



# Algebro-geometric solutions for the generalized nonlinear Schrödinger hierarchy

Qian Li<sup>a</sup>, Tiecheng Xia<sup>a,\*</sup>, Chao Yue<sup>b</sup>.

<sup>a</sup>Department of Mathematics, Shanghai University, Shanghai, 200444 China.

<sup>b</sup>College of Information Engineering, Taishan Medical University, Taian, 271016 China.

Communicated by B. G. Sidharth

---

## Abstract

This paper is dedicated to provide explicit theta function representation of algebro-geometric solutions for the generalized nonlinear Schrödinger hierarchy. Our main tools include zero-curvature equation to derive the generalized nonlinear Schrödinger hierarchy, the hyper-elliptic curve with genus of  $N$ , the Abel-Jacobi coordinates, the meromorphic function, the Baker-Akhiezer functions, and the Dubrovin-type equations for auxiliary divisors. With the help of these tools, the explicit representations of the Baker-Akhiezer functions, the meromorphic function, and the algebro-geometric solutions are obtained for the whole generalized nonlinear Schrödinger hierarchy. ©2016 All rights reserved.

*Keywords:* Algebro-geometric solutions, Abel-Jacobi coordinates, meromorphic function, Dubrovin-type equations, generalized nonlinear Schrödinger hierarchy.

*2010 MSC:* 37K10, 58F07, 35Q35.

---

## 1. Introduction

Algebro-geometric solution (also called quasi-periodic solution or finite gap solution), as an important feature of integrable system, was originally researched on the KdV equation based on the inverse spectral theory [1, 3, 4, 5], subsequently, algebro-geometric method developed in literatures such as [1, 3, 4, 5, 19, 20, 25], in the late 1970s. As a degenerated case of the algebro-geometric solution, the elliptic function solution and multi-soliton solution may be worked out [1, 22, 25]. The inverse spectral theory has been extended to the whole  $(1+1)$  dimensional integrable hierarchy by Gesztesy and Holden, such as the AKNS

---

\*Corresponding author

Email address: [xiatc@shu.edu.cn](mailto:xiatc@shu.edu.cn) (Tiecheng Xia)

hierarchy, the Camassa-Holm hierarchy, and so on [9, 10, 11, 12]. In recent years, Fan and his co-authors [15, 16, 17, 18, 29, 30, 31] investigated the Ruijsenaars-Toda Hierarchy, the Degasperis-Procesi hierarchy, derivative Burgers hierarchy, etc., and obtained their algebro-geometric solutions.

As is well-known, the nonlinear Schrödinger equation is

$$iu_t + u_{xx} + 2|u|^2u = 0, \tag{1.1}$$

it arises from a wide variety of fields, such as plasma physics, the theory of deep water waves, and nonlinear fibre optics, etc. The discovery of the integrability of the equation by the inverse scattering transformation approach has promoted the understanding of the generality of this method to a large extent. So far, there has been much research on the Eq. (1.1). such as its soliton solutions, conserved quantities, Darboux transformation, Bäcklund transformation, and others have been studied in [2, 6, 14, 21, 24].

In this paper, we are going to investigate the generalized nonlinear Schrödinger (GNLS) hierarchy [7], and the first non-trivial member is GNLS equation ( $t_1 = t$ )

$$\begin{aligned} ip_t &= \frac{1}{2}p_{xx} - p^2q + 2i\gamma(pq)_xp + 2\gamma^2p^3q^2, \\ iq_t &= -\frac{1}{2}q_{xx} + pq^2 + 2i\gamma(pq)_xq - 2\gamma^2p^2q^3, \end{aligned} \tag{1.2}$$

where  $\gamma$  is a constant.

Geng discussed the soliton solutions of the equation (1.2) by Darboux transformation [8], Yan studied its  $N$ -Hamiltonian structures and finite-dimensional involutive systems and integrable systems [27], Qin investigated algebro-geometric solutions for equation (1.2) by variable separation method [26]. However, to the best of authors knowledge, the algebro-geometric solutions for the whole GNLS hierarchy are still not presented yet. The main aim of the present paper is to uniformly construct algebro-geometric solutions of the entire GNLS hierarchy.

The present paper is organized as follows. In Section 2, based on the Lenard gradient and the stationary zero-curvature equation, we obtain the GNLS hierarchy associated with a  $2 \times 2$  matrix spectral problem. In Section 3, we introduce a Lax matrix and an algebraic curve  $\mathcal{K}_N$  with genus  $N$ , by which we discuss nonlinear recursion relations of the corresponding homogeneous coefficients, and a direct relation between the elliptic variables and the potentials is established. Then the hierarchy is decomposed into solvable ordinary differential equations. In Section 4, all the flows of the GNLS hierarchy are straighten out under the Abel-Jacobi coordinates. The meromorphic function  $\phi$  and the Baker-Akhiezer function  $\psi$  are introduced on the hyperelliptic curve  $\mathcal{K}_N$ . Then we study the meromorphic function  $\phi$  such that  $\phi$  satisfies nonlinear differential equations, and discuss the properties of the Baker-Akhiezer function  $\psi$ . We present the explicit theta function representations for the meromorphic function and the Baker-Akhiezer function. In particular, we give the algebro-geometric solutions for the whole GNLS hierarchy.

## 2. The GNLS hierarchy

In this section, we shall derive the hierarchy associated with the following spectral problem [7],

$$\psi_x = U\psi, \quad U = \begin{pmatrix} i\lambda - i\gamma pq & p \\ q & -i\lambda + i\gamma pq \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{2.1}$$

where  $p$  and  $q$  are two potentials,  $\gamma$  is a constant,  $\lambda$  is a constant spectral parameter. We now introduce the Lenard gradient sequence  $\{S_j\}$ ,  $j = 0, 1, 2, \dots$  to derive the hierarchy

$$KS_{j-1} = JS_j, \quad j = 1, 2, 3, \dots, \quad S_j|_{(p,q)=0} = 0, \quad S_{-1} = (-1, 0, 0)^T, \quad JS_{-1} = 0, \tag{2.2}$$

where  $S_j = (a_j, b_j, c_j)$  and operators ( $\partial = \partial/\partial x$ )

$$K = \begin{pmatrix} 2p & \partial + 2i\gamma pq & 0 \\ -2q & 0 & \partial - 2i\gamma pq \\ \partial & q & -p \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 2i & 0 \\ 0 & 0 & -2i \\ \partial & q & -p \end{pmatrix}. \tag{2.3}$$

After direct calculation from the recursion relation (2.2) we get

$$S_0 = \begin{pmatrix} 0 \\ ip \\ iq \end{pmatrix}, S_1 = \begin{pmatrix} -\frac{1}{2}pq \\ \frac{1}{2}p_x + i\gamma p^2q \\ -\frac{1}{2}q_x + i\gamma pq^2 \end{pmatrix}, \tag{2.4}$$

$$S_2 = \begin{pmatrix} \frac{i}{4}(p_xq - pq_x) - \gamma p^2q^2 \\ -\frac{i}{4}p_{xx} + \frac{i}{2}p^2q + \frac{3}{2}\gamma p p_xq + \frac{1}{2}\gamma p^2q_x + i\gamma^2 p^3q^2 \\ -\frac{i}{4}q_{xx} + \frac{i}{2}pq^2 - \frac{3}{2}\gamma p q_xq - \frac{1}{2}\gamma q^2p_x + i\gamma^2 p^2q^3 \end{pmatrix}. \tag{2.5}$$

Consider the auxiliary problem:

$$\psi_{t_r} = V^{(r)}\psi, \quad V^{(r)} = \begin{pmatrix} V_{11}^{(r)} & V_{12}^{(r)} \\ V_{21}^{(r)} & -V_{11}^{(r)} \end{pmatrix}, \tag{2.6}$$

where

$$\begin{aligned} V_{11}^{(r)} &= 2i\gamma a_r - i \sum_{j=0}^{r+1} a_{j-1}\lambda^{r+1-j}, \\ V_{12}^{(r)} &= -i \sum_{j=0}^{r+1} b_{j-1}\lambda^{r+1-j}, \\ V_{21}^{(r)} &= -i \sum_{j=0}^{r+1} c_{j-1}\lambda^{r+1-j}. \end{aligned} \tag{2.7}$$

Then the compatibility condition of Eq. (2.1) and Eq. (2.6) is  $U_{t_r} - V_x^{(r)} + [U, V^{(r)}] = 0$ , which is equivalent to

$$p_{t_r} = (2 - 4i\gamma p\partial^{-1}q)b_{r+1} + 4i\gamma p\partial^{-1}pc_{r+1}, \tag{2.8}$$

$$q_{t_r} = 4i\gamma q\partial^{-1}qb_{r+1} + (-2 - 4i\gamma q\partial^{-1}p)c_{r+1}.$$

The first non-trivial member is GNLS equation

$$\begin{aligned} p_{t_1} &= -\frac{i}{2}p_{xx} + ip^2q + 2\gamma(pq)_xp - 2i\gamma^2p^3q^2, \\ q_{t_1} &= \frac{i}{2}q_{xx} - ipq^2 + 2\gamma(pq)_xq + 2i\gamma^2p^2q^3. \end{aligned} \tag{2.9}$$

which is just Eq. (1.2) when  $t_1 = t$ .

### 3. Evolution of elliptic variables

Assume that (2.1) and (2.6) have two basic solutions  $\chi = (\chi_1, \chi_2)^T$  and  $\Phi = (\Phi_1, \Phi_2)^T$ . Then we define a matrix  $W$  of three functions  $g, f, h$  by

$$W = \frac{1}{2}(\Phi\chi^T + \chi\Phi^T)\sigma = \begin{pmatrix} g & f \\ h & -g \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{3.1}$$

which satisfies the Lax equations

$$W_x = [U, W], \quad W_{t_r} = [V^{(r)}, W]. \tag{3.2}$$

Therefore,  $\partial_x \det W = 0, \partial_{t_r} \det W = 0$ . Then Eq. (3.1) and Eq. (3.2) can be written as

$$\begin{aligned} f_x &= 2i(\lambda - \gamma pq)f - 2pg, \\ g_x &= ph - qf, \\ h_x &= 2qg - 2i(\lambda - \gamma pq)h, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} f_{t_r} &= 2fV_{11}^{(r)} - 2gV_{12}^{(r)}, \\ g_{t_r} &= hV_{12}^{(r)} - fV_{21}^{(r)}, \\ h_{t_r} &= 2gV_{21}^{(r)} - 2hV_{11}^{(r)}. \end{aligned} \tag{3.4}$$

Supposing that the functions  $g, f, h$  are finite-order polynomials in  $\lambda$ :

$$f = \sum_{j=0}^{N+1} f_{j-1} \lambda^{N+1-j}, \quad g = \sum_{j=0}^{N+1} g_{j-1} \lambda^{N+1-j}, \quad h = \sum_{j=0}^{N+1} h_{j-1} \lambda^{N+1-j}. \tag{3.5}$$

Substituting (3.5) into (3.3) yields:

$$\begin{aligned} f_{-1} &= 0, h_{-1} = 0, g_{-1} = -1, \\ f_{j,x} &= -2pg_j - 2i\gamma pqf_j + 2if_{j+1}, h_{j,x} = 2qg_j + 2i\gamma pqh_j + 2h_{j+1}, -1 \leq j \leq N-1, \\ g_{j,x} &= ph_j - qf_j, -1 \leq j \leq N, f_{N,x} = 0, g_{N,x} = 0, \end{aligned} \tag{3.6}$$

which is equivalent to

$$KQ_{j-1} = JQ_j (j = 0, 1, 2, \dots, N), \quad JQ_{-1} = 0, \quad KQ_N = 0, \quad Q_j = (g_j, f_j, h_j)^T. \tag{3.7}$$

It is easy to see that  $JQ_{-1} = 0$  has the general solution

$$Q_{-1} = \alpha_0 S_{-1}, \tag{3.8}$$

where  $\alpha_0$  is constant of integration. Without loss of generality, let  $\alpha_0 = 1$ , and acting with the operator  $(J^{-1}K)^{k+1}$  upon (3.8), we obtain from (2.4) and (3.7) that:

$$Q_k = \sum_{j=0}^{k+1} \alpha_j S_{k-j}, \quad k = 0, 1, \dots, N, \tag{3.9}$$

where  $\alpha_0, \alpha_1, \dots, \alpha_{N+1}$  are integral constants. The first few members in (3.9) are

$$Q_0 = \begin{pmatrix} -\alpha_1 \\ ip \\ iq \end{pmatrix}, \tag{3.10}$$

$$Q_1 = \begin{pmatrix} -\frac{1}{2}pq - \alpha_2 \\ \frac{1}{2}px + i\gamma p^2q + ip\alpha_1 \\ -\frac{1}{2}qx + i\gamma pq^2 + iq\alpha_1 \end{pmatrix}, \tag{3.11}$$

where  $\{\alpha_l\}_{l \in N}$  denote integration constants. For subsequent use we introduce the homogeneous coefficients  $\hat{g}_k, \hat{f}_k$  and  $\hat{h}_k$  defined by the vanishing of the integration constants  $\{\alpha_l\}$  for  $l = 1, 2, \dots, k (k \in N_0)$ ,

$$\begin{aligned} \hat{g}_{-1} &= g_{-1} = -1, \hat{h}_{-1} = h_{-1} = 0, \hat{f}_{-1} = f_{-1} = 0, \\ \hat{g}_k &= g_k|_{\alpha_l=0, l=1, \dots, k}, \hat{f}_k = f_k|_{\alpha_l=0, l=1, \dots, k}, \hat{h}_k = h_k|_{\alpha_l=0, l=1, \dots, k}. \end{aligned} \tag{3.12}$$

Defining  $\alpha_0 = 1$ , then we obtain

$$g_k = \sum_{j=-1}^k \alpha_{k-j} \hat{g}_j, \quad f_k = \sum_{j=0}^k \alpha_{k-j} \hat{f}_j, \quad h_k = \sum_{j=0}^k \alpha_{k-j} \hat{h}_j. \tag{3.13}$$

Next, we consider the function  $\det W$ , which is a  $(2N+2)$  th-order polynomial in  $\lambda$  with constant coefficients of the  $x$ -flow and  $t_r$ -flow, we have

$$-\det W = g^2 + fh = \prod_{j=0}^{2N+1} (\lambda - \lambda_j) = R(\lambda). \tag{3.14}$$

We introduce the hyperelliptic curve  $\mathcal{K}_N$  of arithmetic genus  $N$  defined by

$$\mathcal{K}_N : y^2 = \prod_{j=0}^{2N+1} (\lambda - \lambda_j) = R(\lambda). \tag{3.15}$$

The curve  $\mathcal{K}_N$  can be compactified by joining two points at infinity  $P_{\infty\pm}$ , where  $P_{\infty+} \neq P_{\infty-}$ . And the compactification is also denoted by  $\mathcal{K}_N$  for notational simplicity. Here we assume that the zeros  $\lambda_j$  of  $R(\lambda)$  in (3.15) are mutually distinct, which means  $\lambda_j \neq \lambda_k$ , when  $j \neq k, 0 \leq j, k \leq 2N + 1$ . Then the hyper elliptic curve  $\mathcal{K}_N$  becomes nonsingular. We can determine the integration constants  $\alpha_k (0 \leq k \leq N)$  by the constants  $\lambda_0, \dots, \lambda_{2N+1}$  from the following theorem. We shall give some elementary results before giving the theorem.

Let  $\{\lambda_j\}_{j=0,1,\dots,2N+1} \subset \mathbb{C}$  for some  $N \in \mathbb{N}_0$  and  $\xi \in \mathbb{C}, |\xi| < \min\{\lambda_0^{-1}, \dots, \lambda_{2N+1}^{-1}\}$ , and abbreviate  $\underline{\lambda} = (\lambda_0, \dots, \lambda_{2N+1})$ , then

$$\prod_{j=0}^{2N+1} (1 - \lambda_j \xi)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \hat{c}_k(\underline{\lambda}) \xi^k, \tag{3.16}$$

where

$$\begin{aligned} \hat{c}_0(\underline{\lambda}) &= 1, \hat{c}_1(\underline{\lambda}) = \frac{1}{2} \sum_{k=0}^{2N+1} \lambda_j, \hat{c}_2(\underline{\lambda}) = \frac{1}{4} \sum_{j < k} \lambda_j \lambda_k + \frac{3}{8} \sum_{j=0}^{2N+1} \lambda_j^2, \\ \hat{c}_k(\underline{\lambda}) &= \sum_{\substack{j_0, \dots, j_{2N+1}=0 \\ j_0 + \dots + j_{2N+1} = k}} \frac{(2j_0)! \dots (2j_{2N+1})!}{2^{2k} (j_0!)^2 \dots (j_{2N+1}!)^2} \lambda_0^{j_0} \dots \lambda_{2N+1}^{j_{2N+1}}. \end{aligned} \tag{3.17}$$

Similarly, we have

$$\prod_{j=0}^{2N+1} (1 - \lambda_j \xi)^{\frac{1}{2}} = \sum_{k=0}^{\infty} c_k(\underline{\lambda}) \xi^k, \tag{3.18}$$

where

$$\begin{aligned} c_0(\underline{\lambda}) &= 1, c_1(\underline{\lambda}) = -\frac{1}{2} \sum_{k=0}^{2N+1} \lambda_j, c_2(\underline{\lambda}) = \frac{1}{4} \sum_{j < k} \lambda_j \lambda_k - \frac{1}{8} \sum_{j=0}^{2N+1} \lambda_j^2, \\ c_k(\underline{\lambda}) &= \sum_{\substack{j_0, \dots, j_{2N+1}=0 \\ j_0 + \dots + j_{2N+1} = k}} \frac{(2j_0)! \dots (2j_{2N+1})! \lambda_0^{j_0} \dots \lambda_{2N+1}^{j_{2N+1}}}{2^{2k} (j_0!)^2 \dots (j_{2N+1}!)^2 (2j_0 - 1) \dots (2j_{2N+1} - 1)}. \end{aligned} \tag{3.19}$$

A comparison of the coefficients of  $\xi^k$  in the following equation

$$1 = \left( \prod_{j=0}^{2N+1} (1 - \lambda_j \xi)^{\frac{1}{2}} \right) \cdot \left( \prod_{j=0}^{2N+1} (1 - \lambda_j \xi)^{-\frac{1}{2}} \right) = \left( \sum_{k=0}^{\infty} c_k(\underline{\lambda}) \xi^k \right) \cdot \left( \sum_{k=0}^{\infty} \hat{c}_k(\underline{\lambda}) \xi^k \right) \tag{3.20}$$

we get

$$\sum_{j=0}^k c_j(\underline{\lambda}) \hat{c}_{k-j}(\underline{\lambda}) = \delta_{k,0} \quad k \in \mathbb{N}_0. \tag{3.21}$$

**Theorem 3.1.**

$$\alpha_l = c_l(\underline{\lambda}), \quad l = 0, \dots, N. \tag{3.22}$$

*Proof.* It will be convenient to introduce the notion of a degree,  $\text{deg}(\cdot)$ , to effectively distinguish between homogeneous and nonhomogeneous quantities. One defines

$$\text{deg}(p) = 1, \text{deg}(q) = 1, \text{deg}(\partial_x) = 2, \tag{3.23}$$

implying

$$\text{deg}(\hat{f}_k) = 2k + 1, \text{deg}(\hat{h}_k) = 2k + 1, \text{deg}(\hat{g}_k) = 2k, k \in \mathbb{N}_0, \tag{3.24}$$

by using the linear recursion relation (3.6) and induction on  $k$ . Next, dividing  $f(\lambda), h(\lambda), g(\lambda)$  by  $R(\lambda)^{\frac{1}{2}}$  near infinity respectively, one obtains

$$\begin{aligned} \frac{f(\lambda)}{R(\lambda)^{\frac{1}{2}}} &= \sum_{j=0}^{\infty} \hat{c}_j(\underline{\lambda}) \lambda^{-N-1-j} \sum_{j=0}^{N+1} f_{j-1} \lambda^{N+1-j} = \sum_{l=0}^{\infty} \check{f}_{l-1} \lambda^{-l}, \\ \frac{h(\lambda)}{R(\lambda)^{\frac{1}{2}}} &= \sum_{j=0}^{\infty} \hat{c}_j(\underline{\lambda}) \lambda^{-N-1-j} \sum_{j=0}^{N+1} h_{j-1} \lambda^{N+1-j} = \sum_{l=0}^{\infty} \check{h}_{l-1} \lambda^{-l}, \end{aligned} \tag{3.25}$$

$$\frac{g(\lambda)}{R(\lambda)^{\frac{1}{2}}} = \sum_{j=0}^{\infty} \hat{c}_j(\lambda) \lambda^{-N-1-j} \sum_{j=0}^{N+1} g_{j-1} \lambda^{N+1-j} = \sum_{l=0}^{\infty} \check{g}_{l-1} \lambda^{-l},$$

for some coefficients  $\check{f}_l, \check{g}_l, \check{h}_l$  to be determined next. Noticing (3.3) and (3.24), we get

$$\begin{aligned} \check{f}_{k,x} &= -2p\check{g}_k - 2i\gamma pq\check{f}_k + 2i\check{f}_{k+1}, \\ \check{h}_{k,x} &= 2q\check{g}_k + 2i\gamma pq\check{h}_k + 2\check{h}_{k+1}, \\ \check{g}_{k,x} &= p\check{h}_k - q\check{f}_k, k = -1, 0, 1, \dots \end{aligned} \tag{3.26}$$

Here we have chosen that  $\check{f}_{-1} = \hat{f}_{-1} = 0, \check{g}_{-1} = \hat{g}_{-1} = -1, \check{h}_{-1} = \hat{h}_{-1} = 0, \check{f}_0 = \hat{f}_0 = ip, \check{g}_0 = \hat{g}_0 = 0$  and  $\check{h}_0 = \hat{h}_0 = iq$ . Moreover, one can prove inductively that

$$\text{deg}(\check{f}_k) = 2k + 1, \text{deg}(\check{h}_k) = 2k + 1, \text{deg}(\check{g}_k) = 2k. \tag{3.27}$$

Hence,  $\check{g}_l, \check{f}_l$  and  $\check{h}_l$  are equal to  $\hat{g}_l, \hat{f}_l$  and  $\hat{h}_l$ , respectively, for all  $l \in \mathbb{N}_0$ . Thus we proved

$$\begin{aligned} \frac{f(\lambda)}{R(\lambda)^{\frac{1}{2}}} &= \sum_{l=0}^{\infty} \hat{f}_{l-1} \lambda^{-l} = \sum_{l=0}^{\infty} \hat{f}_l \lambda^{-l-1}, \\ \frac{h(\lambda)}{R(\lambda)^{\frac{1}{2}}} &= \sum_{l=0}^{\infty} \hat{h}_{l-1} \lambda^{-l} = \sum_{l=0}^{\infty} \hat{h}_l \lambda^{-l-1}, \\ \frac{g(\lambda)}{R(\lambda)^{\frac{1}{2}}} &= \sum_{l=0}^{\infty} \hat{g}_{l-1} \lambda^{-l}. \end{aligned} \tag{3.28}$$

Using (3.20), we compute that

$$\sum_{m=0}^k c_{k-m}(\lambda) \hat{f}_m = \sum_{m=0}^k c_{k-m}(\lambda) \sum_{l=0}^m f_l \hat{c}_{m-l}(\lambda) = \sum_{l=0}^k f_l \sum_{s=0}^{k-l} c_{k-m} \hat{c}_{k-s-l}(\lambda) \hat{c}_s(\lambda) = f_k, \tag{3.29}$$

where  $k = 0, \dots, N$ , which together with (3.13) implies (3.21) holds. Hence Theorem 3.1 is proved. □

If we write  $f$  and  $h$  as finite products which take the following form

$$\begin{aligned} f &= ip \prod_{j=1}^N (\lambda - \mu_j), \\ h &= iq \prod_{j=1}^N (\lambda - \nu_j), \end{aligned} \tag{3.30}$$

where the roots  $\{\mu_j\}_{j=1}^N$  and  $\{\nu_j\}_{j=1}^N$  are called elliptic variables, from which we obtain

$$g|_{\lambda=\mu_k} = \sqrt{R(\mu_k)}, \quad g|_{\lambda=\nu_k} = \sqrt{R(\nu_k)}. \tag{3.31}$$

Noticing (3.3), (3.29) and (3.30), we get

$$f_x|_{\lambda=\mu_k} = -2pg|_{\lambda=\mu_k} = -ip\mu_{k,x} \prod_{\substack{j=1 \\ j \neq k}}^N (\mu_k - \mu_j), \tag{3.32}$$

$$h_x|_{\lambda=\nu_k} = 2qg|_{\lambda=\nu_k} = -iq\nu_{k,x} \prod_{\substack{j=1 \\ j \neq k}}^N (\nu_k - \nu_j), \tag{3.33}$$

which means

$$\mu_{k,x} = -2i \frac{\sqrt{R(\mu_k)}}{\prod_{\substack{j=1 \\ j \neq k}}^N (\mu_k - \mu_j)}, \nu_{k,x} = 2i \frac{\sqrt{R(\nu_k)}}{\prod_{\substack{j=1 \\ j \neq k}}^N (\nu_k - \nu_j)}, 1 \leq k \leq N. \tag{3.34}$$

In a way similar to the calculation of (3.33), we obtain the evolution of  $\{\mu_j\}$  and  $\{\nu_j\}$  along the  $t_r$  – flow :

$$\mu_{k,t_r} = -2i \frac{\sqrt{R(\mu_k)} V_{12}^{(r)}}{p \prod_{\substack{j=1 \\ j \neq k}}^N (\mu_k - \mu_j)}, \nu_{k,t_r} = 2i \frac{\sqrt{R(\nu_k)} V_{21}^{(r)}}{q \prod_{\substack{j=1 \\ j \neq k}}^N (\nu_k - \nu_j)}. \tag{3.35}$$

where  $1 \leq k \leq N$ .

#### 4. Algebro-geometric solutions

In the section, we shall construct algebro-geometric solutions for the whole GNLS hierarchy (2.8). First we will recall the hyper-elliptic curve  $\mathcal{K}_N$  of arithmetic genus  $N$  defined by

$$\mathcal{K}_N : y^2 - R(\lambda) = 0. \tag{4.1}$$

We can lift the roots  $\{\mu_j\}_{j=1}^N$  and  $\{\nu_j\}_{j=1}^N$  to  $\mathcal{K}_N$  by introducing

$$\hat{\mu}_j(x, t_r) = (\mu_j(x, t_r), -G(\mu_j(x, t_r), x, t_r)), \tag{4.2}$$

$$\hat{\nu}_j(x, t_r) = (\nu_j(x, t_r), G(\nu_j(x, t_r), x, t_r)), \tag{4.3}$$

where  $j = 1, \dots, N, (x, t_r) \in \mathbb{R}^2$ . Now we introduce the Baker-Akhiezer function  $\psi = (\psi_1, \psi_2)^T$  by

$$\begin{aligned} \psi_x(P, x, x_0, t_r, t_{0,r}) &= U(\lambda, x, t_r) \psi(P, x, x_0, t_r, t_{0,r}), \\ \psi_{t_r}(P, x, x_0, t_r, t_{0,r}) &= V^{(r)}(\lambda, x, t_r) \psi(P, x, x_0, t_r, t_{0,r}), \\ W(\lambda, x, t_r) \psi(P, x, x_0, t_r, t_{0,r}) &= y(P) \psi(P, x, x_0, t_r, t_{0,r}), \\ \psi(P, x_0, x_0, t_{0,r}, t_{0,r}) &= 1, (x, t_r) \in \mathbb{R}^2. \end{aligned} \tag{4.4}$$

Closely associated with  $\psi(P, x, x_0, t_r, t_{0,r})$  is the meromorphic function  $\phi(P, x, t_r)$  on  $\mathcal{K}_N$ , defined by

$$\phi(P, x, t_r) = \frac{\psi_1(P, x, x_0, t_r, t_{0,r})}{\psi_2(P, x, x_0, t_r, t_{0,r})}, P \in \mathcal{K}_N, \tag{4.5}$$

which implies from (4.4) that

$$\phi(P, x, t_r) = \frac{y - g}{f} = \frac{h}{y + g}, \tag{4.6}$$

where  $P = (\lambda, y) \in \mathcal{K}_N \setminus \{P_{\infty+}, P_{\infty-}\}, (x, t_r) \in \mathbb{R}^2$ . Hence the divisor of  $\phi(\cdot, x, t_r)$  reads [10]

$$\phi(\cdot, x, t_r) = D_{P_{\infty+}, \hat{\underline{\mu}}(x, t_r)} - D_{P_{\infty-}, \hat{\underline{\nu}}(x, t_r)}, \tag{4.7}$$

where

$$D_{\hat{\underline{\mu}}(x, t_r)} = \sum_{j=1}^N \hat{\mu}_j(x, t_r), D_{\hat{\underline{\nu}}(x, t_r)} = \sum_{j=1}^N \hat{\nu}_j(x, t_r). \tag{4.8}$$

The holomorphic sheet exchange map  $*$  is defined by

$$* : \mathcal{K}_N \rightarrow \mathcal{K}_N, P = (\lambda, y) \rightarrow P^* = (\lambda, -y), P, P^* \in \mathcal{K}_N.$$

With these preparations, we can compute that the meromorphic function  $\phi$  satisfies the Riccati-type non-linear differential equations

$$\phi_x - 2i\gamma pq\phi + p\phi^2 = q - 2i\lambda\phi, \phi_{t_r} = V_{21}^{(r)} - 2V_{11}^{(r)}\phi - V_{12}^{(r)}\phi^2, \tag{4.9}$$

as well as

$$\phi(P)\phi(P^*) = -\frac{h}{f}, \phi(P) + \phi(P^*) = -2\frac{g}{h}, \phi(P) - \phi(P^*) = 2\frac{y}{f}. \tag{4.10}$$

After direct calculation, we can derive the properties of  $\psi(P, x, x_0, t_r, t_{0,r})$

$$\begin{aligned} \psi_1(P, x, x_0, t_r, t_{0,r}) &= \exp\left(\int_{x_0}^x (i\lambda - i\gamma p(x', t_r)q(x', t_r) + p(x', t_r)\phi(P, x', t_r)) dx' \right. \\ &\quad \left. + \int_{t_{0,r}}^{t_r} (V_{11}^{(r)}(\lambda, x_0, s) + V_{12}^{(r)}(\lambda, x_0, s)\phi(P, x_0, s)) ds\right) \\ &= \left[\frac{f(\lambda, x, t_r)}{f(\lambda, x_0, t_{0,r})}\right]^{\frac{1}{2}} \cdot \exp\left[\int_{x_0}^x \frac{y(P)p(x', t_r)}{f(\lambda, x', t_r)} dx' + \int_{t_{0,r}}^{t_r} \frac{y(P)V_{12}^{(r)}(\lambda, x_0, s)}{f(\lambda, x_0, s)} ds\right], \end{aligned} \tag{4.11}$$

and

$$\psi_1(P, x, x_0, t_r, t_{0,r})\psi_1(P^*, x, x_0, t_r, t_{0,r}) = \frac{f(\lambda, x, t_r)}{f(\lambda, x_0, t_{0,r})}, \tag{4.12}$$

$$\psi_2(P, x, x_0, t_r, t_{0,r})\psi_2(P^*, x, x_0, t_r, t_{0,r}) = -\frac{h(\lambda, x, t_r)}{f(\lambda, x_0, t_{0,r})}, \tag{4.13}$$

$$\psi_1(P, x, x_0, t_r, t_{0,r})\psi_2(P^*, x, x_0, t_r, t_{0,r}) + \psi_1(P^*, x, x_0, t_r, t_{0,r})\psi_2(P, x, x_0, t_r, t_{0,r}) = -2\frac{g(\lambda, x, t_r)}{f(\lambda, x_0, t_{0,r})}, \tag{4.14}$$

$$\psi_1(P, x, x_0, t_r, t_{0,r})\psi_2(P^*, x, x_0, t_r, t_{0,r}) - \psi_1(P^*, x, x_0, t_r, t_{0,r})\psi_2(P, x, x_0, t_r, t_{0,r}) = -2\frac{y}{f(\lambda, x_0, t_{0,r})}. \tag{4.15}$$

Next, let us introduce the Riemann surface  $\Gamma$  of the hyperelliptic curve  $\mathcal{K}_N$  and equip  $\Gamma$  with canonical basis cycles:  $a_1, \dots, a_N; b_1, \dots, b_N$ , which are independent and have intersection numbers as follows

$$a_i \circ a_j = 0, b_i \circ b_j = 0, a_i \circ b_j = \delta_{ij}.$$

For the present, we will choose our basis as the following set

$$\tilde{\omega}_l = \frac{\lambda^{l-1}d\lambda}{\sqrt{R(\lambda)}}, 1 \leq l \leq N, \tag{4.16}$$

which are  $N$  linearly independent homomorphic differentials on  $\Gamma$ . By using the cycles  $a$  and  $b$ , the period matrices  $A$  and  $B$  can be constructed from

$$A_{ij} = \int_{a_j} \tilde{\omega}_i, B_{ij} = \int_{b_j} \tilde{\omega}_i.$$

It is possible to show that matrices  $A$  and  $B$  are invertible [13, 23]. Now we define the matrices  $C$  and  $\tau$  by  $C = A^{-1}, \tau = A^{-1}B$ . The matrix  $\tau$  can be shown to be symmetric ( $\tau_{ij} = \tau_{ji}$ ) and it has positive definite imaginary part ( $Im\tau > 0$ ). If we normalize  $\tilde{\omega}_l$  into the new basis  $\omega_j$ ,

$$\omega_j = \sum_{l=1}^N C_{jl}\tilde{\omega}_l, 1 \leq j \leq N, \tag{4.17}$$

then we have

$$\int_{a_i} \omega_j = \sum_{l=1}^N C_{jl} \int_{a_i} \tilde{\omega}_l = \delta_{ji}, \int_{b_i} \omega_j = \tau_{ji}.$$

Define the Abel-Jacobi coordinates

$$\rho_j^{(1)}(x, t_r) = \sum_{k=1}^N \int_{P_0}^{\widehat{\mu}_k(x, t_r)} \omega_j = \sum_{k=1}^N \sum_{l=1}^N C_{jl} \int_{\lambda(P_0)}^{\mu_k} \frac{\lambda^{l-1} d\lambda}{\sqrt{R(\lambda)}}, \tag{4.18}$$

$$\rho_j^{(2)}(x, t_r) = \sum_{k=1}^N \int_{P_0}^{\widehat{\nu}_k(x, t_r)} \omega_j = \sum_{k=1}^N \sum_{l=1}^N C_{jl} \int_{\lambda(P_0)}^{\nu_k} \frac{\lambda^{l-1} d\lambda}{\sqrt{R(\lambda)}}, \tag{4.19}$$

where  $1 \leq j \leq N$ . Without loss of generality, we choose the branch point  $P_0 = (\lambda_{j_0}, 0)$ ,  $j_0 \in \{0, \dots, 2N + 1\}$ , as a convenient base point, and  $\lambda(P_0)$  is its local coordinate. From (3.33), (3.34), (4.18) and (4.19) we obtain the following two lemmas.

**Lemma 4.1.** (Straightening Out of the  $x$ -Flow).

$$\partial_x \rho_j^{(1)} = -2iC_{jN}, \partial_x \rho_j^{(2)} = 2iC_{jN}, 1 \leq j \leq N, \tag{4.20}$$

where

$$\underline{C}_N = (C_{1N}, \dots, C_{NN}), \underline{\rho}^{(i)} = (\rho_1^{(i)}, \dots, \rho_N^{(i)}), i = 1, 2.$$

*Proof.* It can be calculated from (3.33), (4.18), (4.19) that

$$\partial_x \rho_j^{(1)} = \sum_{k=1}^N \sum_{l=1}^N C_{jl} \frac{\mu_k^{l-1} \mu_{k,x}}{\sqrt{R(\mu_k)}} = \sum_{l=1}^N \sum_{k=1}^N \frac{-2iC_{jl} \mu_k^{l-1}}{\prod_{\substack{j=1 \\ j \neq k}}^N (\mu_k - \mu_j)}, \tag{4.21}$$

$$\partial_x \rho_j^{(2)} = \sum_{k=1}^N \sum_{l=1}^N C_{jl} \frac{\nu_k^{l-1} \nu_{k,x}}{\sqrt{R(\nu_k)}} = \sum_{l=1}^N \sum_{k=1}^N \frac{2iC_{jl} \nu_k^{l-1}}{\prod_{\substack{j=1 \\ j \neq k}}^N (\nu_k - \nu_j)}, \tag{4.22}$$

which implies

$$\partial_x \rho_j^{(1)} = -2iC_{jN}, \partial_x \rho_j^{(2)} = 2iC_{jN}, 1 \leq j \leq N, \tag{4.23}$$

with the help of the following equalities

$$\sum_{k=1}^N \frac{\mu_k^{l-1}}{\prod_{\substack{j=1 \\ j \neq k}}^N (\mu_k - \mu_j)} = \begin{cases} \delta_{lN}, & 1 \leq l \leq N, \\ \sum_{i_1 + \dots + i_N = l - N, i_j \geq 0} \mu_1^{i_1} \dots \mu_N^{i_N}, & l > N. \end{cases} \tag{4.24}$$

Thus we complete the proof of the lemma. □

**Lemma 4.2.** (Straightening Out of the  $t_r$ -Flow).

$$\partial_{t_r} \underline{\rho}^{(1)} = 2 \sum_{l=0}^r \gamma_l \underline{C}_{N-r+l}, \tag{4.25}$$

$$\partial_{t_r} \underline{\rho}^{(2)} = -2 \sum_{l=0}^r \gamma_l \underline{C}_{N-r+l}, \tag{4.26}$$

where

$$\underline{C}_j = (C_{1j}, \dots, C_{Nj}), \underline{\rho}^{(i)} = (\rho_1^{(i)}, \dots, \rho_N^{(i)}), 1 \leq j \leq N,$$

and the recursive formula:

$$\gamma_0 = -i, \gamma_1 = i\alpha_1, \gamma_2 = -i(\alpha_1^2 - \alpha_2), \gamma_k = -\sum_{j=1}^k \alpha_j \gamma_{k-j}.$$

*Proof.* From (3.9) we obtain

$$-if_k = \sum_{j=0}^k \alpha_j V_{12,k-j}^{(r)} \tag{4.27}$$

which implies

$$V_{12,k}^{(r)} = \sum_{j=0}^k \gamma_l f_{k-l}. \tag{4.28}$$

From (3.34), (4.18) and (4.28), we have

$$\begin{aligned} \partial_{t,r} \rho_j^{(1)} &= \sum_{k=1}^N \sum_{l=1}^N C_{jl} \frac{\mu_k^{l-1} \mu_{k,t_r}}{\sqrt{R(\mu_k)}} = \sum_{l=1}^N \sum_{k=1}^N \frac{-2iC_{jl} \mu_k^{l-1} V_{12}^{(r)}(\mu_k)}{p \prod_{\substack{j=1 \\ j \neq k}}^N (\mu_k - \mu_j)} \\ &= \sum_{l=1}^N \sum_{k=1}^N \frac{-2iC_{jl} \mu_k^{l-1}}{p \prod_{\substack{j=1 \\ j \neq k}}^N (\mu_k - \mu_j)} \left( \sum_{s=0}^r V_{12,s}^{(r)} \mu_k^{r-s} \right) \\ &= \sum_{l=1}^N \sum_{k=1}^N \frac{-2iC_{jl} \mu_k^{l-1}}{p \prod_{\substack{j=1 \\ j \neq k}}^N (\mu_k - \mu_j)} \sum_{s=0}^r \left( \sum_{t=0}^s \gamma_t f_{s-t} \right) \mu_k^{r-s} \\ &= \sum_{t=0}^r \frac{-2i\gamma_t}{p} \sum_{s=t}^r f_{s-t} \sum_{l=0}^{r-s} C_{j,N-(r-s)+l} \Upsilon_l \\ &= \sum_{t=0}^r \frac{-2i\gamma_t}{p} \sum_{k=0}^{r-t} \sum_{l=0}^k C_{j,N-(r-t)+k} f_l \Upsilon_{k-l}, \end{aligned} \tag{4.29}$$

with

$$\Upsilon_0 = 1, \Upsilon_k = \sum_{j_1+\dots+j_N=k, j_i \geq 0} \mu_1^{j_1} \cdots \mu_N^{j_N}, k \geq 1. \tag{4.30}$$

Then we get

$$\partial_{t,r} \rho_j^{(1)} = \sum_{l=0}^r \frac{-2i\gamma_l}{p} f_0 C_{j,N-r+l} = 2 \sum_{l=0}^r \gamma_l C_{j,N-r+l} \tag{4.31}$$

with the help of the formula [28]

$$\sum_{j_1+j_2=k, j_i \geq 0} \Upsilon_{j_1} \Upsilon_{j_2} = 0, 1 \leq k \leq r, \tag{4.32}$$

where

$$f_0 = ip, f_1 = -ip \sum_{l=1}^N \mu_l, f_k = (-1)^k ip \sum_{j_1 < \dots < j_k, j_i \geq 1} \Upsilon_{j_1} \Upsilon_{j_2} = 0, 1 \leq k \leq N. \tag{4.33}$$

Thus the proof of lemma is completed. □

From the above two lemmas, we have the following theorem.

**Theorem 4.3.**

$$\begin{aligned} \underline{\rho}^{(1)} &= -2i\underline{C}_N x + \underline{\Omega}_r t_r + \underline{\rho}_0^{(1)}, \\ \underline{\rho}^{(2)} &= 2i\underline{C}_N x - \underline{\Omega}_r t_r + \underline{\rho}_0^{(2)}, \end{aligned} \tag{4.34}$$

where  $\underline{\Omega}_r = 2 \sum_{l=0}^r \gamma_l \underline{C}_{j,N-r+l}$ ,  $r = 0, \dots, N - 1$ , constants  $\underline{\rho}_0^{(i)} \in \mathbb{R}^N$ ,  $i = 1, 2$ .

Let  $\mathcal{T}$  be the lattice generated by  $2N$  vectors  $\delta_j, \tau_j$ , where  $\delta_j = (\underbrace{0, \dots, 0}_{j-1}, 1, \underbrace{0, \dots, 0}_{N-j})$  and  $\tau_j = \tau \delta_j$ . The complex torus  $\mathcal{J} = \mathbb{C}^N / \mathcal{T}$  is called Jacobian variety of  $\Gamma$ . Now we introduce the Abel map  $\mathcal{A}(P) : Div(\Gamma) \rightarrow \mathcal{J}$

$$\mathcal{A}(P) = \int_{P_0}^P \underline{\omega}, \mathcal{A} \left( \sum_k n_k P_k \right) = \sum_k n_k \mathcal{A}(P_k), \tag{4.35}$$

where  $P, P_k \in \mathcal{K}_N, \underline{\omega} = (\omega_1, \omega_2, \dots, \omega_N)^\top$ . Consider two special divisors  $\sum_{k=1}^N P_k^{(i)}, i = 1, 2$ , and

$$\mathcal{A} \left( \sum_{k=1}^N P_k^{(i)} \right) = \sum_{k=1}^N \mathcal{A} \left( P_k^{(i)} \right) = \sum_{k=1}^N \int_{P_0}^{P_k^{(i)}} \underline{\omega} = \underline{\rho}^{(i)},$$

with  $P_k^{(1)} = \widehat{\mu}_k(x, t_r)$  and  $P_k^{(2)} = \widehat{\nu}_k(x, t_r)$ , whose components are

$$\sum_{k=1}^N \int_{P_0}^{P_k^{(i)}} \omega_j = \rho_j^{(i)}, 1 \leq j \leq N, i = 1, 2.$$

The Riemann theta function is defined as [10, 13, 23]

$$\theta(P, D) = \theta(\underline{\Lambda} - \mathcal{A}(P) + \mathcal{A}(D)), \tag{4.36}$$

where  $P \in \mathcal{K}_N, D \in Div(\Gamma)$ , and  $\underline{\Lambda} = (\Lambda_1, \dots, \Lambda_N)$  is defined by

$$\Lambda_j = \frac{1}{2}(1 + \tau_{jj}) - \sum_{i=1, i \neq j}^N \int_{a_i} \omega_i \int_{Q_0}^P \omega_j, j = 1, \dots, N.$$

Then according to (4.35) and the definition of Riemann theta function in (4.36), we have

$$\theta(P, D_{\widehat{\mu}(x,t_r)}) = \theta(\underline{\Lambda} - \mathcal{A}(P) + \underline{\rho}^{(1)}), \tag{4.37}$$

$$\theta(P, D_{\widehat{\nu}(x,t_r)}) = \theta(\underline{\Lambda} - \mathcal{A}(P) + \underline{\rho}^{(2)}). \tag{4.38}$$

**Lemma 4.4.** Suppose that  $p(x, t_r), q(x, t_r) \in \mathbb{C}^\infty(\mathbb{R}^2)$  satisfy the hierarchy of nonlinear Eq. (2.8). Let  $\lambda_j \in \mathbb{C} \setminus \{0\}, (0 \leq j \leq 2N + 1)$ , and  $P = (\lambda, y) \in \mathcal{K}_N \setminus (P_{\infty+}, P_{\infty-})$ . Then

$$\phi_{\zeta \rightarrow 0} \equiv \begin{cases} -\frac{iq}{2}\zeta + O(\zeta^2) \text{ as } P \rightarrow P_{\infty+}, \\ \frac{2i}{p}\zeta^{-1} + \frac{px}{p^2} + 2i\gamma q + O(\zeta) \text{ as } P \rightarrow P_{\infty-}, \end{cases} \quad \zeta = \lambda^{-1}. \tag{4.39}$$

*Proof.* Introducing the local coordinate  $\zeta = \lambda^{-1}$  near  $P_{\infty\pm}$ , from Theorem 3.1 we have

$$\begin{aligned} y &= \mp \prod_{j=0}^{2N+1} (\lambda - \lambda_j)^{\frac{1}{2}} = \mp \zeta^{-N-1} \prod_{j=0}^{2N+1} (1 - \zeta \lambda_j)^{\frac{1}{2}}, \\ &\underset{\zeta \rightarrow 0}{=} \mp \zeta^{-N-1} (1 + \alpha_1 \zeta + \alpha_2 \zeta^2 + \alpha_3 \zeta^3 + O(\zeta^4)) \text{ as } P \rightarrow P_{\infty\pm}. \end{aligned} \tag{4.40}$$

From (3.5) and (3.6), we can derive

$$\begin{aligned}
 f^{-1} &= (f_0\lambda^N + f_1\lambda^{N-1} + \dots + f_N)^{-1} \\
 &\stackrel{\zeta \rightarrow 0}{=} f_0^{-1}\zeta^N [1 - f_0^{-1}f_1\zeta + O(\zeta^2)] \\
 &\stackrel{\zeta \rightarrow 0}{=} -\frac{i}{p}\zeta^N (1 + (\frac{ip_x}{2p} - \gamma pq - \alpha_1)\zeta + O(\zeta^2)) \quad \text{as } P \rightarrow P_{\infty\pm},
 \end{aligned}
 \tag{4.41}$$

$$\begin{aligned}
 g &= g_{-1}\lambda^{N+1} + g_0\lambda^N + g_1\lambda^{N-1} + g_2\lambda^{N-2} + O(\lambda^{N-3}) \\
 &\stackrel{\zeta \rightarrow 0}{=} \zeta^{-N-1}(g_{-1} + g_0\zeta + g_1\zeta^2 + g_2\zeta^3 + O(\zeta^4)) \quad \text{as } P \rightarrow P_{\infty\pm}.
 \end{aligned}
 \tag{4.42}$$

Then according to the definition of  $\phi$  in (4.6), we finally obtain that

$$\begin{aligned}
 \phi(P, x, t_r) &= \frac{y-g}{f} \\
 &\stackrel{\zeta \rightarrow 0}{=} \frac{i}{p}\zeta^{-1} [\mp(1 + \alpha_1\zeta + \alpha_2\zeta^2 + \alpha_3\zeta^3 + O(\zeta^4)) \\
 &\quad + (g_{-1} + g_0\zeta + g_1\zeta^2 + g_2\zeta^3 + O(\zeta^4))] \\
 &\quad \cdot (1 + (\frac{ip_x}{2p} - \gamma pq - \alpha_1)\zeta + O(\zeta^2)) \\
 &\stackrel{\zeta \rightarrow 0}{=} \begin{cases} -\frac{iq}{2}\zeta + O(\zeta^2) \quad \text{as } P \rightarrow P_{\infty+}, \\ \frac{2i}{p}\zeta^{-1} + \frac{p_x}{p^2} + 2i\gamma q + O(\zeta) \quad \text{as } P \rightarrow P_{\infty-}. \end{cases}
 \end{aligned}
 \tag{4.43}$$

Thus proves the lemma. □

**Lemma 4.5.** *Under the same supposition in Lemma 4.4, in the special case of (3.9) when  $\alpha_0 = 1, \alpha_k = 0, 1 \leq k \leq N$ , we have that*

$$\psi_1(P, x, x_0, t_r, t_{0,r}) \stackrel{\zeta \rightarrow 0}{=} \begin{cases} \exp[i\zeta^{-1}(x - x_0) + i\zeta^{-r-1}(t_r - t_{0,r}) + O(1)] \quad \text{as } P \rightarrow P_{\infty+}, \\ \exp[-i\zeta^{-1}(x - x_0) - i\zeta^{-r-1}(t_r - t_{0,r}) + O(1)] \quad \text{as } P \rightarrow P_{\infty-}. \end{cases}$$

*Proof.* In the special case of (3.9) when  $\alpha_0 = 1, \alpha_k = 0, 1 \leq k \leq N$ , we derive that  $f_j = b_j, h_j = c_j, -1 \leq j$ . Introducing the local coordinate  $\zeta = \lambda^{-1}$  near  $P_{\infty\pm}$ , we obtain from (4.39) that

$$\begin{aligned}
 &\exp(\int_{x_0}^x (i\lambda - i\gamma p(P, x', t_r)q(P, x', t_r) + p(x', t_r)\phi(P, x', t_r))dx') \\
 &\stackrel{\zeta \rightarrow 0}{=} \begin{cases} \exp[\int_{x_0}^x (i\zeta^{-1} - i\gamma p(P, x', t_r)q(P, x', t_r) \\ \quad + p(P, x', t_r)(-\frac{iq}{2}\zeta + O(\zeta^2))dx')] \quad \text{as } P \rightarrow P_{\infty+} \\ \exp[\int_{x_0}^x (i\zeta^{-1} - i\gamma p(P, x', t_r)q(P, x', t_r) \\ \quad + p(P, x', t_r)(-\frac{2i}{p(P, x', t_r)}\zeta^{-1} + \frac{p_x}{p^2} + 2i\gamma q + O(\zeta)))dx'] \quad \text{as } P \rightarrow P_{\infty-} \end{cases} \\
 &\stackrel{\zeta \rightarrow 0}{=} \begin{cases} \exp(i\zeta^{-1}(x - x_0) + O(1)) \quad \text{as } P \rightarrow P_{\infty+}, \\ \frac{p(x, t_r)}{p(x_0, t_r)} \exp(-i\zeta^{-1}(x - x_0) + O(1)) \quad \text{as } P \rightarrow P_{\infty-}. \end{cases}
 \end{aligned}
 \tag{4.44}$$

Then in the special case of (3.9), combining (2.8), (3.4), (3.29), (4.6), (4.40) and (4.41) yields

$$\begin{aligned}
 &\int_{t_0,r}^{t_r} (V_{11}^{(r)}(\lambda, x_0, s) + V_{12}^{(r)}(\lambda, x_0, s)\phi(P, x_0, s))ds \\
 &= \exp \left[ \int_{t_0,r}^{t_r} \left( \frac{y}{f(\lambda, x_0, s)} V_{12}^{(r)}(\lambda, x_0, s) + \frac{1}{2} \frac{f_{t_r}(\lambda, x_0, s)}{f(\lambda, x_0, s)} \right) ds \right] \\
 &\stackrel{\zeta \rightarrow 0}{=} \exp \left[ \int_{t_0,r}^{t_r} \left( \frac{\sum_{j=0}^r b_j(\zeta^{-1}, x_0, s)\zeta^{j-r}}{\sum_{l=0}^{\infty} \hat{f}_l(\zeta^{-1}, x_0, s)\zeta^{l+1}} + \frac{f_{t_r}(\zeta^{-1}, x_0, s)}{2f(\zeta^{-1}, x_0, s)} + O(\zeta) \right) ds \right] \\
 &\stackrel{\zeta \rightarrow 0}{=} \exp \left[ \int_{t_0,r}^{t_r} \left( \pm i\zeta^{-r-1} \frac{\sum_{j=0}^r b_j(\zeta^{-1}, x_0, s)\zeta^j}{\sum_{j=0}^{\infty} \hat{f}_j(\zeta^{-1}, x_0, s)\zeta^l} + \frac{p_{t_r}(x_0, s)}{2p(x_0, s)} + O(\zeta) \right) ds \right] \\
 &\stackrel{\zeta \rightarrow 0}{=} \exp \left[ \int_{t_0,r}^{t_r} \left( \pm i\zeta^{-r-1} (1 - \zeta^{r+1} \frac{b_{r+1}(\zeta^{-1}, x_0, s)}{\hat{f}_0(\zeta^{-1}, x_0, s)}) + \frac{p_{t_r}(x_0, s)}{2p(x_0, s)} + O(\zeta) \right) ds \right] \\
 &\stackrel{\zeta \rightarrow 0}{=} \exp \left[ \int_{t_0,r}^{t_r} \left( \pm i\zeta^{-r-1} \mp i \frac{b_{r+1}(\zeta^{-1}, x_0, s)}{\hat{f}_0(\zeta^{-1}, x_0, s)} + \frac{p_{t_r}(x_0, s)}{2p(x_0, s)} + O(\zeta) \right) ds \right] \\
 &\stackrel{\zeta \rightarrow 0}{=} \begin{cases} \exp(i\zeta^{-r-1}(t_r - t_{0,r}) + O(1)) \quad \text{as } P \rightarrow P_{\infty+}, \\ \exp(-i\zeta^{-r-1}(t_r - t_{0,r}) + O(1)) \quad \text{as } P \rightarrow P_{\infty-}. \end{cases}
 \end{aligned}
 \tag{4.45}$$

Hence, combining with the definition of  $\psi_1$  in (4.11), we can arrive at (4.38). Next, we shall derive the representation of  $\phi, \psi, p$  and  $q$  in terms of the Riemann theta function. Let  $\omega_{P_{\infty+}, P_{\infty-}}^{(3)}$  be the normal differential of the third kind holomorphic on  $\mathcal{K}_N \setminus \{P_{\infty+}, P_{\infty-}\}$  with simple poles at  $P_{\infty+}$  and  $P_{\infty-}$  and residues 1 and -1, respectively, which can be expressed as

$$\omega_{P_{\infty+}, P_{\infty-}}^{(3)} = \frac{1}{y} \prod_{j=1}^N (\lambda - \beta_j) d\lambda, \tag{4.46}$$

where  $\beta_j \in \mathbb{C}, j = 1, \dots, N$  are constants that are determined by

$$\int_{a_j} \omega_{P_{\infty+}, P_{\infty-}}^{(3)} = 0, j = 1, \dots, N.$$

If the local coordinate near  $P_{\infty\pm}$  is given by  $\zeta = \lambda^{-1}$ , then we have the asymptotic expansions of  $\omega_{P_{\infty+}, P_{\infty-}}^{(3)}$  near  $P_{\infty\pm}$ :

$$\begin{aligned} \omega_{P_{\infty+}, P_{\infty-}}^{(3)} &\underset{\zeta \rightarrow 0}{=} \mp \zeta^{N+1} \prod_{j=0}^{2N+1} (1 - \zeta \lambda_j)^{-\frac{1}{2}} \cdot (-\zeta^{-N-2}) \prod_{j=1}^N (1 - \zeta \beta_j) d\zeta \\ &\underset{\zeta \rightarrow 0}{=} \pm \zeta^{-1} \prod_{j=0}^{2N+1} (1 - \zeta \lambda_j)^{-\frac{1}{2}} \cdot \prod_{j=1}^N (1 - \zeta \beta_j) d\zeta \\ &\underset{\zeta \rightarrow 0}{=} \pm (\zeta^{-1} + O(1)) \text{ as } P \rightarrow P_{\infty\pm}. \end{aligned} \tag{4.47}$$

Therefore,

$$\int_{P_0}^P \omega_{P_{\infty+}, P_{\infty-}}^{(3)} \underset{\zeta \rightarrow 0}{=} \pm (\ln \zeta - \ln \omega_0 + O(\zeta)) \text{ as } P \rightarrow P_{\infty\pm} \tag{4.48}$$

for some constant  $\omega_0 \in \mathbb{C}$ . Next, let  $\omega_{P_{\infty\pm}, r}^{(2)}, r \in \mathbb{N}_0$ , be normalized differentials of the second kind with a unique pole at  $P_{\infty\pm}$  and principal part near  $P_{\infty\pm}$  is  $-\zeta^{-2-r} d\zeta$ , and satisfying

$$\int \omega_{P_{\infty\pm}, r}^{(2)} = 0, j = 1, \dots, N. \tag{4.49}$$

Then we can define  $\Omega_0^{(2)}$  and  $\Omega_{r-1}^{(2)}$  by

$$\Omega_0^{(2)} = \omega_{P_{\infty+, 0}}^{(2)} - \omega_{P_{\infty-, 0}}^{(2)}, \tag{4.50}$$

$$\Omega_{r-1}^{(2)} = \sum_{s=0}^{r-1} \alpha_{r-1-s} (s+1) (\omega_{P_{\infty+, s}}^{(2)} - \omega_{P_{\infty-, s}}^{(2)}), \tag{4.51}$$

where  $\alpha_{r-1-s}$  are the integral constants in (3.9). Therefore,

$$\int_{a_j} \Omega_0^{(2)} = 0, \int_{a_j} \Omega_{r-1}^{(2)} = 0, j = 1, \dots, N. \tag{4.52}$$

$$\int_{P_0}^P \Omega_0^{(2)} \underset{\zeta \rightarrow 0}{=} \pm (\zeta^{-1} + e_{0,0} + e_{0,1} \zeta + O(\zeta^2)) \text{ as } P \rightarrow P_{\infty\pm}, \tag{4.53}$$

$$\int_{P_0}^P \Omega_{r-1}^{(2)} \underset{\zeta \rightarrow 0}{=} \pm \left( \sum_{s=0}^{r-1} \alpha_{r-1-s} \zeta^{-1-s} + e_{r-1,0} + O(\zeta) \right) \text{ as } P \rightarrow P_{\infty\pm} \tag{4.54}$$

for some constants  $e_{0,0}, e_{0,1}, e_{r-1,0} \in \mathbb{C}$ . If  $D_{\hat{\mu}(x, t_r)}$  and  $D_{\hat{\nu}(x, t_r)}$  in (4.7) are assumed to be nonspecial [13], then according to the Riemann's vanishing theorem [10, 13], the definition and asymptotic properties of the meromorphic function  $\phi(P, x, t_r), \phi$  has the expressions of the following type:

$$\phi(P, x, t_r) = C(x, t_r) \frac{\theta(P, D_{\hat{\mu}(x, t_r)})}{\theta(P, D_{\hat{\nu}(x, t_r)})} \exp \left( \int_{P_0}^P \omega_{P_{\infty+}, P_{\infty-}}^{(3)} \right), \tag{4.55}$$

where  $C(x, t_r)$  is independent of  $P \in \mathcal{K}_N$ . □

**Theorem 4.6.** Let  $P = (\lambda, y) \in \mathcal{K}_N \setminus (P_{\infty+}, P_{\infty-}), (x, t_r), (x_0, t_{0,r}) \in M$  where  $M \subseteq \mathbb{R}^2$  is open and connected. Suppose  $p, q \in C^\infty(M)$  satisfy the hierarchy of nonlinear equations (2.8), and assume that  $\lambda_j, 0 \leq j \leq 2N + 1$  in (3.15) satisfy  $\lambda_j \in \mathbb{C} \setminus \{0\}$  and  $\lambda_j \neq \lambda_k$  as  $j \neq k$ . Moreover, suppose that  $D_{\widehat{\underline{\mu}}(x,t_r)}$ , or equivalently,  $D_{\widehat{\underline{\nu}}(x,t_r)}$  is nonspecial for  $(x, t_r)$ . Then  $\phi, \psi_1, \psi_2$  admit the following representation:

$$\phi(P, x, t_r) = C_0 \frac{\theta(P_{\infty+}, D_{\widehat{\underline{\mu}}(x,t_r)})\theta(P, D_{\widehat{\underline{\nu}}(x,t_r)})}{\theta(P_{\infty-}, D_{\widehat{\underline{\nu}}(x,t_r)})\theta(P, D_{\widehat{\underline{\mu}}(x,t_r)})} \cdot \exp \left( \int_{P_0}^P \omega_{P_{\infty+}, P_{\infty-}}^{(3)} + 2ie_{0,0}x + 2ie_{r-1,0}t_r \right), \tag{4.56}$$

$$\begin{aligned} \psi_1(P, x, t_r, x_0, t_{r,0}) &= \frac{\theta(P_{\infty+}, D_{\widehat{\underline{\mu}}(x_0,t_{0,r})})\theta(P, D_{\widehat{\underline{\mu}}(x,t_r)})}{\theta(P_{\infty+}, D_{\widehat{\underline{\mu}}(x,t_r)})\theta(P, D_{\widehat{\underline{\mu}}(x_0,t_{0,r})})} \cdot \exp \left( i \left( \int_{P_0}^P \Omega_0^{(2)} - e_{0,0} \right) (x - x_0) \right) \\ &\cdot \exp \left( i \left( \int_{P_0}^P i\Omega_{r-1}^{(2)} - e_{r-1,0} \right) (t_r - t_{0,r}) \right), \end{aligned} \tag{4.57}$$

$$\begin{aligned} \psi_2(P, x, t_r, x_0, t_{0,r}) &= C_0 \frac{\theta(P_{\infty+}, D_{\widehat{\underline{\mu}}(x_0,t_{0,r})})\theta(P, D_{\widehat{\underline{\nu}}(x,t_r)})}{\theta(P_{\infty-}, D_{\widehat{\underline{\nu}}(x,t_r)})\theta(P, D_{\widehat{\underline{\mu}}(x_0,t_{0,r})})} \\ &\cdot \exp \left( i \left( \int_{P_0}^P \Omega_0^{(2)} + e_{0,0} \right) (x - x_0) + i \left( \int_{P_0}^P \Omega_{r-1}^{(2)} + e_{r-1,0} \right) (t_r - t_{0,r}) \right) \\ &\cdot \exp \left( \int_{P_0}^P \omega_{P_{\infty+}, P_{\infty-}}^{(3)} + 2ie_{0,0}x_0 + 2ie_{r-1,0}t_{0,r} \right), \end{aligned} \tag{4.58}$$

where

$$C_0 = \frac{-2i}{q(x_0, t_{0,r})\omega_0} \frac{\theta(P_{\infty-}, D_{\widehat{\underline{\mu}}(x_0,t_{0,r})})}{\theta(P_{\infty+}, D_{\widehat{\underline{\nu}}(x_0,t_{0,r})})} \exp(-2ie_{0,0}x_0 - 2ie_{r-1,0}t_{0,r}), \tag{4.59}$$

$$p(x, t_r) = p(x_0, t_{0,r}) \frac{\theta(P_{\infty+}, D_{\widehat{\underline{\mu}}(x_0, t_{0,r})})\theta(P_{\infty-}, D_{\widehat{\underline{\mu}}(x,t_r)})}{\theta(P_{\infty-}, D_{\widehat{\underline{\mu}}(x_0,t_{0,r})})\theta(P_{\infty+}, D_{\widehat{\underline{\mu}}(x,t_r)})} \exp(-2ie_{0,0}x_0 - 2ie_{r-1,0}t_{0,r}), \tag{4.60}$$

$$q(x, t_r) = q(x_0, t_{0,r}) \frac{\theta(P_{\infty-}, D_{\widehat{\underline{\nu}}(x_0, t_{0,r})})\theta(P_{\infty+}, D_{\widehat{\underline{\nu}}(x,t_r)})}{\theta(P_{\infty+}, D_{\widehat{\underline{\nu}}(x_0,t_{0,r})})\theta(P_{\infty-}, D_{\widehat{\underline{\nu}}(x,t_r)})} \exp(2ie_{0,0}x_0 + 2ie_{r-1,0}t_{0,r}), \tag{4.61}$$

$$p(x_0, t_{0,r})q(x_0, t_{0,r}) = \frac{4}{\omega_0^2} \cdot \frac{\theta(P_{\infty+}, D_{\widehat{\underline{\nu}}(x_0, t_{0,r})})\theta(P_{\infty-}, D_{\widehat{\underline{\mu}}(x_0,t_{0,r})})}{\theta(P_{\infty-}, D_{\widehat{\underline{\nu}}(x_0,t_{0,r})})\theta(P_{\infty+}, D_{\widehat{\underline{\mu}}(x_0,t_{0,r})})}. \tag{4.62}$$

*Proof.* First, we shall consider the theta function representation (4.57) for  $\psi_1$ . Without loss of generality, in the following, we only consider the special case of (3.9) for  $\alpha_0 = 1, \alpha_k = 0, 1 \leq k \leq N + 1$ . We temporarily assume that  $\mu_j(x, t_r) \neq \mu_{j'}(x, t_r)$  as  $j \neq j'$  and  $(x, t_r) \in \widetilde{M}$  for appropriate  $\widetilde{M} \subseteq M$  and define the right-hand side of (4.57) to be  $\widetilde{\psi}_1$ . In order to prove  $\psi_1 = \widetilde{\psi}_1$ , we investigate the local zeros and poles of  $\psi_1$  defined by (4.11). From (3.33), (3.34), (4.2), and (4.6), we have

$$v(x', t_r)\phi(P, x', t_r) \underset{P \rightarrow \widehat{\mu}_j(x', t_r)}{=} -\partial_{x'} \ln(\lambda - \mu_j(x', t_r)) + O(1), \tag{4.63}$$

$$V_{12}^{(r)}(\lambda, x_0, s)\phi(P, x_0, s) \underset{P \rightarrow \widehat{\mu}_j(x_0, s)}{=} -\partial_s \ln(\lambda - \mu_j(x_0, s)) + O(1), \tag{4.64}$$

$$\psi_1(P, x, t_r, x_0, t_{r,0}) = \begin{cases} (\lambda - \mu_j(x, t_r))O(1), & P \rightarrow \widehat{\mu}_j(x, t_r) \neq \widehat{\mu}_j(x_0, t_{r,0}), \\ O(1), & P \rightarrow \widehat{\mu}_j(x, t_r) = \widehat{\mu}_j(x_0, t_{r,0}), \\ (\lambda - \mu_j(x_0, t_{r,0}))^{-1}O(1), & P \rightarrow \widehat{\mu}_j(x_0, t_{r,0}) \neq \widehat{\mu}_j(x, t_r), \end{cases} \tag{4.65}$$

with  $P = (\lambda, y) \in \mathcal{K}_N$ ,  $(x, t_r), (x_0, t_{0,r}) \in \widetilde{M}$ , and  $O(1) \neq 0$ . Hence  $\psi_1$  and  $\widetilde{\psi}_1$  share the same singularities and zeros on  $\mathcal{K}_N \setminus (P_{\infty+}, P_{\infty-})$ , which are all simple by hypothesis (4.63). Next, we study the behavior of  $\psi_1$  and  $\widetilde{\psi}_1$  near  $P_{\infty\pm}$ , taking into account (4.38), (4.53), (4.54) and (4.57), then shows that  $\psi_1$  and  $\widetilde{\psi}_1$  have identical exponential behavior up to order  $O(1)$  near  $P_{\infty\pm}$ . Thus,  $\psi_1$  and  $\widetilde{\psi}_1$  share the same singularities and zeros. Then the Riemann-Roch-type uniqueness proves that  $\psi_1 = \widetilde{\psi}_1$ . Hence (4.57) holds subject to (4.63). Inserting (4.48) into (4.55) and comparing with (4.39) one finds

$$q = C(x, t_r) \frac{2i \theta(P_{\infty+}, D_{\widehat{\nu}(x, t_r)})}{\omega_0 \theta(P_{\infty+}, D_{\widehat{\mu}(x, t_r)}), p = -\frac{2i \theta(P_{\infty-}, D_{\widehat{\mu}(x, t_r)})}{C(x, t_r) \omega_0 \theta(P_{\infty-}, D_{\widehat{\nu}(x, t_r)}}. \quad (4.66)$$

Re-examining the asymptotic behavior of  $\psi_1$  near  $P_{\infty-}$  yields

$$\begin{aligned} \psi_1(P, x, t_r, x_0, t_{0,r}) &= \frac{p(x, t_r)}{p(x_0, t_{0,r})} \exp(-i\zeta^{-1}(x - x_0) + O(1)) \cdot \frac{p(x_0, t_r)}{p(x_0, t_{0,r})} \exp(-i\zeta^{-1-r}(t_r - t_{0,r}) + O(1)) \\ &= \frac{p(x, t_r)}{p(x_0, t_{0,r})} \exp[-i\zeta^{-1}(x - x_0) - i\zeta^{-r-1}(t_r - t_{0,r}) + O(1)]. \end{aligned} \quad (4.67)$$

A comparison of (4.57) and (4.68) then proves (4.60). Inserting (4.60) into the second equation of (4.67), we have

$$C(x, t_r) = \frac{-2i \theta(P_{\infty+}, D_{\widehat{\mu}(x, t_r)}) \theta(P_{\infty-}, D_{\widehat{\mu}(x_0, t_{0,r})})}{q(x_0, t_{0,r}) \omega_0 \theta(P_{\infty-}, D_{\widehat{\nu}(x, t_r)}) \theta(P_{\infty+}, D_{\widehat{\nu}(x_0, t_{0,r})})} \cdot \exp(2ie_{0,0}(x - x_0) + 2ie_{r-1,0}(t_r - t_{0,r})), \quad (4.68)$$

together with the first equation of (4.67) yields (4.61), (4.62). Given  $C(x, t_r)$ , we can determine  $\phi$  in (4.5) from (4.55) and  $\psi_2$  in (4.58) from  $\widetilde{\psi}_2 = \phi\psi_1$ . So we complete the prove of the theorem on  $\widetilde{M}$ . Finally the extension of all these results from  $\widetilde{M}$  to  $M$  then follows by continuity of the Abel map  $\mathcal{A}(P)$  and nonspecialty of  $D_{\widehat{\mu}(x, t_r)}$  on  $M$ . Hence, we obtain the algebro-geometric solutions of the whole GNLS hierarchy.  $\square$

### Acknowledgements:

This Work is in part supported by the Natural Science Foundation of China (Grant No. 11271008), the First-class Discipline of University in Shanghai and the Shanghai Univ. Leading Academic Discipline Project (A.13-0101-12-004), and the Natural Science Foundation of Shandong Province (Grant Nos. ZR2012AM021 and ZR2012AQ011).

### References

- [1] M. J. Ablowitz, D. J. Kaup, A. C. Newell, H. Segur, *The inverse scattering transform-Fourier analysis for nonlinear problems*, Studies in Appl. Math., **53** (1974), 249–315.1
- [2] M. J. Ablowitz, H. Segur, *Solitons and the inverse scattering transform*, SIAM Studies in Applied Mathematics, Philadelphia, (1981).1
- [3] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'Skii, A. R. Its, V. B. Matveev, *Algebro-geometric approach to nonlinear integrable equations*, Springer-Verlag, Berlin, (1994).1
- [4] B. A. Dubrovin, *Completely integrable Hamiltonian systems associated with matrix operators and Abelian varieties*, Funct. Anal. Appl., **11** (1977), 265–277.1
- [5] B. A. Dubrovin, *Matrix finite-gap operators*, Revs. Sci. Tech., **23** (1983), 33–78.1
- [6] L. D. Faddeev, L. A. Takhtajan, *Hamiltonian method in the theory of solitons*, Springer-Verlag, Berlin, (1987).1
- [7] X. Geng, *A hierarchy of nonlinear evolution equations, Its Hamiltonian structure and classical integrable system*, Phys. A: Statist. Mech. Appl., **180** (1992), 241–251.1, 2
- [8] X. Geng, H. W. Tam, *Darboux transformation and soliton solutions for generalized nonlinear Schrödinger equations*, J. Phys. Soc. Japan, **68** (1999), 1508–1512.1
- [9] F. Gesztesy, H. Holden, *Algebro-geometric solutions of the Camassa-Holm hierarchy*, Rev. Mat. Iberoamericana, **19** (2003), 73–142.1
- [10] F. Gesztesy, H. Holden, *Soliton equations and their algebro-geometric solutions*, Cambridge University Press, Cambridge, (2003).1, 4, 4, 4
- [11] F. Gesztesy, H. Holden, *Real-valued algebro-geometric solutions of the Camassa-Holm hierarchy*, Philos. Trans. R. Soc. Lond. Ser. A, Math. Phys. Eng. Sci., **366** (2008), 1025–1054.1

- [12] F. Gesztesy, R. Ratneseelan, *An alternative approach to algebro-geometric solutions of the AKNS hierarchy*, Rev. Math. Phys., **10** (1998), 345–391. 1
- [13] P. Griffiths, J. Harris, *Principles of algebraic geometry*, John Wiley & Sons, Inc., New York, (1994). 4, 4, 4
- [14] C. H. Gu, *Soliton theory and its application*, Springer-Verlag, Berlin, (1995). 1
- [15] Y. Hou, E. Fan, Z. J. Qiao, *The algebro-geometric solutions for the modified Camassa-Holm hierarchy*, **2012** (2012), 59 pages. 1
- [16] Y. Hou, E. Fan, Z. Qiao, Z. Wang, *Algebro-geometric solutions for the derivative Burgers hierarchy*, J. Nonlinear Sci., **25** (2015), 1–35. 1
- [17] Y. Hou, E. Fan, P. Zhao, *Algebro-geometric solutions for the Hunter-Saxton hierarchy*, Z. Angew. Math. Phys., **65** (2014), 487–520. 1
- [18] Y. Hou, P. Zhao, E. Fan, Z. Qiao, *Algebro-geometric solutions for Degasperis-Procesi hierarchy*, SIAM J. Math. Anal., **45** (2013), 1216–1266. 1
- [19] I. M. Krichever, *An algebraic-geometric construction of the Zaharov-Sabat equations and their periodic solutions*, Dokl. Akad. Nauk SSSR, **227** (1976), 291–294. 1
- [20] I. M. Krichever, *Integration of nonlinear equations by the methods of algebraic geometry*, Funkcional. Anal. i Priložen., **11** (1977), 15–31. 1
- [21] V. B. Matveev, M. A. Salle, *Darboux transformations and solitons*, Springer-Verlag, Berlin, (1991). 1
- [22] V. B. Matveev, M. I. Yavor, *Solutions presque périodiques et à  $N$ -solitons de l'équation hydrodynamique non linéaire de Kaup*, Ann. Inst. H. Poincaré Sect. A, **31** (1979), 25–41. 1
- [23] D. Mumford, *Tata lectures on Theta II*, (Birkhäuser Boston, Inc., Boston, (1984). 4, 4
- [24] A. C. Newell, *Solitons in mathematics and physics*, SIAM, Philadelphia, (1985). 1
- [25] S. P. Novikov, S. V. Manakov, L. P. Pitaevskii, V. E. Zakharov, *Theory of solitons, the inverse scattering methods*, Consultants Bureau, New York and London, (1984). 1
- [26] Z. Y. Qin, *Quasi-periodic solutions for an extension of AKNS hierarchy and their reductions*, Chaos, Solitons Fractals, **25** (2005), 585–599. 1
- [27] Z. Y. Yan, H. Q. Zhang, *A hierarchy of generalized AKNS equations,  $N$ -Hamiltonian structures and finite-dimensional involutive systems and integrable systems*, J. Math. Phys., **42** (2001), 330–339. 1
- [28] Y. Zhai, X. Geng, *Straightening out of the flows for the Hu hierarchy and its algebro-geometric solutions*, J. Math. Anal. Appl., **397** (2013), 561–576. 4
- [29] P. Zhao, E. Fan, Y. Hou, *Algebro-geometric solutions for the Ruijsenaars-Toda Hierarchy*, Chaos, Solitons Fractals, **54** (2013), 8–25. 1
- [30] P. Zhao, E. Fan, Y. Hou, *The algebro-geometric initial value problem for the relativistic Lotka-Volterra hierarchy and quasi-periodic solutions*, preprint arXiv: 1209.4250. 1
- [31] P. Zhao, E. Fan, Y. Hou, *Algebro-geometric solutions and their reductions for the Fokas-Lenells hierarchy*, J. Nonlinear Math. Phys., **20** (2013), 355–393. 1