fix(N)\cap\Omega$, Ceng, Wang and Yao [3] introduced the following relaxed extragradient method:

$$\begin{cases} y^n = Proj_{\mathbb{C}}(x^n - \xi\mathbb{G}x^n), \\ x^{n+1} = \zeta_n u + \beta_n x^n + \gamma_n NProj_{\mathbb{C}}(y^n - \eta\mathbb{F}y^n), n \geq 0. \end{cases} \tag{1.3}$$

They proved the strong convergence of the above method to some element in $Fix(N) \cap \Omega$.

On the other hand, in many problems, it is needed to find a solution with minimum norm. A typical example is the least-squares solution to the constrained linear inverse problem, see [14].

It is our purpose in this paper that we construct two methods, one implicit and one explicit, to find the minimum norm element in $Fix(N) \cap \Omega$; namely, the unique solution x^* to the quadratic minimization problem:

$$x^* = \arg \min_{x \in Fix(N) \cap \Omega} \|x\|^2.$$

We obtain two strong convergence theorems.

2. Preliminaries

Let $\mathbb{C}$ be a nonempty closed convex subset of $\mathbb{H}$. The following lemmas are useful for our main results.

Lemma 2.1. *Given $x \in \mathbb{H}$ and $z \in \mathbb{C}$.*

(i) $z = Proj_{\mathbb{C}} x$ if and only if there holds the relation:

$$\langle x - z, y - z \rangle \leq 0 \quad \text{for all } y \in \mathbb{C}.$$

(ii) $z = Proj_{\mathbb{C}} x$ if and only if there holds the relation:

$$\|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2 \quad \text{for all } y \in \mathbb{C}.$$

(iii) There holds the relation

$$\langle Proj_{\mathbb{C}} x - Proj_{\mathbb{C}} y, x - y \rangle \geq \|Proj_{\mathbb{C}} x - Proj_{\mathbb{C}} y\|^2 \quad \text{for all } x, y \in \mathbb{H}.$$

Consequently, $Proj_{\mathbb{C}}$ is nonexpansive and monotone.

Lemma 2.2 ([3])**.** *Let $\mathbb{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. Let the mapping $\mathbb{F} : \mathbb{C} \to \mathbb{H}$ be ζ-inverse strongly monotone. Then, we have*

$$\|(I - \eta\mathbb{F})x - (I - \eta\mathbb{F})y\|^2 \leq \|x - y\|^2 + \eta(\eta - 2\zeta)\|\mathbb{F}x - \mathbb{F}y\|^2, \forall x, y \in \mathbb{C}.$$

In particular, if $0 \leq \eta \leq 2\zeta$, then $I - \eta\mathbb{F}$ is nonexpansive.

Lemma 2.3 ([3])**.** *x^* is a solution of SVI if and only if x^* is a fixed point of the mapping $\mathbb{U} : \mathbb{C} \to \mathbb{C}$ defined by*

$$\mathbb{U}(x) = Proj_{\mathbb{C}}[Proj_{\mathbb{C}}(x - \xi\mathbb{G}x) - \eta\mathbb{F}Proj_{\mathbb{C}}(x - \xi\mathbb{G}x)], \forall x \in \mathbb{C},$$

where $y^* = Proj_{\mathbb{C}}(x^* - \xi\mathbb{G}x^*)$.

In particular, if the mappings $\mathbb{F}, \mathbb{G} : \mathbb{C} \to \mathbb{H}$ are ζ-inverse strongly monotone and δ-inverse strongly monotone, respectively, then the mapping $\mathbb{U}$ is a nonexpansive mapping provided $\eta \in (0, 2\zeta)$ and $\xi \in (0, 2\delta)$.

Lemma 2.4 ([18])**.** *Let $\{x^n\}$ and $\{y^n\}$ be bounded sequences in a Banach space $\mathbb{X}$ and let $\{\delta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$. Suppose $x^{n+1} = (1 - \delta_n)y^n + \delta_n x^n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y^{n+1} - y^n\| - \|x^{n+1} - x^n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y^n - x^n\| = 0$.*

Lemma 2.5 ([10])**.** *Let $\mathbb{C}$ be a closed convex subset of a real Hilbert space $\mathbb{H}$ and let $N : \mathbb{C} \to \mathbb{C}$ be a nonexpansive mapping. Then, the mapping $I - N$ is demiclosed. That is, if $\{x^n\}$ is a sequence in $\mathbb{C}$ such that $x^n \to x^*$ weakly and $(I - N)x^n \to y$ strongly, then $(I - N)x^* = y$.*

Lemma 2.6 ([23]). *Assume* $\{a^n\}$ *is a sequence of nonnegative real numbers such that*

$$a^{n+1} \leq (1-\gamma_n)a^n + \delta_n\gamma_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty} a^n = 0$.

3. Main results

In this section we will introduce two schemes for finding the unique point x^* which solves the quadratic minimization

$$\|x^*\|^2 = \min_{x\in Fix(N)\cap\Omega} \|x\|^2. \tag{3.1}$$

Let $\mathbb{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. Let $\mathbb{R} : \mathbb{C} \to \mathbb{H}$ be a κ-contraction. Let the mappings $\mathbb{F}, \mathbb{G} : \mathbb{C} \to \mathbb{H}$ be ζ-inverse strongly monotone and δ-inverse strongly monotone, respectively. Suppose $\eta \in (0, 2\zeta)$ and $\xi \in (0, 2\delta)$. Let $N : \mathbb{C} \to \mathbb{C}$ be a nonexpansive mapping.

For each $t \in (0,1)$, we study the following mapping T^t given by

$$T^t x = NProj_{\mathbb{C}}[t\mathbb{R}(x) + (1-t)Proj_{\mathbb{C}}(I-\eta\mathbb{F})Proj_{\mathbb{C}}(I-\xi\mathbb{G})x], \forall x \in \mathbb{C}.$$

Since the mappings N, $Proj_{\mathbb{C}}$, $I - \eta\mathbb{F}$ and $I - \xi\mathbb{G}$ are nonexpansive, we can check easily that $\|T^t x - T^t y\| \leq [1-(1-\kappa)t]\|x-y\|$ which implies that T^t is a contraction. Then there exists a unique fixed point x^t of T^t in $\mathbb{C}$ such that

$$\begin{cases} z^t = Proj_{\mathbb{C}}(x^t - \xi\mathbb{G}x^t), \\ y^t = Proj_{\mathbb{C}}(z^t - \eta\mathbb{F}z^t), \\ x^t = NProj_{\mathbb{C}}[t\mathbb{R}(x^t) + (1-t)y^t]. \end{cases} \tag{3.2}$$

In particular, if we take $\mathbb{R} \equiv 0$, then (3.2) reduces to

$$\begin{cases} z^t = Proj_{\mathbb{C}}(x^t - \xi\mathbb{G}x^t), \\ y^t = Proj_{\mathbb{C}}(z^t - \eta\mathbb{F}z^t), \\ x^t = NProj_{\mathbb{C}}[(1-t)y^t]. \end{cases} \tag{3.3}$$

We next prove that the implicit methods (3.2) and (3.3) both converge.

Theorem 3.1. *Suppose* $\Gamma := Fix(N) \cap \Omega \neq \emptyset$. *Then the net* $\{x^t\}$ *generated by the implicit method* (3.2) *converges in norm, as* $t \to 0^+$, *to the unique solution* x^* *of the following variational inequality*

$$x^* \in \Gamma, \quad \langle (I-\mathbb{R})x^*, x - x^* \rangle \geq 0, \quad x \in \Gamma. \tag{3.4}$$

In particular, if we take $\mathbb{R} = 0$, *then the net* $\{x^t\}$ *defined by* (3.3) *converges in norm, as* $t \to 0^+$, *to the minimum norm element in* Γ, *namely, the unique solution* x^* *to the quadratic minimization problem:*

$$x^* = \arg\min_{x\in\Gamma} \|x\|^2. \tag{3.5}$$

Proof. First, we prove that $\{x^t\}$ is bounded. Take $u \in \Gamma$. From Lemma 2.3, we have $u = Nu$ and

$$u = Proj_{\mathbb{C}}[Proj_{\mathbb{C}}(u - \xi\mathbb{G}u) - \eta\mathbb{F}Proj_{\mathbb{C}}(u - \xi\mathbb{G}u)].$$

Put $v = Proj_{\mathbb{C}}(u - \xi\mathbb{G}u)$. Then $u = Proj_{\mathbb{C}}(v - \eta\mathbb{F}v)$. From Lemma 2.2, we note that

$$\|z^t - v\| = \|Proj_{\mathbb{C}}(x^t - \xi\mathbb{G}x^t) - Proj_{\mathbb{C}}(u - \xi\mathbb{G}u)\| \leq \|x^t - u\|,$$

and

$$\|y^t - u\| = \|Proj_{\mathbb{C}}(z^t - \eta\mathbb{F}z^t) - Proj_{\mathbb{C}}(v - \eta\mathbb{F}v)\| \leq \|z^t - v\|.$$

It follows from (3.2) that

$$\begin{aligned}
\|x^t - u\| &= \|NProj_{\mathbb{C}}[t\mathbb{R}(x^t) + (1-t)y^t] - NProj_{\mathbb{C}}u\| \\
&\leq \|t(\mathbb{R}(x^t) - u) + (1-t)(y^t - u)\| \\
&\leq t\|\mathbb{R}(x^t) - \mathbb{R}(u)\| + t\|\mathbb{R}(u) - u\| + (1-t)\|y^t - u\| \\
&\leq t\kappa\|x^t - u\| + t\|\mathbb{R}(u) - u\| + (1-t)\|x^t - u\| \\
&= [1 - (1-\kappa)t]\|x^t - u\| + t\|\mathbb{R}(u) - u\|,
\end{aligned}$$

that is,

$$\|x^t - u\| \leq \frac{\|\mathbb{R}(u) - u\|}{1 - \kappa}.$$

Hence, $\{x^t\}$ is bounded and so are $\{y^t\}, \{z^t\}$ and $\{\mathbb{R}(x^t)\}$. Now we can choose a constant $M > 0$ such that

$$\begin{aligned}
\sup_t \Big\{ &2\|\mathbb{R}(x^t) - u\|\|y^t - u\| + \|\mathbb{R}(x^t) - u\|^2, 2\xi\|x^t - z^t - (u - v)\|, \\
&2\eta\|z^t - y^t + (u - v)\|, \|y^t - \mathbb{R}(x^t)\|^2 \Big\} \leq M.
\end{aligned}$$

Since $\mathbb{F}$ is ζ-inverse strongly monotone and $\mathbb{G}$ is δ-inverse strongly monotone, we have from Lemma 2.2 that

$$\begin{aligned}
\|y^t - u\|^2 &= \|(I - \eta\mathbb{F})z^t - (I - \eta\mathbb{F})v\|^2 \\
&\leq \|z^t - v\|^2 + \eta(\eta - 2\zeta)\|\mathbb{F}z^t - \mathbb{F}v\|^2,
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
\|z^t - v\|^2 &= \|(I - \xi\mathbb{G})x^t - (I - \xi\mathbb{G})u\|^2 \\
&\leq \|x^t - u\|^2 + \xi(\xi - 2\delta)\|\mathbb{G}x^t - \mathbb{G}u\|^2.
\end{aligned} \tag{3.7}$$

Combining (3.6) with (3.7) to get

$$\begin{aligned}
\|y^t - u\|^2 &= \|(I - \eta\mathbb{F})z^t - (I - \eta\mathbb{F})v\|^2 \\
&\leq \|x^t - u\|^2 + \eta(\eta - 2\zeta)\|\mathbb{F}z^t - \mathbb{F}v\|^2 \\
&\quad + \xi(\xi - 2\delta)\|\mathbb{G}x^t - \mathbb{G}u\|^2.
\end{aligned} \tag{3.8}$$

From (3.2) and (3.8), we have

$$\begin{aligned}
\|x^t - u\|^2 &\leq \|(1-t)(y^t - u) + t(\mathbb{R}(x^t) - u)\|^2 \\
&= (1-t)^2\|y^t - u\|^2 + 2t(1-t)\langle \mathbb{R}(x^t) - u, y^t - u\rangle + t^2\|\mathbb{R}(x^t) - u\|^2 \\
&= \|y^t - u\|^2 + tM \\
&\leq \|x^t - u\|^2 + \eta(\eta - 2\zeta)\|\mathbb{F}z^t - \mathbb{F}v\|^2 \\
&\quad + \xi(\xi - 2\delta)\|\mathbb{G}x^t - \mathbb{G}u\|^2 + tM,
\end{aligned} \tag{3.9}$$

that is,

$$\eta(2\zeta-\eta)\|\mathbb{F}z^t-\mathbb{F}v\|^2+\xi(2\delta-\xi)\|\mathbb{G}x^t-\mathbb{G}u\|^2\le tM\to 0.$$

Since $\eta(2\zeta-\eta)>0$ and $\xi(2\delta-\xi)>0$, we derive

$$\lim_{t\to 0}\|\mathbb{F}z^t-\mathbb{F}v\|=0 \text{ and } \lim_{t\to 0}\|\mathbb{G}x^t-\mathbb{G}u\|=0. \tag{3.10}$$

From Lemma 2.1 and (3.2), we obtain

$$\begin{aligned}\|z^t-v\|^2&=\|Proj_{\mathbb{C}}(x^t-\xi\mathbb{G}x^t)-Proj_{\mathbb{C}}(u-\xi\mathbb{G}u)\|^2\\ &\le\langle(x^t-\xi\mathbb{G}x^t)-(u-\xi\mathbb{G}u),z^t-v\rangle\\ &=\frac{1}{2}\Big(\|(x^t-\xi\mathbb{G}x^t)-(u-\xi\mathbb{G}u)\|^2+\|z^t-v\|^2\\ &\quad-\|(x^t-u)-\xi(\mathbb{G}x^t-\mathbb{G}u)-(z^t-v)\|^2\Big)\\ &\le\frac{1}{2}\Big(\|x^t-u\|^2+\|z^t-v\|^2-\|(x^t-z^t)-\xi(\mathbb{G}x^t-\mathbb{G}u)-(u-v)\|^2\Big)\\ &=\frac{1}{2}\Big(\|x^t-u\|^2+\|z^t-v\|^2-\|x^t-z^t-(u-v)\|^2\\ &\quad+2\xi\langle x^t-z^t-(u-v),\mathbb{G}x^t-\mathbb{G}u\rangle-\xi^2\|\mathbb{G}x^t-\mathbb{G}u\|^2\Big),\end{aligned}$$

and

$$\begin{aligned}\|y^t-u\|&=\|Proj_{\mathbb{C}}(z^t-\eta\mathbb{F}z^t)-Proj_{\mathbb{C}}(v-\eta\mathbb{F}v)\|^2\\ &\le\langle z^t-\eta\mathbb{F}z^t-(v-\eta\mathbb{F}v),y^t-u\rangle\\ &=\frac{1}{2}\Big(\|z^t-\eta\mathbb{F}z^t-(v-\eta\mathbb{F}v)\|^2+\|y^t-u\|^2\\ &\quad-\|z^t-\eta\mathbb{F}z^t-(v-\eta\mathbb{F}v)-(y^t-u)\|^2\Big)\\ &\le\frac{1}{2}\Big(\|z^t-v\|^2+\|y^t-u\|^2-\|z^t-y^t+(u-v)\|^2\\ &\quad+2\eta\langle\mathbb{F}z^t-\mathbb{F}v,z^t-y^t+(u-v)\rangle-\eta^2\|\mathbb{F}z^t-\mathbb{F}v\|^2\Big)\\ &\le\frac{1}{2}\Big(\|x^t-u\|^2+\|y^t-u\|^2-\|z^t-y^t+(u-v)\|^2\\ &\quad+2\eta\langle\mathbb{F}z^t-\mathbb{F}v,z^t-y^t+(u-v)\rangle\Big).\end{aligned}$$

Thus, we have

$$\|z^t-v\|^2\le\|x^t-u\|^2-\|x^t-z^t-(u-v)\|^2+M\|\mathbb{G}x^t-\mathbb{G}u\|, \tag{3.11}$$

and

$$\|y^t-u\|^2\le\|x^t-u\|^2-\|z^t-y^t+(u-v)\|^2+M\|\mathbb{F}z^t-\mathbb{F}v\|. \tag{3.12}$$

By (3.9) and (3.11), we have

$$\begin{aligned}\|x^t-u\|^2&\le\|y^t-v\|^2+tM\\ &\le\|z^t-v\|^2+tM\\ &\le\|x^t-u\|^2-\|x^t-z^t-(u-v)\|^2+(\|\mathbb{G}x^t-\mathbb{G}u\|+t)M.\end{aligned}$$

It follows that

$$\|x^t - z^t - (u - v)\|^2 \leq (\|\mathbb{G}x^t - \mathbb{G}u\| + t)M.$$

Since $\|\mathbb{G}x^t - \mathbb{G}u\| \to 0$, we deduce that

$$\lim_{t\to 0} \|x^t - z^t - (u - v)\| = 0. \tag{3.13}$$

From (3.9) and (3.12), we have

$$\begin{aligned}\|x^t - u\|^2 &\leq \|y^t - u\|^2 + tM \\ &\leq \|x^t - u\|^2 - \|z^t - y^t + (u - v)\|^2 + (\|\mathbb{F}z^t - \mathbb{F}v\| + t)M.\end{aligned}$$

It follows that

$$\|z^t - y^t + (u - v)\|^2 \leq (\|\mathbb{F}z^n - \mathbb{F}v\| + t)M,$$

which implies that

$$\lim_{t\to 0} \|z^t - y^t + (u - v)\| = 0. \tag{3.14}$$

Thus, combining (3.13) with (3.14), we deduce that

$$\lim_{t\to 0} \|x^t - y^t\| = 0. \tag{3.15}$$

We note that

$$\begin{aligned}\|x^t - Ny^t\| &= \|NProj_{\mathbb{C}}[t\mathbb{R}(x^t) + (1 - t)y^t] - NProj_{\mathbb{C}}y^t\| \\ &\leq tM \to 0.\end{aligned}$$

Hence,

$$\|Ny^t - y^t\| \leq \|Ny^t - x^t\| + \|x^t - y^t\| \to 0.$$

Therefore,

$$\|x^t - Nx^t\| \to 0. \tag{3.16}$$

At the same time, from (3.2) and Lemma 2.3, we have

$$\begin{aligned}\|x^t - \mathbb{U}(x^t)\| &= \|NProj_{\mathbb{C}}[t\mathbb{R}(x^t) + (1 - t)\mathbb{U}(x^t)] - NProj_{\mathbb{C}}[\mathbb{U}(x^t)]\| \\ &\leq tM \to 0. \end{aligned} \tag{3.17}$$

Next we show that $\{x^t\}$ is relatively norm compact as $t \to 0$. Let $\{t^n\} \subset (0, 1)$ be a sequence such that $t^n \to 0$ as $n \to \infty$. Put $x^n := x^{t^n}$ and $y^n := y^{t^n}$. From (3.15)-(3.17), we have

$$\|x^n - y^n\| \to 0,\ \|x^n - Nx^n\| \to 0 \textit{ and } \|x^n - \mathbb{U}(x^n)\| \to 0. \tag{3.18}$$

From (3.2), we get

$$\begin{aligned}\|x^t - u\|^2 &= \|NProj_{\mathbb{C}}[t\mathbb{R}(x^t) + (1 - t)y^t] - Nu\|^2 \\ &\leq \|y^t - u - ty^t + t\mathbb{R}(x^t)\|^2 \\ &= \|y^t - u\|^2 - 2t\langle y^t, y^t - u\rangle + 2t\langle \mathbb{R}(x^t), y^t - u\rangle + t^2\|y^t - \mathbb{R}(x^t)\|^2 \\ &= \|y^t - u\|^2 - 2t\langle y^t - u, y^t - u\rangle - 2t\langle u, y^t - u\rangle \\ &\quad + 2t\langle \mathbb{R}(x^t) - \mathbb{R}(u), y^t - u\rangle + 2t\langle \mathbb{R}(u), y^t - u\rangle + t^2\|y^t - \mathbb{R}(x^t)\|^2 \\ &\leq [1 - 2(1 - \kappa)t]\|x^t - u\|^2 + 2t\langle \mathbb{R}(u) - u, y^t - u\rangle \\ &\quad + t^2\|y^t - \mathbb{R}(x^t)\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\|x^t-u\|^2 &\le \frac{1}{1-\kappa}\langle u-\mathbb{R}(u),u-y^t\rangle+\frac{t}{2(1-\kappa)}\|y^t-\mathbb{R}(x^t)\|^2\\ &\le \frac{1}{1-\kappa}\langle u-\mathbb{R}(u),u-y^t\rangle+\frac{t}{2(1-\kappa)}M.\end{aligned}$$

In particular,

$$\|x^n-u\|^2 \le \frac{1}{1-\kappa}\langle u-\mathbb{R}(u),u-y^n\rangle+\frac{t^n}{2(1-\kappa)}M,\quad u\in\Gamma. \tag{3.19}$$

By the boundedness of $\{x^n\}$, without loss of generality, we assume that $\{x^n\}$ converges weakly to a point $x^*\in C$. It is clear that $y^n\to x^*$ weakly. From (3.18) we can use Lemma 2.5 to get $x^*\in\Gamma$. We substitute x^* for u in (3.19) to get

$$\|x^n-x^*\|^2 \le \frac{1}{1-\kappa}\langle x^*-\mathbb{R}(x^*),x^*-y^n\rangle+\frac{t^n}{2(1-\kappa)}M.$$

So the weak convergence of $\{y^n\}$ to x^* implies that $x^n\to x^*$ strongly. We prove the relative norm compactness of the net $\{x^t\}$ as $t\to 0$. In (3.19), we take the limit as $n\to\infty$ to get

$$\|x^*-u\|^2 \le \frac{1}{1-\kappa}\langle u-\mathbb{R}(u),u-x^*\rangle,\quad u\in\Gamma. \tag{3.20}$$

Which implies that x^* solves the following variational inequality

$$x^*\in\Gamma,\quad \langle (I-\mathbb{R})u,u-x^*\rangle\ge 0,\quad u\in\Gamma.$$

It equals to its dual variational inequality

$$x^*\in\Gamma,\ \langle (I-\mathbb{R})x^*,u-x^*\rangle\ge 0,\ u\in\Gamma.$$

Therefore, $x^*=(Proj_\Gamma\mathbb{R})x^*$. This shows that x^* is the unique fixed point in Γ of the contraction $Proj_\Gamma\mathbb{R}$. This is sufficient to conclude that the entire net $\{x^t\}$ converges in norm to x^* as $t\to 0$.

Setting $\mathbb{R}=0$, then (3.20) is reduced to

$$\|x^*-u\|^2\le\langle u,u-x^*\rangle,\quad u\in\Gamma.$$

Equivalently,

$$\|x^*\|^2\le\langle x^*,u\rangle,\quad u\in\Gamma.$$

This implies that

$$\|x^*\|\le\|u\|,\quad u\in\Gamma.$$

Therefore, x^* is the minimum norm element in Γ. This completes the proof. □

Below we introduce an explicit scheme for finding the minimum-norm element in Γ.

Theorem 3.2. *Suppose $\Gamma := Fix(N)\cap\Omega\ne\emptyset$. For given $x_0\in C$ arbitrarily, let the sequences $\{x^n\}$, $\{y^n\}$ and $\{z^n\}$ be generated iteratively by*

$$\begin{cases} z^n=P_C(x^n-\xi\mathbb{G}x^n),\\ y^n=P_C(z^n-\eta\mathbb{F}z^n),\\ x^{n+1}=\delta_n x^n+(1-\delta_n)NProj_{\mathbb{C}}[\zeta_n\mathbb{R}(x^n)+(1-\zeta_n)y^n],n\ge 0,\end{cases} \tag{3.21}$$

where $\{\zeta_n\}$ and $\{\delta_n\}$ are two sequences in $[0,1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \zeta_n = 0$ *and* $\sum_{n=0}^{\infty} \zeta_n = \infty$*;*

(ii) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1.$

Then the sequence $\{x^n\}$ *converges strongly to* x^* *which is the unique solution of variational inequality* (3.4)*. In particular, if* $\mathbb{R} = 0$*, then* x^* *is the minimum norm element in* Γ*.*

Proof. First, we prove that the sequences $\{x^n\}$, $\{y^n\}$ and $\{z^n\}$ are bounded.

Let $v = Proj_{\mathbb{C}}(u - \xi\mathbb{G}u)$ and $u = Proj_{\mathbb{C}}(v - \eta\mathbb{F}v)$. From (3.21), we get

$$\begin{aligned}\|y^n - u\| &= \|Proj_{\mathbb{C}}(z^n - \eta\mathbb{F}z^n) - Proj_{\mathbb{C}}(v - \eta\mathbb{F}v)\| \\ &\leq \|z^n - v\| \\ &= \|Proj_{\mathbb{C}}(x^n - \xi\mathbb{G}x^n) - Proj_{\mathbb{C}}(u - \xi\mathbb{G}u)\| \\ &\leq \|x^n - u\|,\end{aligned}$$

and

$$\begin{aligned}\|x^{n+1} - u\| &= \|\delta_n(x^n - u) + (1-\delta_n)(NProj_{\mathbb{C}}[\zeta_n\mathbb{R}(x^n) + (1-\zeta_n)y^n] - u)\| \\ &\leq \delta_n\|x^n - u\| + (1-\delta_n)\|\zeta_n(\mathbb{R}(x^n) - u) + (1-\zeta_n)(y^n - u)\| \\ &\leq \delta_n\|x^n - u\| + (1-\delta_n)[\zeta_n\|\mathbb{R}(x^n) - \mathbb{R}(u)\| + \zeta_n\|\mathbb{R}(u) - u\| + (1-\zeta_n)\|y^n - u\|] \\ &\leq \delta_n\|x^n - u\| + (1-\delta_n)[\zeta_n\kappa\|x^n - u\| + \zeta_n\|\mathbb{R}(u) - u\| + (1-\zeta_n)\|x^n - u\|] \\ &= [1 - (1-\kappa)(1-\delta_n)\zeta_n]\|x^n - u\| + \zeta_n(1-\delta_n)\|\mathbb{R}(u) - u\| \\ &\leq \max\{\|x^n - u\|, \frac{\|\mathbb{R}(u) - u\|}{1-\kappa}\}.\end{aligned}$$

By induction, we obtain, for all $n \geq 0$,

$$\|x^n - u\| \leq \max\left\{\|x_0 - u\|, \frac{\|\mathbb{R}(u) - u\|}{1-\kappa}\right\}.$$

Hence, $\{x^n\}$ is bounded. Consequently, we deduce that $\{y^n\}$,$\{z^n\}$, $\{\mathbb{R}(x^n)\}$, $\{\mathbb{F}z^n\}$ and $\{\mathbb{G}x^n\}$ are all bounded. Let $M > 0$ is a constant such that

$$\begin{aligned}\sup_n \Big\{&\|y^n\| + \|\mathbb{R}(x^n)\|, 2\|y^n - \mathbb{R}(x^n)\|\|y^n - u\| + \|y^n - \mathbb{R}(x^n)\|^2, \\ &(\|x^n - u\| + \|x^{n+1} - u\|), 2\xi\|x^n - z^n - (u - v)\|, 2\eta\|z^n - y^n + (u - v)\|, \\ &2\eta\|\mathbb{F}z^n - \mathbb{F}v\|\|z^n - y^n + (u - v)\|, 2\xi\|x^n - z^n - (u - v)\|\|\mathbb{G}x^n - \mathbb{G}u\|\|\Big\} \leq M.\end{aligned}$$

Next we show $\lim_{n\to\infty} \|y^n - Ny^n\| = 0$.

Define $x^{n+1} = \delta_n x^n + (1-\delta_n)u^n$ for all $n \geq 0$. It follows from (3.21) that

$$\begin{aligned}\|u^{n+1} - u^n\| &= \|NProj_{\mathbb{C}}[\zeta_{n+1}\mathbb{R}(x^{n+1}) + (1-\zeta_{n+1})y^{n+1}] - NProj_{\mathbb{C}}[\zeta_n\mathbb{R}(x^n) + (1-\zeta_n)y^n]\| \\ &\leq \|\zeta_{n+1}\mathbb{R}(x^{n+1}) + (1-\zeta_{n+1})y^{n+1} - \zeta_n\mathbb{R}(x^n) - (1-\zeta_n)y^n\| \\ &\leq \|y^{n+1} - y^n\| + \zeta_{n+1}(\|y^{n+1}\| + \|\mathbb{R}(x^{n+1})\|) + \zeta_n(\|y^n\| + \|\mathbb{R}(x^n)\|) \\ &\leq \|Proj_{\mathbb{C}}(z^{n+1} - \eta\mathbb{F}z^{n+1}) - Proj_{\mathbb{C}}(z^n - \eta\mathbb{F}z^n)\| + M(\zeta_{n+1} + \zeta_n) \\ &\leq \|z^{n+1} - z^n\| + M(\zeta_{n+1} + \zeta_n) \\ &= \|Proj_{\mathbb{C}}(x^{n+1} - \xi\mathbb{G}x^{n+1}) - Proj_{\mathbb{C}}(x^n - \xi\mathbb{G}x^n)\| + M(\zeta_{n+1} + \zeta_n) \\ &\leq \|x^{n+1} - x^n\| + M(\zeta_{n+1} + \zeta_n).\end{aligned}$$

This together with (i) imply that

$$\limsup_{n\to\infty}\Big(\|u^{n+1}-u^n\|-\|x^{n+1}-x^n\|\Big)\le 0.$$

Hence by Lemma 2.4, we get $\lim_{n\to\infty}\|u^n-x^n\|=0$. Consequently,

$$\lim_{n\to\infty}\|x^{n+1}-x^n\|=\lim_{n\to\infty}(1-\delta_n)\|u^n-x^n\|=0.$$

By the convexity of the norm $\|\cdot\|$, we have

$$\begin{aligned}\|x^{n+1}-u\|^2&=\|\delta_n(x^n-u)+(1-\delta_n)(u^n-u)\|^2\\&\le\delta_n\|x^n-u\|^2+(1-\delta_n)\|u^n-u\|^2\\&\le\delta_n\|x^n-u\|^2+(1-\delta_n)\|y^n-u-\zeta_n(y^n-\mathbb{R}(x^n))\|^2\\&\le\delta_n\|x^n-u\|^2+(1-\delta_n)\|y^n-u\|^2+\zeta_nM.\end{aligned}\tag{3.22}$$

From Lemma 2.2 and (3.21), we have

$$\begin{aligned}\|y^n-u\|^2&\le\|z^n-v\|^2+\eta(\eta-2\zeta)\|\mathbb{F}z^n-\mathbb{F}v\|^2\\&\le\|x^n-u\|^2+\xi(\xi-2\delta)\|\mathbb{G}x^n-\mathbb{G}u\|^2+\eta(\eta-2\zeta)\|\mathbb{F}z^n-\mathbb{F}v\|^2.\end{aligned}\tag{3.23}$$

Substituting (3.23) into (3.22), we have

$$\begin{aligned}\|x^{n+1}-u\|^2&\le\delta_n\|x^n-u\|^2+(1-\delta_n)[\|x^n-u\|^2+\xi(\xi-2\delta)\|\mathbb{G}x^n-\mathbb{G}u\|^2\\&\quad+\eta(\eta-2\zeta)\|\mathbb{F}z^n-\mathbb{F}v\|^2]+\zeta_nM\\&=\|x^n-u\|^2+(1-\delta_n)\xi(\xi-2\delta)\|\mathbb{G}x^n-\mathbb{G}u\|^2\\&\quad+(1-\delta_n)\eta(\eta-2\zeta)\|\mathbb{F}z^n-\mathbb{F}v\|^2+\zeta_nM.\end{aligned}$$

Therefore,

$$\begin{aligned}(1-\delta_n)\eta(2\zeta-\eta)\|\mathbb{F}z^n-\mathbb{F}v\|^2&+(1-\delta_n)\xi(2\delta-\xi)\|\mathbb{G}x^n-\mathbb{G}u\|^2\\&\le\|x^n-u\|-\|x^{n+1}-u\|+\zeta_nM\\&\le(\|x^n-u\|+\|x^{n+1}-u\|)\|x^n-x^{n+1}\|+\zeta_nM\\&\le(\|x^n-x^{n+1}\|+\zeta_n)M.\end{aligned}$$

Since $\liminf_{n\to\infty}(1-\delta_n)\eta(2\zeta-\eta)>0$, $\liminf_{n\to\infty}(1-\delta_n)\xi(2\delta-\xi)>0$, $\|x^n-x^{n+1}\|\to 0$ and $\zeta_n\to 0$, we derive

$$\lim_{n\to\infty}\|\mathbb{F}z^n-\mathbb{F}v\|=0 \text{ and } \lim_{n\to\infty}\|\mathbb{G}x^n-\mathbb{G}u\|=0.$$

From Lemma 2.1 and (3.21), we obtain

$$\begin{aligned}\|z^n-v\|^2&=\|P_C(x^n-\xi\mathbb{G}x^n)-P_C(u-\xi\mathbb{G}u)\|^2\\&\le\langle(x^n-\xi\mathbb{G}x^n)-(u-\xi\mathbb{G}u),z^n-v\rangle\\&=\frac{1}{2}\Big(\|(x^n-\xi\mathbb{G}x^n)-(u-\xi\mathbb{G}u)\|^2+\|z^n-v\|^2-\|(x^n-u)-\xi(\mathbb{G}x^n-\mathbb{G}u)-(z^n-v)\|^2\Big)\\&\le\frac{1}{2}\Big(\|x^n-u\|^2+\|z^n-v\|^2-\|(x^n-z^n)-\xi(\mathbb{G}x^n-\mathbb{G}u)-(u-v)\|^2\Big)\\&=\frac{1}{2}\Big(\|x^n-u\|^2+\|z^n-v\|^2-\|x^n-z^n-(u-v)\|^2\\&\quad+2\xi\langle x^n-z^n-(u-v),\mathbb{G}x^n-\mathbb{G}u\rangle-\xi^2\|\mathbb{G}x^n-\mathbb{G}u\|^2\Big),\end{aligned}$$

and

$$\begin{aligned}\|y^n-u\| &= \|Proj_{\mathbb{C}}(z^n-\eta\mathbb{F}z^n)-Proj_{\mathbb{C}}(v-\eta\mathbb{F}v)\|^2\\ &\le \langle z^n-\eta\mathbb{F}z^n-(v-\eta\mathbb{F}v), y^n-u\rangle\\ &= \frac{1}{2}\Big(\|z^n-\eta\mathbb{F}z^n-(v-\eta\mathbb{F}v)\|^2+\|y^n-u\|^2\\ &\quad -\|z^n-\eta\mathbb{F}z^n-(v-\eta\mathbb{F}v)-(y^n-u)\|^2\Big)\\ &\le \frac{1}{2}\Big(\|z^n-v\|^2+\|y^n-u\|^2-\|z^n-y^n+(u-v)\|^2\\ &+2\eta\langle\mathbb{F}z^n-\mathbb{F}v, z^n-y^n+(u-v)\rangle-\eta^2\|\mathbb{F}z^n-\mathbb{F}v\|^2\Big)\\ &\le \frac{1}{2}\Big(\|x^n-u\|^2+\|y^n-u\|^2-\|z^n-y^n+(u-v)\|^2\\ &\quad +2\eta\langle\mathbb{F}z^n-\mathbb{F}v, z^n-y^n+(u-v)\rangle\Big).\end{aligned}$$

Thus, we deduce

$$\begin{aligned}\|z^n-v\|^2 &\le \|x^n-u\|^2-\|x^n-z^n-(u-v)\|^2\\ &+2\xi\|x^n-z^n-(u-v)\|\|\mathbb{G}x^n-\mathbb{G}u\|\\ &\le \|x^n-u\|^2-\|x^n-z^n-(u-v)\|^2+M\|\mathbb{G}x^n-\mathbb{G}u\| \end{aligned}\tag{3.24}$$

and

$$\|y^n-u\|^2 \le \|x^n-u\|^2-\|z^n-y^n+(u-v)\|^2+M\|\mathbb{F}z^n-\mathbb{F}v\|. \tag{3.25}$$

By (3.22) and (3.24), we have

$$\begin{aligned}\|x^{n+1}-u\|^2 &\le \delta_n\|x^n-u\|^2+(1-\delta_n)\|y^n-u\|^2+\zeta_nM\\ &\le \delta_n\|x^n-u\|^2+(1-\delta_n)\|z^n-v\|^2+\zeta_nM\\ &\le \delta_n\|x^n-u\|^2+(1-\delta_n)[\|x^n-u\|^2-\|x^n-z^n-(u-v)\|^2\\ &\quad +M\|\mathbb{G}x^n-\mathbb{G}u\|]+\zeta_nM\\ &\le \|x^n-u\|^2-(1-\delta_n)\|x^n-z^n-(u-v)\|^2+(\|\mathbb{G}x^n-\mathbb{G}u\|+\zeta_n)M.\end{aligned}$$

It follows that

$$(1-\delta_n)\|x^n-z^n-(u-v)\|^2 \le (\|x^{n+1}-x^n\|+\|\mathbb{G}x^n-\mathbb{G}u\|+\zeta_n)M.$$

Since $\liminf_{n\to\infty}(1-\delta_n)>0$, $\zeta_n\to 0$, $\|x^{n+1}-x^n\|\to 0$ and $\|\mathbb{G}x^n-\mathbb{G}u\|\to 0$, we deduce that

$$\lim_{n\to\infty}\|x^n-z^n-(u-v)\|=0. \tag{3.26}$$

From (3.22) and (3.25), we have

$$\begin{aligned}\|x^{n+1}-u\|^2 &\le \delta_n\|x^n-u\|^2+(1-\delta_n)\|y^n-u\|^2+\zeta_nM\\ &\le \delta_n\|x^n-u\|^2+(1-\delta_n)[\|x^n-u\|^2-\|z^n-y^n+(u-v)\|^2\\ &\quad +M\|\mathbb{F}z^n-\mathbb{F}v\|]+\zeta_nM\\ &\le \|x^n-u\|^2-(1-\delta_n)\|z^n-y^n+(u-v)\|^2+(\|\mathbb{F}z^n-\mathbb{F}v\|+\zeta_n)M.\end{aligned}$$

It follows that

$$(1-\delta_n)\|z^n-y^n+(u-v)\|^2 \le (\|x^{n+1}-x^n\|+\|\mathbb{F}z^n-\mathbb{F}v\|+\zeta_n)M,$$

which implies that

$$\lim_{n\to\infty} \|z^n - y^n + (u - v)\| = 0. \tag{3.27}$$

Thus, from (3.26) and (3.27), we deduce that

$$\lim_{n\to\infty} \|x^n - y^n\| = 0.$$

Hence,

$$\|Ny^n - u^n\| = \|NProj_{\mathbb{C}} y^n - NProj_{\mathbb{C}}[\zeta_n \mathbb{R}(x^n) + (1-\zeta_n)y^n]\| \le \zeta_n M \to 0.$$

Therefore,

$$\|Ny^n - y^n\| \le \|Ny^n - u^n\| + \|u^n - x^n\| + \|x^n - y^n\| \to 0.$$

Next we prove

$$\limsup_{n\to\infty} \langle x^* - \mathbb{R}(x^*), x^* - y^n\rangle \le 0,$$

where $x^* = P_\Gamma \mathbb{R}(x^*)$.

Indeed, we can choose a subsequence $\{y_{n_i}\}$ of $\{y^n\}$ such that

$$\limsup_{n\to\infty} \langle x^* - \mathbb{R}(x^*), x^* - y^n\rangle = \lim_{i\to\infty} \langle x^* - \mathbb{R}(x^*), x^* - y_{n_i}\rangle.$$

Without loss of generality, we may further assume that $y_{n_i} \to z$ weakly, then it is clear that $z \in \Gamma$. Therefore,

$$\limsup_{n\to\infty} \langle x^* - \mathbb{R}(x^*), x^* - y^n\rangle = \lim_{i\to\infty} \langle x^* - \mathbb{R}(x^*), x^* - z\rangle \le 0.$$

From (3.21), we have

$$\begin{aligned}
\|x^{n+1} - x^*\|^2 &\le \delta_n \|x^n - x^*\|^2 + (1-\delta_n)\|\zeta_n(\mathbb{R}(x^n) - x^*) + (1-\zeta_n)(y^n - x^*)\|^2 \\
&\le \delta_n \|x^n - x^*\|^2 + (1-\delta_n)[(1-\zeta_n)^2\|y^n - x^*\|^2 \\
&\quad + 2\zeta_n(1-\zeta_n)\langle \mathbb{R}(x^n) - x^*, y^n - x^*\rangle + {\zeta_n}^2\|\mathbb{R}(x^n) - x^*\|^2] \\
&= \delta_n \|x^n - x^*\|^2 + (1-\delta_n)[(1-\zeta_n)^2\|y^n - x^*\|^2 \\
&\quad + 2\zeta_n(1-\zeta_n)\langle \mathbb{R}(x^n) - \mathbb{R}(x)^*, y^n - x^*\rangle \\
&\quad + 2\zeta_n(1-\zeta_n)\langle \mathbb{R}(x^*) - x^*, y^n - x^*\rangle + {\zeta_n}^2\|\mathbb{R}(x^n) - x^*\|^2] \\
&\le [1 - 2(1-\kappa)(1-\delta_n)\zeta_n]\|x^n - x^*\|^2 \\
&\quad + 2\zeta_n(1-\zeta_n)(1-\delta_n)\langle \mathbb{R}(x^*) - x^*, y^n - x^*\rangle + (1-\delta_n){\zeta_n}^2 M \\
&= (1-\gamma^n))\|x^n - x^*\|^2 + \delta^n\gamma^n,
\end{aligned}$$

where $\gamma^n = 2(1-\kappa)(1-\delta_n)\zeta_n$ and $\delta^n/ = \frac{(1-\zeta_n)}{1-\kappa}\langle \mathbb{R}(x)^* - x^*, y^n - x^*\rangle + \frac{\zeta_n M}{2(1-\kappa)}$. It is clear that $\sum_{n=0}^{\infty} \gamma^n = \infty$ and $\limsup_{n\to\infty} \delta \le 0$. Hence, all conditions of Lemma 2.6 are satisfied. Therefore, we immediately deduce that $x^n \to x^*$.

Finally, if we take $\mathbb{R} = 0$, by the similar argument as that Theorem 3.1, we deduce immediately that x^* is a solution of (3.5). This completes the proof. □

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