



Hybrid projection algorithms for total asymptotically strict quasi- ϕ -pseudo-contractions

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Abstract

The purpose of this article is to prove strong convergence theorems for total asymptotically strict quasi- ϕ -pseudo-contractions by using a hybrid projection algorithm in Banach spaces. As applications, we apply our main results to find minimizers of proper, lower semicontinuous, convex functionals and solutions of equilibrium problems. ©2015 All rights reserved.

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1. Introduction

Fixed point theory, as an important branch of nonlinear analysis theory, has been applied in the study of nonlinear phenomena. The theory itself is a beautiful mixture of analysis, topology, and geometry. Lots of problems arising in economics, engineering, and physics can be studied by fixed point techniques.

Constructing iterative algorithms to approximate fixed points of nonlinear mappings is always one of the main concerns of fixed point theory. The simplest and oldest iterative algorithm is the Picard iterative algorithm. It is known that T , where T stands for a contractive mapping, admits a unique fixed point and the sequence generated by the Picard iterative algorithm can converge to the unique fixed point. However, for more general nonexpansive mappings, the Picard iterative algorithm fails to converge to fixed points of nonexpansive mappings even when they admit fixed points. The Mann iterative algorithm has been studied for approximating fixed points of nonexpansive mappings and their extensions. However, It is known that the Mann iterative algorithm only has weak convergence, even for nonexpansive mappings in infinite-dimensional

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Hilbert spaces; for more details, see [24, 31] and the reference therein. To obtain the strong convergence of the Mann iterative algorithm so-called hybrid projection algorithms have been considered; for more details, see [1, 11, 12, 15, 16, 17, 28, 29, 40, 41, 42] and the references therein.

In 2007, Marino and Xu [15] established a strong convergence theorem for fixed points of strict pseudo-contraction based on hybrid projection algorithms in Hilbert spaces. In 2010, Zhou and Gao [42] studied a new projection algorithm for strict quasi- ϕ -pseudocontractions and obtained a strong convergence theorem. In 2011, Qin, Wang, and Cho [22] introduced a new nonlinear mapping, which was called asymptotically strict quasi- ϕ -pseudocontraction, and proved a strong convergence theorem for fixed points of an asymptotically strict quasi- ϕ -pseudocontraction in some Banach space. In 2012, Qin, Agarwal, Cho, and Kang [19] established strong convergence theorems for common fixed points of a family of generalized asymptotically quasi- ϕ -nonexpansive mappings in the framework of Banach spaces. In the same year, Qin, Wang, Kang [23] proved strong convergence theorems for fixed points of asymptotically strict quasi- ϕ -pseudo-contractions, in the intermediate, sense in a real Banach space.

In this paper, we will introduce a new nonlinear mapping, total asymptotically strict quasi- ϕ -pseudo-contraction, and give a strong convergence theorem by a hybrid projection algorithm in a real Banach space. The results presented in this paper mainly improve the known corresponding results announced in the literature sources listed within this work.

2. Preliminaries

Throughout this paper, we assume that E is a real Banach space with the dual E^* , C is a nonempty closed convex subset of E , and $J : E \rightarrow 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}, \quad x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing of elements between E and E^* . We note that in a Hilbert space H , J is the identity operator. The following facts are well known: (1) if E^* is strictly convex then J is single valued; (2) if E^* is uniformly smooth then J is uniformly continuous on bounded subsets of E ; (3) if E^* is a reflexive and smooth Banach space, then J is single valued and demicontinuous.

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n \rightarrow \infty} \|\frac{x_n + y_n}{2}\| = 1$. Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E . Then the Banach space E is said to be smooth provided

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2.1)$$

exists for all $x, y \in U_E$. It is also said to be uniformly smooth if the limit (2.1) is attained uniformly for all $x, y \in U_E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E . It is also well known that E is uniformly smooth if and only if E^* is uniformly convex.

Let E be a smooth Banach space. The Lyapunov functional $\phi : E \times E \rightarrow \mathbb{R}$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \quad (2.2)$$

It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad \forall x, y \in E. \quad (2.3)$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle, \quad \forall x, y, z \in E. \quad (2.4)$$

Observe that in a Hilbert space H , (2.2) is reduced to $\phi(x, y) = \|x - y\|^2$, for all $x, y \in H$. If E is a reflexive, strictly convex, and smooth Banach space, then, for all $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$, then $x = y$. From (2.3), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J , we see that $Jx = Jy$. It follows that $x = y$; see [10, 34] for more details.

Let E be a reflexive, strictly convex and smooth Banach space and let C be a nonempty closed and convex subset of E . The generalized projection [3, 4, 14] $\Pi_C : E \rightarrow C$ is a mapping that assigns to an arbitrary point $x \in E$, the minimum point of the functional $\phi(x, y)$; that is, $\Pi_C x = \bar{x}$, where \bar{x} is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x).$$

The existence and uniqueness of the operator Π_C follow from the properties of the Lyapunov functional $\phi(x, y)$ and the strict monotonicity of the mapping J ; see, [3, 4, 10, 14]. In Hilbert spaces, $\Pi_C = P_C$, where $P_C : H \rightarrow C$ is the metric projection from a Hilbert space H onto a nonempty, closed, and convex subset C of H .

Let $T : C \rightarrow C$ be a mapping, the set of fixed points of T is denoted by $F(T)$; that is, $F(T) := \{x \in C : Tx = x\}$. A point p is said to be an asymptotic fixed point of T [25] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$.

Next, we recall the following definitions.

(1) T is called relatively nonexpansive [7, 8, 9] if $\widehat{F}(T) = F(T) \neq \emptyset$, and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \forall p \in F(T).$$

The asymptotic behavior of a relatively nonexpansive mapping was studied in [7, 8, 9].

(2) T is said to be relatively asymptotically nonexpansive if $\widehat{F}(T) = F(T) \neq \emptyset$, and

$$\phi(p, T^n x) \leq (1 + k_n)\phi(p, x), \quad \forall x \in C, \forall p \in F(T), \forall n \geq 1,$$

where $\{k_n\} \subset [0, \infty)$ is a sequence such that $k_n \rightarrow 0$ as $n \rightarrow \infty$. The class of relatively asymptotically nonexpansive mappings was first introduced in Su and Qin [32], see also, Agarwal, Cho, and Qin [2], and Qin et al. [21].

(3) T is said to be hemi-relatively nonexpansive if $F(T) \neq \emptyset$, and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \forall p \in F(T).$$

The class of hemi-relatively nonexpansive mappings was considered in Su, Wang and Xu [33], Wang, Kang and Cho [38], Phuangphoo and Kumam [18], and Wang and Kumam [39].

(4) T is said to be asymptotically quasi- ϕ -nonexpansive if $F(T) \neq \emptyset$, and there exists a sequence $\{k_n\} \subset [0, \infty)$ with $k_n \rightarrow 0$ as $n \rightarrow \infty$ such that

$$\phi(p, T^n x) \leq (1 + k_n)\phi(p, x), \quad \forall x \in C, \forall p \in F(T), \forall n \geq 1.$$

The class of asymptotically quasi- ϕ -nonexpansive mappings was considered in Zhou, Gao, and Tan [43], Qin, Cho, and Kang [20] and Saewan, Kumam and Kim [30].

(5) T is said to be generalized asymptotically quasi- ϕ -nonexpansive if $F(T) \neq \emptyset$, and there exist two sequences $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \rightarrow 0$, and $\{\nu_n\}$ with $\nu_n \rightarrow 0$ as $n \rightarrow \infty$ such that

$$\phi(p, T^n x) \leq (1 + \mu_n)\phi(p, x) + \nu_n, \quad \forall x \in C, \forall p \in F(T), \forall n \geq 1.$$

The class of generalized asymptotically quasi- ϕ -nonexpansive mappings was first considered in Qin, Agarwal, Cho and Kang [19].

Remark 2.1. According to the comparison with the definition above, the following facts can be obtained easily.

(a) The class of hemi-relatively mappings and the class of asymptotically quasi- ϕ -nonexpansive mappings are more general than the class of relatively nonexpansive mappings and the class of relatively asymptotically nonexpansive mappings. In fact, hemi-relatively nonexpansive mappings and asymptotically quasi- ϕ -nonexpansive do not require $F(T) = \widehat{F}(T)$.

(b) The class of generalized asymptotically quasi- ϕ -nonexpansive mappings is more general than the class of asymptotically quasi- ϕ -nonexpansive mappings.

(6) T is said to be a strict quasi- ϕ -pseudo-contraction if $F(T) \neq \emptyset$, and there exists a constant $k \in [0, 1)$ such that

$$\phi(p, Tx) \leq \phi(p, x) + k\phi(x, Tx), \quad \forall x \in C, \forall p \in F(T).$$

(7) T is said to be an asymptotically strict quasi- ϕ -pseudo-contraction if $F(T) \neq \emptyset$, and there exists a sequence $\{\mu_n\} \subset [0, \infty)$ with $\mu \rightarrow 0$ as $n \rightarrow \infty$ and a constant $k \in [0, 1)$ such that

$$\phi(p, T^n x) \leq (1 + \mu_n)\phi(p, x) + k\phi(x, T^n x), \quad \forall x \in C, \forall p \in F(T), \forall n \geq 1.$$

The class of asymptotically strict quasi- ϕ -pseudo-contractions was first considered in Qin, Wang, and Cho [22].

(8) T is said to be an asymptotically strict quasi- ϕ -pseudo-contraction in the intermediate sense if $F(T) \neq \emptyset$, and there exists a sequence $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \rightarrow 0$ as $n \rightarrow \infty$ and a constant $k \in [0, 1)$ such that

$$\limsup_{n \rightarrow \infty} \sup_{p \in F(T), x \in C} (\phi(p, T^n x) - (1 + \mu_n)\phi(p, x) - k\phi(x, T^n x)) \leq 0. \quad (2.5)$$

Put

$$\nu_n = \max\{0, \sup_{p \in F(T), x \in C} (\phi(p, T^n x) - (1 + \mu_n)\phi(p, x) - k\phi(x, T^n x))\},$$

which follows that $\nu_n \rightarrow 0$ as $n \rightarrow \infty$. Then, (2.5) is reduced to the following:

$$\phi(p, T^n x) \leq (1 + \mu_n)\phi(p, x) + k\phi(x, T^n x) + \nu_n, \quad \forall p \in F(T), \forall x \in C, \forall n \geq 1.$$

The class of asymptotically strict quasi- ϕ -pseudo-contractions in the intermediate sense was first considered in Qin, Wang, and Kang [23].

(9) The mapping T is said to be asymptotically regular on C if for any bounded subset K of C ,

$$\lim_{n \rightarrow \infty} \sup_{x \in K} \{\|T^{n+1}x - T^n x\|\} = 0.$$

In this paper, we introduce and consider the following new nonlinear mapping: total asymptotically strict quasi- ϕ -pseudo-contractions.

(10) T is said to be a total asymptotically strict quasi- ϕ -pseudo-contraction if $F(T) \neq \emptyset$, and there exist two sequences $\{\mu_n\} \subset [0, \infty)$ and $\{\nu_n\} \subset [0, \infty)$ with $\mu_n \rightarrow 0$ and $\nu_n \rightarrow 0$ as $n \rightarrow \infty$ and a constant $\kappa \in [0, 1)$ such that

$$\phi(p, T^n x) \leq \phi(p, x) + \kappa\phi(x, T^n x) + \mu_n\varphi(\phi(p, x)) + \nu_n, \quad \forall x \in C, p \in F(T), \quad (2.6)$$

where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a continuous and strictly increasing function with $\varphi(0) = 0$.

Remark 2.2. The following facts can be obtained from the above definitions.

(a) If the sequence $\mu_n \equiv 0$, the class of asymptotically strict quasi- ϕ -pseudo-contractions is reduced to the class of strict quasi- ϕ -pseudo-contractions.

(b) If $k = 0$, the class of asymptotically strict quasi- ϕ -pseudo-contractions is reduced to the class of asymptotically quasi- ϕ -nonexpansive mappings.

(c) The class of asymptotically strict quasi- ϕ -pseudo-contractions in the intermediate sense is a generalization of the class of asymptotically strict quasi- ϕ -pseudo-contractions. In fact, if $k = 0$ and $\mu \equiv 0$, the class of asymptotically strict quasi- ϕ -pseudo-contractions in the intermediate sense is reduced to the class of asymptotically quasi- ϕ -nonexpansive mappings in the intermediate sense.

(d) The class of total asymptotically strict quasi- ϕ -pseudo-contractions is reduced to the class of asymptotically strict quasi- ϕ -pseudo-contractions in the intermediate sense if $\varphi(x) \equiv x$ for all $x \in [0, \infty)$ and

$$\nu_n = \max\{0, \sup_{p \in F(T), x \in C} (\phi(p, T^n x) - (1 + \mu_n)\phi(p, x) - k\phi(x, T^n x))\}.$$

The definition of the closedness of T is needed in the process of proof.

(11) T is said to be closed if for any sequence $\{x_n\} \subset C$ with $x_n \rightarrow x \in C$ and $Tx_n \rightarrow y \in C$ as $n \rightarrow \infty$, then $Tx = y$.

Next, we give an example of the class of total asymptotically strict quasi- ϕ -pseudo-contractions.

Example 2.3. Let C be a unit ball in real Hilbert space l^2 and let $T : C \rightarrow C$ be a mapping defined by

$$T : (x_1, x_2, x_2, \dots) \rightarrow (0, x_1^2, a_2x_2, a_3x_3, \dots), \quad (x_1, x_2, x_3, \dots) \in l^2,$$

where $\{a_i\}$ is a sequence in $(0,1)$ such that $\prod_{i=2}^\infty a_i = \frac{1}{2}$. Then, T is a total asymptotically strict quasi- ϕ -pseudo-contraction.

Proof. It is proved in Goebel and Kirk [13] that

- (i) $\|Tx - Ty\| \leq 2\|x - y\|, \quad \forall x, y \in C;$
- (ii) $\|T^n x - T^n y\| \leq 2 \prod_{j=2}^\infty a_j \|x - y\|, \quad \forall x, y \in C, \forall n \geq 2.$

Denote by $k_1^{\frac{1}{2}} = 2, k_n^{\frac{1}{2}} = 2 \prod_{j=2}^n a_j, n \geq 2$, then

$$\lim_{n \rightarrow \infty} k_n = \lim_{n \rightarrow \infty} (2 \prod_{j=2}^n a_j)^2 = 1.$$

Letting $\mu_n = (k_n - 1)$, for all $n \geq 1, \varphi(t) = t^2$, for all $t \geq 0, \kappa \in [0, 1)$ and $\{\nu_n\}$ be a nonnegative sequence with $\nu_n \rightarrow 0$ as $n \rightarrow \infty$, then we have

$$\|T^n x - T^n y\|^2 \leq \|x - y\|^2 + \kappa \|x - y - (T^n x - T^n y)\|^2 + \mu_n \varphi(\|x - y\|) + \nu_n \quad \forall x, y \in C, n \geq 1.$$

Since $0 \in C$ and $0 \in F(T)$, this follows that $F(T) \neq \emptyset$. From the above inequality, we have

$$\|p - T^n y\|^2 \leq \|p - y\|^2 + \kappa \|y - T^n y\|^2 + \mu_n \varphi(\|p - y\|) + \nu_n. \quad \forall p \in F(T), y \in C.$$

It is well-known that l^2 is a real Hilbert space, then $\phi(x, y) = \|x - y\|^2$ for all $x, y \in l^2$. Therefore, we have

$$\phi(p, T^n y) \leq \phi(p, y) + \kappa \phi(y, T^n y) + \mu_n \varphi(\phi(p, y)) + \nu_n. \quad \forall p \in F(T), y \in C.$$

This show that the mapping T is a total asymptotically strict quasi- ϕ -pseudo-contraction. □

In order to prove our main results, we also need the following lemmas:

Lemma 2.4 ([14]). *Let E be a uniformly convex and smooth Banach space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in E . If $\phi(x_n, y_n) \rightarrow 0$ and $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \rightarrow 0$ as $n \rightarrow \infty$.*

Lemma 2.5 ([3]). *Let E be a reflexive, strictly convex, and smooth Banach space. Let C be a nonempty, closed, and convex subset of E , and $x \in E$ then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 2.6 ([3]). *Let C be a nonempty, closed, and convex subset of a smooth Banach space E and $x \in E$ then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.7. *Let E be a uniformly convex and smooth Banach space, let C be a nonempty, closed and convex subset of E . Suppose $T : C \rightarrow C$ is a closed and total asymptotically strict quasi- ϕ -pseudo-contraction. Then, $F(T)$ is closed and convex.*

Proof. First, we show that $F(T)$ is closed. Let $\{p_n\}$ be a sequence in $F(T)$ such that $p_n \rightarrow p$ as $n \rightarrow \infty$. We see that $p \in F(T)$. Indeed, from the definition of T , we have

$$\phi(p_n, T^n p) \leq \phi(p_n, p) + \kappa \phi(p, T^n p) + \mu_n \varphi(\phi(p_n, p)) + \nu_n.$$

In addition, we have from (2.6) that

$$\phi(p_n, T^n p) = \phi(p_n, p) + \phi(p, T^n p) + 2\langle p_n - p, Jp - JT^n p \rangle.$$

It follows that

$$\phi(p_n, p) + \phi(p, T^n p) + 2\langle p_n - p, Jp - JT^n p \rangle \leq \phi(p_n, p) + \kappa \phi(p, T^n p) + \mu_n \varphi(\phi(p_n, p)) + \nu_n,$$

which implies that

$$\phi(p, T^n p) \leq \frac{\mu_n}{1 - \kappa} \varphi(\phi(p_n, p)) + \frac{2}{1 - \kappa} \langle p - p_n, Jp - JT^n p \rangle + \frac{\nu_n}{1 - \kappa}.$$

from $\lim_{n \rightarrow \infty} p_n = p$, $\lim_{n \rightarrow \infty} \mu_n = \lim_{n \rightarrow \infty} \nu_n = 0$ and the above inequality, it follows that

$$\lim_{n \rightarrow \infty} \phi(p, T^n p) = 0.$$

From Lemma 2.4, we have $T^n p \rightarrow p$ as $n \rightarrow \infty$. This implies that $TT^n p = T^{n+1} p \rightarrow p$ as $n \rightarrow \infty$. From the closedness of T , we obtain that $p \in F(T)$, that is, $F(T)$ is closed.

Next, we show that $F(T)$ is convex. Let $p_1, p_2 \in F(T)$ and $p_t = tp_1 + (1 - t)p_2$, where $t \in (0, 1)$. We see that $p_t = Tp_t$. Indeed, we have from the definition of T that

$$\phi(p_1, T^n p_t) \leq \phi(p_1, p_t) + \kappa \phi(p_t, T^n p_t) + \mu_n \varphi(\phi(p_1, p_t)) + \nu_n,$$

$$\phi(p_2, T^n p_t) \leq \phi(p_2, p_t) + \kappa \phi(p_t, T^n p_t) + \mu_n \varphi(\phi(p_2, p_t)) + \nu_n.$$

By virtue of (2.6), we obtain that

$$\phi(p_t, T^n p_t) \leq \frac{\mu_n}{1 - \kappa} \varphi(\phi(p_1, p_t)) + \frac{2}{1 - \kappa} \langle p_t - p_1, Jp_t - JT^n p_t \rangle + \frac{\nu_n}{1 - \kappa}, \tag{2.7}$$

$$\phi(p_t, T^n p_t) \leq \frac{\mu_n}{1 - \kappa} \varphi(\phi(p_2, p_t)) + \frac{2}{1 - \kappa} \langle p_t - p_2, Jp_t - JT^n p_t \rangle + \frac{\nu_n}{1 - \kappa}. \tag{2.8}$$

Multiplying t and $(1 - t)$ on both the sides of (2.7) and (2.8), respectively, yields that

$$\phi(p_t, T^n p_t) \leq \frac{t\mu_n}{1 - \kappa} \varphi(\phi(p_1, p_t)) + \frac{(1 - t)\mu_n}{1 - \kappa} \varphi(\phi(p_2, p_t)) + \frac{\nu_n}{1 - \kappa}.$$

It follows that

$$\lim_{n \rightarrow \infty} \phi(p_t, T^n p_t) = 0.$$

In view of Lemma 2.4, we see that $T^n p_t \rightarrow p_t$ as $n \rightarrow \infty$. This implies that $TT^n p_t = T^{n+1} p_t \rightarrow p_t$ as $n \rightarrow \infty$. From the closedness of T , we obtain that $p_t \in F(T)$, that is, $F(T)$ is convex. Therefore, $F(T)$ is closed and convex. □

3. Main results

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a uniformly convex and smooth Banach space E . let $T : C \rightarrow C$ be a closed and total asymptotically strict quasi- ϕ -pseudo-contraction with two sequences $\{\mu_n\} \subset [0, \infty)$, $\{\nu_n\} \subset [0, \infty)$ such that $\mu_n \rightarrow 0$, $\nu_n \rightarrow 0$ as $n \rightarrow \infty$, and a constant $\kappa \in [0, 1)$. Assume that T is asymptotically regular on C and $F(T)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily,} \\ C_1 = C, \\ x_1 = \Pi_{C_1}x_0 \\ C_{n+1} = \{u \in C_n : \phi(x_n, T^n x_n) \leq \frac{2}{1-\kappa} \langle x_n - u, Jx_n - JT^n x_n \rangle + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \end{cases} \tag{3.1}$$

where $\theta_n = \mu_n \frac{M_n}{1-\kappa} + \frac{\nu_n}{1-\kappa}$, $M_n = \sup\{\varphi(\phi(p, x_n)) : p \in F(T)\}$. Then the sequence $\{x_n\}$ converges strongly to $\bar{x} = \Pi_{F(T)}x_0$, where $\Pi_{F(T)}$ is the generalized projection of E onto $F(T)$.

Proof. The proof is split into six steps.

Step 1: Show that $\Pi_{F(T)}x_0$ is well defined for any $x_0 \in E$.

By Lemma 2.7, we know that $F(T)$ is a closed and convex. Therefore, in view of the assumption of $F(T) \neq \emptyset$, $\Pi_{F(T)}x_0$ is well defined for any $x_0 \in E$.

Step 2: Show that C_n is closed and convex for each $n \geq 1$.

From the structure of C_n in (3.1), it is obvious that C_n is closed for each $n \geq 1$. Therefore, we only show that C_n is convex for each $n \geq 1$. This can be proved by induction on n . For $n = 1$, it is obvious that $C_1 = C$ is convex. Suppose that C_n is convex for some $n \in \mathbb{N}$. Next, we show that C_{n+1} is also convex for the same n . Let $w_1, w_2 \in C_{n+1}$ and $w_t = tw_1 + (1-t)w_2$, where $t \in (0, 1)$. It follows that

$$\phi(x_n, T^n x_n) \leq \frac{2}{1-\kappa} \langle x_n - w_1, Jx_n - JT^n x_n \rangle + \theta_n \tag{3.2}$$

and

$$\phi(x_n, T^n x_n) \leq \frac{2}{1-\kappa} \langle x_n - w_2, Jx_n - JT^n x_n \rangle + \theta_n, \tag{3.3}$$

where $w_1, w_2 \in C_n$. Multiplying t and $(1-t)$ on both the sides of (3.2) and (3.3), respectively, implies that

$$\phi(x_n, T^n x_n) \leq \frac{2}{1-\kappa} \langle x_n - w_t, Jx_n - JT^n x_n \rangle + \theta_n,$$

where $w_t \in C_n$. It follows that $w_t \in C_{n+1}$, that is, C_{n+1} is convex for the same n . Therefore, C_n is closed and convex for each $n \geq 1$.

Step 3: Show that $F(T) \subset C_n$ for each $n \geq 1$.

It is obvious that $F(T) \subset C = C_1$. Suppose that $F(T) \subset C_n$ for some $n \in \mathbb{N}$. We see that $F(T) \subset C_{n+1}$ for the same n . Indeed, For any $p \in F(T) \subset C_n$, we see that

$$\phi(p, T^n x_n) \leq \phi(p, x) + \kappa\phi(x_n, T^n x_n) + \mu_n\varphi(\phi(p, x_n)) + \nu_n. \tag{3.4}$$

On the other hand, we obtain from (2.6) that

$$\phi(p, T^n x_n) = \phi(p, x_n) + \phi(x_n, T^n x_n) + 2\langle p - x_n, Jx_n - JT^n x_n \rangle. \tag{3.5}$$

Combining (3.4) with (3.5), we have

$$\begin{aligned} \phi(x_n, T^n x_n) &\leq \frac{\mu_n}{1-\kappa} \varphi(\phi(p, x_n)) + \frac{2}{1-\kappa} \langle x_n - p, Jx_n - JT^n x_n \rangle + \frac{\nu_n}{1-\kappa} \\ &\leq \frac{\mu_n}{1-\kappa} M_n + \frac{2}{1-\kappa} \langle x_n - p, Jx_n - JT^n x_n \rangle + \frac{\nu_n}{1-\kappa} \\ &= \frac{2}{1-\kappa} \langle x_n - p, Jx_n - JT^n x_n \rangle + \theta_n, \end{aligned}$$

which implies that $p \in C_{n+1}$, that is, $F(T) \subset C_{n+1}$ for the same n . By the mathematical induction principle, $F(T) \subset C_n$ for each $n \geq 1$.

Step 4: Show that $\{x_n\}$ is a Cauchy sequence.

From $x_n = \Pi_{C_n} x_0$, one sees

$$\langle x_n - z, Jx_0 - Jx_n \rangle \geq 0, \quad \forall z \in C_n. \tag{3.6}$$

Since $F(T) \subset C_n$ for all $n \geq 1$, we arrive at

$$\langle x_n - w, Jx_0 - Jx_n \rangle \geq 0, \quad \forall w \in F(T). \tag{3.7}$$

From Lemma 2.5, one has

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(w, x_0) - \phi(w, x_n) \leq \phi(w, x_0)$$

for each $w \in F(T)$ and $n \geq 1$. Therefore, the sequence $\phi(x_n, x_0)$ is bounded. On the other hand, noticing that $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, one has

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0)$$

for all $n \geq 0$. Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing. It follows that the limit of $\{\phi(x_n, x_0)\}$ exists. By the construction of C_n , one has that $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_0 \in C_n$ for any positive integer $m \geq n$. It follows that

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \\ &\leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0). \end{aligned} \tag{3.8}$$

Letting $m, n \rightarrow \infty$ in (3.8), one has $\phi(x_m, x_n) \rightarrow 0$. It follows from Lemma 2.4 that $x_m - x_n \rightarrow 0$ as $m, n \rightarrow \infty$. Hence $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, one can assume that $x_n \rightarrow \bar{x} \in C$ as $n \rightarrow \infty$.

Step 5: Show that $\bar{x} \in F(T)$.

By utilizing the construction of C_n and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, T^n x_n) \leq \frac{2}{1+\kappa} \langle x_n - x_{n+1}, Jx_n - JT^n x_n \rangle + \theta_n, \tag{3.9}$$

Since $\lim_{n \rightarrow \infty} \|x_n - x_{n+1}\| = 0$ and $\lim_{n \rightarrow \infty} \theta_n = 0$, we have from (3.9) that

$$\lim_{n \rightarrow \infty} \phi(x_n, T^n x_n) \rightarrow 0.$$

In view of Lemma 2.4, we arrive at

$$\lim_{n \rightarrow \infty} \|x_n - T^n x_n\| = 0. \tag{3.10}$$

Note that $x_n \rightarrow \bar{x}$ as $n \rightarrow \infty$ and

$$\|T^n x_n - \bar{x}\| \leq \|T^n x_n - x_n\| + \|x_n - \bar{x}\|.$$

It follows from the above inequality that

$$T^n x_n \rightarrow \bar{x}, \tag{3.11}$$

as $n \rightarrow \infty$. Observe that

$$\|T^{n+1} x_n - \bar{x}\| \leq \|T^{n+1} x_n - T^n x_n\| + \|T^n x_n - \bar{x}\|. \tag{3.12}$$

By using (3.11), (3.12) and the asymptotic regularity of T , we have

$$T^{n+1} x_n \rightarrow \bar{x},$$

as $n \rightarrow \infty$, that is, $TT^n x_n \rightarrow \bar{x}$. From the closedness of T , we obtain that $\bar{x} = T\bar{x}$.

Step 6: Show that $\bar{x} = \Pi_{F(T)}x_0$.

Notice from (3.7), that

$$\langle x_n - w, Jx_0 - Jx_n \rangle \geq 0, \quad \forall w \in F(T).$$

Taking the limit in the above inequality yields

$$\langle \bar{x} - w, Jx_0 - J\bar{x} \rangle \geq 0, \quad \forall w \in F(T).$$

Hence, we obtain from Lemma 2.6 that $\bar{x} = \Pi_{F(T)}x_0$. This completes the proof. □

Based on Theorem 3.1, we have the following corollary.

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a uniformly convex and smooth Banach space E . Let $T : C \rightarrow C$ be a closed and asymptotically strict quasi- ϕ -pseudo-contraction in the intermediate sense with a sequence $\{\mu_n\} \subset [0, \infty)$ such that $\mu_n \rightarrow 0$ as $n \rightarrow \infty$, and a constant $\kappa \in [0, 1)$. Assume that T is asymptotically regular on C and $F(T)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily,} \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0 \\ C_{n+1} = \{u \in C_n : \phi(x_n, T^n x_n) \leq \frac{2}{1-\kappa} \langle x_n - u, Jx_n - JT^n x_n \rangle + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where $\theta_n = \mu_n \frac{M_n}{1-\kappa} + \frac{\nu_n}{1-\kappa}$, $M_n = \sup\{\phi(p, x_n) : p \in F(T)\}$ and

$$\nu_n = \max\{0, \sup_{p \in F(T), x \in C} (\phi(p, T^n x) - (1 + \mu_n)\phi(p, x) - \kappa\phi(x, T^n x))\}.$$

Then the sequence $\{x_n\}$ converges strongly to $\bar{x} = \Pi_{F(T)}x_0$, where $\Pi_{F(T)}$ is the generalized projection of E onto $F(T)$.

Proof. Putting $\varphi(x) = x$ for all $x \in [0, \infty)$ and

$$\nu_n = \max\{0, \sup_{p \in F(T), x \in C} (\phi(p, T^n x) - (1 + \mu_n)\phi(p, x) - \kappa\phi(x, T^n x))\},$$

the conclusion can be obtained from Theorem 3.1. □

Let C be a nonempty, closed, and convex subset of a Hilbert space H , a mapping $T : C \rightarrow C$ is said to be a total asymptotically strict quasi-pseudo-contraction if $F(T) \neq \emptyset$, and there exist two sequences $\{\mu_n\} \subset [0, \infty)$, $\{\nu_n\} \subset [0, \infty)$ with $\mu_n \rightarrow 0$ and $\nu_n \rightarrow 0$ as $n \rightarrow \infty$ and a constant $\kappa \in [0, 1)$ such that

$$\|T^n x - p\|^2 \leq \|x - p\|^2 + \kappa \|x - T^n x\|^2 + \mu_n \varphi(\|x - p\|) + \nu_n,$$

where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a continuous and strictly increasing function with $\varphi(0) = 0$.

In the framework of Hilbert spaces, we have the following result for a total asymptotically strict quasi-pseudo-contraction.

Corollary 3.3. *Let C be a nonempty, closed and convex subset of a Hilbert space H . Let $T : C \rightarrow C$ be a closed and total asymptotically strict quasi-pseudo-contraction with two sequences $\{\mu_n\} \subset [0, \infty)$, $\{\nu_n\} \subset [0, \infty)$ such that $\mu_n \rightarrow 0$ and $\nu_n \rightarrow 0$ as $n \rightarrow \infty$, and a constant $\kappa \in [0, 1)$. Assume that T is asymptotically regular on C and $F(T)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily,} \\ C_1 = C, \\ x_1 = P_{C_1} x_0 \\ C_{n+1} = \{u \in C_n : \|x_n - T^n x_n\| \leq \frac{2}{1-\kappa} \langle x_n - u, x_n - T^n x_n \rangle + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \end{cases}$$

where $\theta_n = \mu_n \frac{M_n}{1-\kappa} + \frac{\nu_n}{1-\kappa}$, $M_n = \sup\{\varphi(\|p - x_n\|) : p \in F(T)\}$. Then the sequence $\{x_n\}$ converges strongly to $\bar{x} = P_{F(T)} x_0$, where $P_{F(T)}$ is the metric projection of E onto $F(T)$.

Remark 3.4. Since the class of the total asymptotically strict quasi- ϕ -pseudo-contractions includes the class of asymptotically strict quasi- ϕ -pseudo-contractions in the intermediate sense, the class of asymptotically strict quasi-pseudo-contractions, the class of strict quasi- ϕ -pseudo-contractions, the class of generalized asymptotically quasi-nonexpansive mappings, the class of asymptotically quasi- ϕ -nonexpansive mappings, the class of relatively asymptotically nonexpansive mappings, and the class of hemi-relatively nonexpansive mappings as special cases. So, Theorem 3.1 improves many current results, see Su, and Qin [32], Su, Wang, and Xu [33], Wang, Kang, and Cho [38], Zhou, Gao, and Tan [43], Qin, Cho, and Kang [20], Qin, Wang, and Kang [23], Qin, Wang, and Cho [22].

4. Applications

4.1. Application to optimization problems

In this part, we consider minimizers of proper, lower semicontinuous, and convex functionals. Let T be a mapping of E into 2^{E^*} . The effective domain of T is denoted by $D(T)$, that is, $D(T) = \{x \in E : Tx \neq \emptyset\}$. The range of T is denoted by $R(T)$, that is, $R(T) = \cup\{Tx : x \in D(T)\}$. A multi-valued operator $T : E \rightarrow 2^{E^*}$ with graph $G(T) = \{(x, x^*) : x^* \in Tx\}$ is said to be monotone if for any $x, y \in D(T)$, $x^* \in Tx$ and $y^* \in Ty$,

$$\langle x - y, x^* - y^* \rangle \geq 0.$$

A monotone operator T is said to be maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone operator. If E is reflexive and strictly convex, then a monotone operator T is maximal if and only if $R(J + rT) = E^*$ for all $r > 0$, see [27, 35] for more details.

Let E be a Banach space with the dual E^* . For a proper lower semicontinuous convex function $f : E \rightarrow (-\infty, \infty]$, the subdifferential mapping $\partial f \subset E \times E^*$ of f is defined as follows:

$$\partial f(x) = \{x^* \in E^* : f(y) \geq f(x) + \langle y - x, x^* \rangle, \forall y \in E\}, \quad \forall x \in E.$$

From Rockafellar [26], we know that ∂f is maximal monotone operator, and $0 \in \partial f(v)$ if and only if $f(v) = \min_{x \in E} f(x)$. For each $r > 0$, and $x \in E$, there exists a unique $x_r \in D(\partial f)$ such that

$$Jx \in Jx_r + r\partial f x_r.$$

If $J_r x = x_r$, then we can define a single valued mapping $J_r : E \rightarrow D(\partial f)$ by

$$J_r = (J + r\partial f)^{-1}J,$$

which is said to be the resolvent of ∂f . We affirm that $(\partial f)^{-1}0 = F(J_r)$ for all $r > 0$. In fact,

$$\begin{aligned} u \in F(J_r) &\Leftrightarrow u = J_r u = (J + r\partial f)^{-1}Ju \Leftrightarrow Ju \in Ju + r\partial f u \\ &\Leftrightarrow 0 \in r\partial f u \Leftrightarrow 0 \in \partial f u \Leftrightarrow u \in (\partial f)^{-1}0, \quad \forall r > 0. \end{aligned}$$

It is well known that if ∂f is a maximal monotone operator, then $(\partial f)^{-1}0$ is closed and convex. In view of Lemma 4.2 of Wang, Kang, Cho [38], we learn that J_r is a closed hemi-relatively nonexpansive mapping. Notice that every hemi-relatively nonexpansive mapping is a total asymptotically strict quasi- ϕ -pseudo-contraction. In view of Theorem 3.1, the following theorem is obtained immediately.

Theorem 4.1. *Let C be a nonempty, closed and convex subset of a uniformly convex and smooth Banach space E . Let $f : E \rightarrow (-\infty, \infty]$ be a proper, lower semicontinuous, and convex function, ∂f the subdifferential mapping of f , J_r the resolvent of ∂f . Assume that $(\partial f)^{-1}(0)$ is nonempty. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily,} \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0 \\ C_{n+1} = \{u \in C_n : \phi(x_n, J_r x_n) \leq 2\langle x_n - u, Jx_n - J J_r x_n \rangle\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{4.1}$$

where $r > 0$. Then the sequence $\{x_n\}$ converges strongly to $\bar{x} = \Pi_{(\partial f)^{-1}(0)} x_0$, where $\Pi_{(\partial f)^{-1}(0)}$ is the generalized projection of E onto $(\partial f)^{-1}(0)$.

4.2. Application to equilibrium problems

In this part, we consider the problem for finding a solution to equilibrium problems. Let C be a nonempty, closed, and convex subset of a Banach space E . Let $f : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying the following conditions:

- (A1) $f(x, y) = 0$ for all $x \in C$;
- (A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
- (A3) $\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$ for all $x, y, z \in C$;
- (A4) $f(x, \cdot)$ is convex and lower semicontinuous for all $x \in C$.

The mathematical model related to equilibrium problems is to find $\bar{x} \in C$ such that

$$f(\bar{x}, y) \geq 0, \quad \forall y \in C. \tag{4.2}$$

The set of solutions to equilibrium problems (4.2) is denoted by $EP(f)$. The following lemma can be obtained in Blum and Oettli [6]:

Lemma 4.2. *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E , and let f be a fibunction from $C \times C$ to \mathbb{R} satisfying (A1)-(A4), and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C.$$

The following lemma can be found in Takahashi and Zembayashi [37]:

Lemma 4.3. *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E , and let f be a fibunction from $C \times C$ to \mathbb{R} satisfying (A1)-(A4). For $r > 0$ and $x \in E$, define a mapping $T_r : E \rightarrow C$ as follows:*

$$T_r x = \{z \in C : f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C\}, \quad \forall x \in E.$$

Then, the following hold:

- (1) T_r is single-valued;
- (2) T_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle;$$

- (3) $R(T_r) = EP(f)$;
- (4) $EP(f)$ is closed and convex.

Motivate by Takahashi et al. [36] in a Hilbert space, we obtain the following lemma:

Lemma 4.4. *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E , and let f be a fibunction from $C \times C$ to \mathbb{R} satisfying (A1)-(A4). Let A_f be a multi-valued mapping of E into E^* defined by*

$$A_f x = \begin{cases} \{x^* \in E^* : f(x, y) \geq \langle y - x, z^* \rangle, \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

Then, $EP(f) = A_f^{-1}0$ and A_f is a maximal monotone operator with $D(A_f) \subset C$. Furthore, for any $x \in E$ and $r > 0$, the resolvent T_r of f coincides with the resolvent of A_f ; i.e.,

$$T_r x = (J + rA_f)^{-1} Jx.$$

Proof. First, we show that $EP(f) = A_f^{-1}0$. In fact, we have that

$$\begin{aligned} u \in EP(f) &\Leftrightarrow f(u, y) \geq 0, \quad \forall y \in C \\ &\Leftrightarrow f(z, y) \geq \langle y - u, 0^* \rangle, \quad \forall y \in C \\ &\Leftrightarrow 0^* \in A_f u \\ &\Leftrightarrow u \in A_f^{-1}0. \end{aligned}$$

We show that A_f is monotone. Let $(x_1, z_1^*), (x_2, z_2^*) \in A_f$. Then, we have, for all $y \in C$,

$$f(x_1, y) \geq \langle y - x_1, z_1^* \rangle \quad \text{and} \quad f(x_2, y) \geq \langle y - x_2, z_2^* \rangle$$

and hence

$$f(x_1, x_2) \geq \langle x_2 - x_1, z_1^* \rangle \quad \text{and} \quad f(x_2, x_1) \geq \langle x_1 - x_2, z_2^* \rangle.$$

It follows by applying (A2) that

$$0 \geq f(x_1, x_2) + f(x_2, x_1) \geq \langle x_2 - x_1, z_1^* \rangle + \langle x_1 - x_2, z_2^* \rangle = -\langle x_1 - x_2, z_1^* - z_2^* \rangle.$$

This implies that A_f is montone. Next, we show that A_f is maximal monotone. To prove that A_f is maximal monotone, it is sufficient to show that $R(J + rA_f) = E^*$ for all $r > 0$. Let $x \in E$ and $r > 0$. Hence, in view of Lemma 4.2, there exists $z \in C$ such that

$$f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C.$$

Therefore we obtain that

$$f(z, y) \geq \langle y - z, \frac{1}{r}(Jx - Jz) \rangle, \quad \forall y \in C.$$

In view of the definition of A_f , we have

$$A_f z \ni \frac{1}{r}(Jx - Jz),$$

which implies that $Jx \in Jz + rA_f z$. Hence $E^* \subset R(J + rA_f)$. So, $R(J + rA_f) = E^*$ and at the same time, $Jx \in Jz + rA_f z$ implies that $T_r x = (J + rA_f)^{-1} Jx$ for all $x \in E$ and $r > 0$. This completes the proof. \square

Theorem 4.5. *Let C be a nonempty, closed and convex subset of a uniformly convex and smooth Banach space E . Let f be a bifunction from $C \times C$ to \mathbb{R} satisfying (A1)-(A4) and T_r be defined as Lemma 4.3 for $r > 0$. Assume that $EP(f)$ is nonempty. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily,} \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0 \\ C_{n+1} = \{u \in C_n : \phi(x_n, T_r x_n) \leq 2\langle x_n - u, Jx_n - JT_r x_n \rangle\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{cases}$$

Then the sequence $\{x_n\}$ converges strongly to $\bar{x} = \Pi_{EP(f)} x_0$, where $\Pi_{EP(f)}$ is the generalized projection of E onto $EP(f)$.

Proof. From Lemma 4.4, we know that T_r be regarded as the resolvent of A_f for $r > 0$. By using Theorem 4.1, we have that the sequence $\{x_n\}$ converges strongly to $\bar{x} = \Pi_{(A_f)^{-1}(0)} x_0$. From Lemma 4.4, we get $EP(f) = A_f^{-1}(0)$. So, the sequence $\{x_n\}$ converges strongly to $\bar{x} = \Pi_{EP(f)} x_0$. \square

5. Numerical Examples

In this section, we give an numerical example of a maximal monotone operator to illustrate our result.

Example 5.1. Let $E = \mathbb{R}$, $C = [0, 1]$, $Tx = \frac{1}{2}x$, $Ax = x$. From the definition of T , it is obvious that 0 is the unique fixed point of T , that is, $F(T) = \{0\}$. Since

$$|Tx - Ty|^2 + |(Id - T)x - (Id - T)y|^2 = |\frac{1}{2}x - \frac{1}{2}y|^2 + |x - \frac{1}{2}x - (y - \frac{1}{2}y)|^2 = \frac{1}{2}|x - y|^2 \leq |x - y|^2,$$

it follows that T is a firmly nonexpansive mapping. From [5], we have $A = T^{-1} - Id$ is maximally monotone and the resolvent operator $J_A = T$. On the other hand, we have

$$\phi(0, J_A x) = \phi(0, Tx) = |0|^2 - \langle 0, JT x \rangle + |Tx|^2 = \frac{1}{4}x^2 \leq x^2 = |0|^2 - \langle 0, Jx \rangle + |x|^2 = \phi(0, x),$$

it follows from Lemma 4.2 of Wang, Kang, Cho [38] that T is a closed hemi-relatively nonexpansive mapping. From (2.2), we compute that

$$\phi(x_n, J_A x_n) = |x_n|^2 - 2\langle x_n, J_A x_n \rangle + |J_A x_n|^2 = \frac{1}{4}x_n^2 \quad (5.1)$$

and

$$2\langle x_n - u, Jx_n - JJ_A x_n \rangle = 2\langle x_n - u, Jx_n - J\frac{1}{2}x_n \rangle = x_n^2 - x_n u. \quad (5.2)$$

From (5.1), (5.2), and the C_{n+1} of the algorithm (4.1) can be evolved into the following:

$$C_{n+1} = \{u \in C_n : u \leq \frac{3}{4}x_n\}.$$

Therefore, the algorithm (4.1) can be simplified as

$$\begin{cases} x_0 \in \mathbb{R} \text{ chosen arbitrarily,} \\ C_1 = C = [0, 1], \\ x_1 = \Pi_{C_1} x_0 \\ C_{n+1} = \{u \in C_n : u \leq \frac{3}{4}x_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0 = \frac{3}{4}x_n. \end{cases} \quad (5.3)$$

So, the sequence $\{x_n\}$ converges strongly to $\bar{x} = \Pi_{A^{-1}(0)}x_0 = 0$ by using Theorem 4.1.

Take initial point $x_0 \in (1, +\infty)$ arbitrarily, the numerical experiment result using software Matlab 7.0 is given in Figure 1, which shows that the iteration process of the sequence $\{x_n\}$ converges to 0.

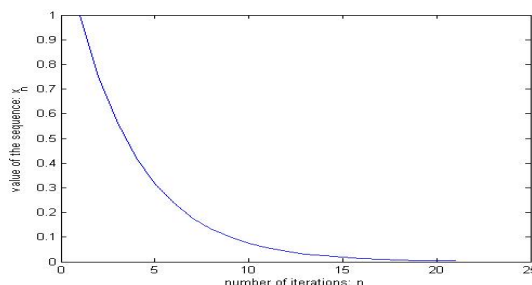


Figure 1: $x_0 \in (1, +\infty)$, the convergence process of the sequence $\{x_n\}$ in Example 5.1

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