Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 3376–3387
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Certain inequalities involving generalized fractional k-integral operators

K. S. Nisar[a], M. Al-Dhaifallah[b], M. S. Abouzaid[c], P. Agarwal[d,*]

[a]Department of Mathematics, College of Arts and Science, Prince Sattam bin Abdulaziz University, Wadi Aldawaser, 11991, Saudi Arabia.

[b]Department of Electrical Engineering, College of Engineering-Wadi AlDawaser, Prince Sattam bin Abdulaziz University, Saudi Arabia.

[c]Department of Mathematics, Faculty of Science, Kafrelsheikh University, Egypt.

[d]Department of Mathematics, Anand International College of Engineering, Jaipur-303012, India.

Communicated by A. Atangana

Abstract

Recently, fractional k-integral operators have been investigated in the literature by some authors. Here, we focus to prove some new fractional integral inequalities involving generalized fractional k-integral operator due to Sarikaya et al. for the cases of synchronous functions as well as of functions bounded by integrable functions are considered.

Keywords: Coincidence point, common fixed point, contraction, implicit relation, partial metric space.
2010 MSC: 47H10, 54H25.

1. Introduction

In 1882, P. L. Chebyshev[12] was established the Chebyshev functional (1.1), which has attracted many researcher's attention due mainly to diverse applications in numerical quadrature, transform theory, probability and statistical problems. Among those applications, the functional (1.1) has also been employed to yield a number of integral inequalities (see, *e.g.*, [1, 2, 3, 5, 9, 13, 14, 15, 16, 17, 20, 23, 25, 29, 33, 34]). This is defined as (see [12]):

*Corresponding author

Email addresses: ksnisar1@gmail.com (K. S. Nisar), m.aldhaifallah@psau.edu.sa (M. Al-Dhaifallah), moheb_abouzaid@hotmail.com (M. S. Abouzaid), goyal.praveen2011@gmail.com (P. Agarwal)

Received 2016-04-05

$$
\begin{aligned}
T(f,g,p,q) = &\int_a^b q(x)\,dx \int_a^b p(x)\,f(x)\,g(x)\,dx \\
&+ \int_a^b p(x)dx \int_a^b q(x)\,f(x)\,g(x)dx \\
&- \left(\int_a^b q(x)\,f(x)dx\right)\left(\int_a^b p(x)\,g(x)dx\right) \\
&- \left(\int_a^b p(x)\,f(x)dx\right)\left(\int_a^b q(x)\,g(x)dx\right),
\end{aligned}
\tag{1.1}
$$

where $f,\, g\, :\, [a,\, b] \to \mathbb{R}$ are two integrable functions on $[a, b]$ and $p(x)$ and $q(x)$ are positive integrable functions on $[a,\, b]$. If f and g are *synchronous* on $[a, b]$, *i.e.*,

$$(f(x) - f(y))\,(g(x) - g(y)) \geq 0 \tag{1.2}$$

for any $x,\, y \in [a,\, b]$, then we have (see [27]):

$$T(f, g, p, q) \geq 0. \tag{1.3}$$

The inequality in (1.2) is reversed if f and g are *asynchronous* on $[a,\, b]$, *i.e.*,

$$(f(x) - f(y))\,(g(x) - g(y)) \leq 0 \tag{1.4}$$

for any $x,\, y \in [a,\, b]$. If $p(x) = q(x)$ for any $x,\, y \in [a,\, b]$, we get the Chebyshev inequality (see [12]).

Ostrowski [30] established the following generalization of the Chebyshev inequality:

If f and g are two differentiable and synchronous functions on $[a,\, b]$, and p is a positive integrable function on $[a,\, b]$ with $|f'(x)| \geq m$ and $|g'(x)| \geq r$ for $x \in [a,\, b]$, then we have

$$T(f, g, p) = T(f, g, p, p) \geq m\,r\,T(x - a, x - a, p) \geq 0. \tag{1.5}$$

If f and g are asynchronous on $[a,\, b]$, then we have

$$T(f, g, p) \leq m\,r\,T(x - a, x - a, p) \leq 0. \tag{1.6}$$

If f and g are two differentiable functions on $[a,\, b]$ with $|f'(x)| \leq M$ and $|g'(x)| \leq R$ for $x \in [a,\, b]$ and p is a positive integrable function on $[a,\, b]$, then we have

$$|T(f, g, p)| \leq M\,R\,T(x - a, x - a, p) \leq 0. \tag{1.7}$$

Here, we begin with the following definitions.

Definition 1.1. Let $k > 0$, then the generalized $k-$gamma and $k-$beta functions defined by [18]:

$$\Gamma_k(x) = \begin{array}{c} lim \\ n \to \infty \end{array} \frac{n!\,k^n (nk)^{\frac{x}{k}} - 1}{(x)_{n,k}}, \tag{1.8}$$

where $(x)_{n,k}$ is a Pochhammer k-symbol defined by

$$(x)_{n,k} = x(x + k)(x + 2k) \cdots (x + (n - 1)k)\,, \qquad (n \geq 1).$$

Definition 1.2. The Mellin transform of the exponential function $e^{-\frac{t^k}{k}}$ is the $k-$ gamma function defined as:

$$\Gamma_k = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}}\,dt, \qquad \Re(x) > 0.$$

Clearly

$$\Gamma(x) = \begin{array}{c} lim \\ k \to 1 \end{array} \Gamma_k(x) = k^{\frac{x}{k} - 1} \;\; and \;\; \Gamma_k(x + k) = x\Gamma_k(x).$$

The inequalities involving fractional integral operators has gained considerable popularity and importance during the past few years. In literature point of view many fractional integral operators already proved their importance. Very recently, fractional operator, whose derivative has no singular kernel introduced by Caputo and Fabrizio [10, 26]. Motivated by above work many researchers applied new derivative in certain real world problems (see, e.g., [4, 6, 7, 8, 11, 19, 21, 22]). In the sequel, recently, k-extensions of some familiar fractional integral operator like Riemann-Liouville have been investigated by many authors in interesting and useful manners (see [31, 32]). Here, we begin with the following.

Definition 1.3. If $k > 0$, let $f \in L_1(a, b)$, then the Riemann-Liouville k-fractional integral $R^{\alpha}_{a,k}$ of order $a \geq 0$ and $\alpha > 0$ for a real-valued continuous function $f(t)$ is defined by ([28], see also [32]):

$$R^{\alpha}_{a,k}\{f(t)\} = \frac{1}{k\Gamma_k(\alpha)}\int_a^t (t-\tau)^{\frac{\alpha}{k}-1} f(\tau)d\tau, \quad (\alpha > 0, \tau > a). \tag{1.9}$$

For $k = 1$, equation (1.9) reduces to the classical Riemann-liouville fractional integral.

Definition 1.4. If $k > 0$, let $f \in L_{1,r}[a, b]$ then the generalized Riemann-Liouville k-fractional integral $R^{\alpha,r}_{a,k}$ of order $a \geq 0, \alpha > 0$ and $r \in \mathbb{R}\backslash\{-1\}$ for a real-valued continuous function $f(t)$ is defined by ([31]):

$$R^{\alpha,r}_{a,k}\{f(t)\} = \frac{(1+r)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t (t^{r+1} - \tau^{r+1})^{\frac{\alpha}{k}-1} f(\tau)d\tau, \quad (t \in [a, b]), \tag{1.10}$$

where Γ_k is the Euler gamma k-function.

For $a = 0$, it is easy to see that

$$R^{\alpha,r}_{a,k}\{f(t)\} = R^{\alpha,r}_{k}\{f(t)\}.$$

The (1.10) has the following properties

$$R^{\alpha,r}_{a,k}\left\{R^{\beta,r}_{a,k} f(t)\right\} = R^{\alpha+\beta,r}_{a,k}\{f(t)\} = R^{\beta,r}_{a,k}\left\{R^{\alpha,r}_{a,k} f(t)\right\} \tag{1.11}$$

and

$$R^{\alpha,r}_{a,k}\{1\} = \frac{(t^{r+1} - \tau^{r+1})^{\frac{\alpha}{k}-1}}{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}, \quad \alpha > 0. \tag{1.12}$$

Here, our purpose is to prove k-calculus analogous of some classical integral inequalities and prove k-generalizations of the Chebyshev integral inequalities by using the generalized Riemann-Liouville fractional k-integral operator. For our object we consider the case of synchronous functions as well as the case of functions bounded by integrable functions.

We organize the paper as follows: in Section 2, we prove two inequalities involving a generalized Riemann-Liouville $k-$fractional integral operators for synchronous functions and Section 3 contains some new inequalities involving generalized fractional k-integral operator in the case where the functions are bounded by integrable functions and not necessary increasing or decreasing as are the synchronous functions.

2. Inequalities involving generalized fractional k-integral operator for synchronous functions

This section begins by presenting two inequalities involving generalized fractional k-integral operator (1.10) stated in Lemmas 2.1 and 2.2.

Lemma 2.1. *Let f and g be two continuous and synchronous functions on $[0, \infty)$ and $u, v : [0, \infty) \to [0, \infty)$ be continuous functions. Then the following inequality holds true:*

$$\begin{aligned} &R^{\alpha,r}_{k}\{u\}(t)\, R^{\alpha,r}_{k}\{v\,f\,g\}(t) + R^{\alpha,r}_{k}\{v\}(t)\, R^{\alpha,r}_{k}\{u\,f\,g\}(t) \\ &\quad \geq R^{\alpha,r}_{k}\{u\,f\}(t)\, R^{\alpha,r}_{k}\{v\,g\}(t) + R^{\alpha,r}_{k}\{v\,f\}(t)\, R^{\alpha,r}_{k}\{u\,g\}(t) \end{aligned} \tag{2.1}$$

for all $t > 0, k > 0$, $\alpha > 0$ and $r \in \mathbb{R}\backslash\{-1\}$.

Proof. Let f and g be two continuous and synchronous functions on $[0, \infty)$. Then, for all τ, $\rho \in (0,t)$ with $t > 0$, we have

$$(f(\tau) - f(\rho))\,(g(\tau) - g(\rho)) \geq 0 \tag{2.2}$$

or, equivalently,

$$f(\tau)g(\tau) + f(\rho)g(\rho) \geq f(\tau)g(\rho) + f(\rho)g(\tau). \tag{2.3}$$

Now, multiplying both sides of (2.3) by $\frac{(1+r)^{1-\frac{\alpha}{k}}\,(t^{r+1}-\tau^{r+1})^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}u(\tau)$, and integrating the resulting inequality with respect to τ from 0 to t, and using (1.10), we get

$$R_k^{\alpha,r}\left\{u\,f\,g\right\}(t) + f(\rho)g(\rho)R_k^{\alpha,r}\left\{u\right\}(t) \geq g(\rho)R_k^{\alpha,r}\left\{u\,f\right\}(t) + f(\rho)R_k^{\alpha,r}. \tag{2.4}$$

Next, multiplying both sides of (2.4) by $\frac{(1+r)^{1-\frac{\alpha}{k}}\,(t^{r+1}-\rho^{r+1})^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}v(\rho)$ and integrating the resulting inequality with respect to ρ from 0 to t and using (1.10), we are led to the desired result (2.1). □

Lemma 2.2. *Let f and g be two continuous and synchronous functions on $[0, \infty)$ and let u, $v : [0,\infty) \to [0,\infty)$ be continuous functions. Then the following inequality holds true:*

$$\begin{aligned} &R_k^{\beta,r}\left\{v\right\}(t)\,R_k^{\alpha,r}\left\{u\,f\,g\right\}(t) + R_k^{\beta,r}\left\{v\,f\,g\right\}(t)\,R_k^{\alpha,r}\left\{u\right\}(t) \\ &\quad \geq R_k^{\beta,r}\left\{v\,g\right\}(t)\,R_k^{\alpha,r}\left\{u\,f\right\}(t) + R_k^{\beta,r}\left\{v\,f\right\}(t)\,R_k^{\alpha,r}\left\{u\,g\right\}(t) \end{aligned} \tag{2.5}$$

for all $t > 0, k > 0,\ \alpha > 0$ and $r \in \mathbb{R}\backslash\{-1\}$.

Proof. Multiplying both sides of (2.4) by $\frac{(1+r)^{1-\frac{\beta}{k}}\,(t^{r+1}-\rho^{r+1})^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}v(\rho)$, which remains nonnegative under the conditions in (2.5) and integrating the resulting inequality with respect to ρ from 0 to t and using (1.10), we get the desired result (2.5). □

Theorem 2.3. *Let f and g be two continuous and synchronous functions on $[0,\infty)$ and let $l, m, n : [0,\infty) \to [0,\infty)$ be continuous functions. Then the following inequality holds true:*

$$\begin{aligned} &2R_k^{\alpha,r}\left\{l\right\}(t)\left[R_k^{\alpha,r}\left\{m\right\}(t)\,R_k^{\alpha,r}\left\{n\,f\,g\right\}(t) + R_k^{\alpha,r}\left\{n\right\}(t)\,R_k^{\alpha,r}\left\{m\,f\,g\right\}(t)\right] \\ &\qquad + 2R_k^{\alpha,r}\left\{m\right\}(t)\,R_k^{\alpha,r}\left\{n\right\}(t)\,R_k^{\alpha,r}\left\{l\,f\,g\right\}(t) \\ &\quad \geq R_k^{\alpha,r}\left\{l\right\}(t)\left[R_k^{\alpha,r}\left\{m\,f\right\}(t)\,R_k^{\alpha,r}\left\{n\,g\right\}(t) + R_k^{\alpha,r}\left\{n\,f\right\}(t)\,R_k^{\alpha,r}\left\{m\,g\right\}(t)\right] \\ &\qquad + R_k^{\alpha,r}\left\{m\right\}(t)\left[R_k^{\alpha,r}\left\{l\,f\right\}(t)\,R_k^{\alpha,r}\left\{n\,g\right\}(t) + R_k^{\alpha,r}\left\{n\,f\right\}(t)\,R_k^{\alpha,r}\left\{l\,g\right\}(t)\right] \\ &\qquad + R_k^{\alpha,r}\left\{n\right\}(t)\left[R_k^{\alpha,r}\left\{l\,f\right\}(t)\,R_k^{\alpha,r}\left\{m\,g\right\}(t) + R_k^{\alpha,r}\left\{m\,f\right\}(t)\,R_k^{\alpha,r}\left\{l\,g\right\}(t)\right] \end{aligned} \tag{2.6}$$

for all $t > 0, k > 0,\ \alpha > 0$ and $r \in \mathbb{R}\backslash\{-1\}$.

Proof. By setting $u = m$ and $v = n$ in Lemma 2.1, we get

$$\begin{aligned} &R_k^{\alpha,r}\left\{m\right\}(t)\,R_k^{\alpha,r}\left\{n\,f\,g\right\}(t) + R_k^{\alpha,r}\left\{n\right\}(t)\,R_k^{\alpha,r}\left\{m\,f\,g\right\}(t) \\ &\quad \geq R_k^{\alpha,r}\left\{m\,f\right\}(t)\,R_k^{\alpha,r}\left\{n\,g\right\}(t) + R_k^{\alpha,r}\left\{n\,f\right\}(t)\,R_k^{\alpha,r}\left\{m\,g\right\}(t). \end{aligned} \tag{2.7}$$

Since $R_k^{\alpha,r}\left\{l\right\}(t) \geq 0$ under the given conditions, multiplying both sides of (2.7) by $R_k^{\alpha,r}\left\{l\right\}(t)$, we have

$$\begin{aligned} &R_k^{\alpha,r}\left\{l\right\}(t)\,\left[R_k^{\alpha,r}\left\{m\right\}(t)\,R_k^{\alpha,r}\left\{n\,f\,g\right\}(t) + R_k^{\alpha,r}\left\{n\right\}(t)\,R_k^{\alpha,r}\left\{m\,f\,g\right\}(t)\right] \\ &\quad \geq R_k^{\alpha,r}\left\{l\right\}(t)\,\left[R_k^{\alpha,r}\left\{m\,f\right\}(t)\,R_k^{\alpha,r}\left\{n\,g\right\}(t) + R_k^{\alpha,r}\left\{n\,f\right\}(t)\,R_k^{\alpha,r}\left\{m\,g\right\}(t)\right]. \end{aligned} \tag{2.8}$$

Similarly replacing u, v by l, n and u, v by l, m, respectively, in (2.1), and then multiplying both sides of the resulting inequalities by $R_k^{\alpha,r}\left\{m\right\}(t)$ and $R_k^{\alpha,r}\left\{n\right\}(t)$ both of which are nonnegative under the given assumptions, respectively, we get the following inequalities:

$$\begin{aligned} &R_k^{\alpha,r}\left\{m\right\}(t)\,\left[R_k^{\alpha,r}\left\{l\right\}(t)\,R_k^{\alpha,r}\left\{n\,f\,g\right\}(t) + R_k^{\alpha,r}\left\{n\right\}(t)\,R_k^{\alpha,r}\left\{l\,f\,g\right\}(t)\right] \\ &\quad \geq R_k^{\alpha,r}\left\{m\right\}(t)\,\left[R_k^{\alpha,r}\left\{l\,f\right\}(t)\,R_k^{\alpha,r}\left\{n\,g\right\}(t) + R_k^{\alpha,r}\left\{n\,f\right\}(t)\,R_k^{\alpha,r}\left\{l\,g\right\}(t)\right] \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &R_k^{\alpha,r}\{n\}(t)\,\left[R_k^{\alpha,r}\{l\}(t)\,R_k^{\alpha,r}\{m\,f\,g\}(t)+R_k^{\alpha,r}\{m\}(t)\,R_k^{\alpha,r}\{l\,f\,g\}(t)\right] \\ &\quad\geq R_k^{\alpha,r}\{n\}(t)\,\left[R_k^{\alpha,r}\{l\,f\}(t)\,R_k^{\alpha,r}\{m\,g\}(t)+R_k^{\alpha,r}\{m\,f\}(t)\,R_k^{\alpha,r}\{l\,g\}(t)\right]. \end{aligned} \tag{2.10}$$

Finally, by adding (2.8), (2.9) and (2.10), sides by sides, we get the desired result (2.6). □

Theorem 2.4. *Let f and g be two continuous and synchronous functions on* $[0,\infty)$ and let $l, m, n : [0,\infty)\to[0,\infty)$ *be continuous functions. Then the following inequality holds true:*

$$\begin{aligned} &R_k^{\alpha,r}\{l\}(t)\Big[2R_k^{\alpha,r}\{m\}(t)\,R_k^{\beta,r}\{n\,f\,g\}(t)+R_k^{\alpha,r}\{n\}(t)\,R_k^{\beta,r}\{m\,f\,g\}(t) \\ &\qquad+R_k^{\beta,r}\{n\}(t)\,R_k^{\alpha,r}\{m\,f\,g\}(t)\Big]+R_k^{\alpha,r}\{l\,f\,g\}(t)\Big[R_k^{\alpha,r}\{m\}(t)\,R_k^{\beta,r}\{n\}(t) \\ &\qquad+R_k^{\alpha,r}\{n\}(t)\,R_k^{\beta,r}\{m\}(t)\Big] \\ &\quad\geq R_k^{\alpha,r}\{l\}(t)\Big[R_k^{\alpha,r}\{m\,f\}(t)\,R_k^{\beta,r}\{n\,g\}(t)+R_k^{\alpha,r}\{m\,g\}(t)\,R_k^{\beta,r}\{n\,f\}(t)\Big] \\ &\qquad+R_k^{\alpha,r}\{m\}(t)\Big[R_k^{\alpha,r}\{l\,f\}(t)\,R_k^{\beta,r}\{n\,g\}(t)+R_k^{\alpha,r}\{l\,g\}(t)\,R_k^{\beta,r}\{n\,f\}(t)\Big] \\ &\qquad+R_k^{\alpha,r}\{n\}(t)\Big[R_k^{\alpha,r}\{l\,f\}(t)\,R_k^{\beta,r}\{m\,g\}(t)+R_k^{\alpha,r}\{l\,g\}(t)\,R_k^{\beta,r}\{m\,f\}(t)\Big] \end{aligned} \tag{2.11}$$

for all $t>0, k>0,\ \alpha>0, \beta>0$ *and* $r\in\mathbb{R}\backslash\{-1\}$.

Proof. Setting $u=m$ and $v=n$ in (2.5), we have

$$\begin{aligned} &R_k^{\beta,r}\{n\}(t)\,R_k^{\alpha,r}\{m\,f\,g\}(t)+R_k^{\beta,r}\{n\,f\,g\}(t)\,R_k^{\alpha,r}\{m\}(t) \\ &\quad\geq R_k^{\beta,r}\{n\,g\}(t)\,R_k^{\alpha,r}\{m\,f\}(t)+R_k^{\beta,r}\{n\,f\}(t)\,R_k^{\alpha,r}\{m\,g\}(t). \end{aligned} \tag{2.12}$$

Multiplying both sides of (2.12) by $R_k^{\alpha,r}\{l\}(t)$, after a little simplification, we get

$$\begin{aligned} &R_k^{\alpha,r}\{l\}(t)\,\left[R_k^{\beta,r}\{n\}(t)\,R_k^{\alpha,r}\{m\,f\,g\}(t)+R_k^{\beta,r}\{n\,f\,g\}(t)\,R_k^{\alpha,r}\{m\}(t)\right] \\ &\quad\geq R_k^{\alpha,r}\{l\}(t)\,\left[R_k^{\beta,r}\{n\,g\}(t)\,R_k^{\alpha,r}\{m\,f\}(t)+R_k^{\beta,r}\{n\,f\}(t)\,R_k^{\alpha,r}\{m\,g\}(t)\right]. \end{aligned} \tag{2.13}$$

Now, by replacing u, v by l, n and u, v by l, m in (2.5), respectively, and then multiplying both sides of the resulting inequalities by $R_k^{\alpha,r}\{m\}(t)$ and $R_k^{\alpha,r}\{n\}(t)$, respectively, we get the following two inequalities

$$\begin{aligned} &R_k^{\alpha,r}\{m\}(t)\,\left[R_k^{\beta,r}\{n\}(t)\,R_k^{\alpha,r}\{l\,f\,g\}(t)+R_k^{\beta,r}\{n\,f\,g\}(t)\,R_k^{\alpha,r}\{l\}(t)\right] \\ &\quad\geq R_k^{\alpha,r}\{m\}(t)\,\left[R_k^{\beta,r}\{n\,g\}(t)\,R_k^{\alpha,r}\{l\,f\}(t)+R_k^{\beta,r}\{n\,f\}(t)\,R_k^{\alpha,r}\{l\,g\}(t)\right] \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} &R_k^{\alpha,r}\{n\}(t)\,\left[R_k^{\beta,r}\{m\}(t)\,R_k^{\alpha,r}\{l\,f\,g\}(t)+R_k^{\beta,r}\{m\,f\,g\}(t)\,R_k^{\alpha,r}\{l\}(t)\right] \\ &\quad\geq R_k^{\alpha,r}\{n\}(t)\,\left[R_k^{\beta,r}\{m\,g\}(t)\,R_k^{\alpha,r}\{l\,f\}(t)+R_k^{\beta,r}\{m\,f\}(t)\,R_k^{\alpha,r}\{l\,g\}(t)\right]. \end{aligned} \tag{2.15}$$

Finally we find that the Inequality (2.11) follows by adding the Inequalities (2.13), (2.14) and (2.15), sides by sides. □

3. Inequalities involving generalized fractional k-integral operator for bounded functions

In this section we obtain some new inequalities involving fractional k-integral operator in the case where the functions are bounded by integrable functions and not necessary increasing or decreasing as are the synchronous functions.

Theorem 3.1. *Let f be an integrable function on $[0,\infty)$ and $u,v:[0,\infty)\to[0,\infty)$ be continuous functions. Assume that:*

(H_1) *There exist two integrable functions φ_1,φ_2 on $[0,\infty)$ such that*

$$\varphi_1(t)\leq f(t)\leq\varphi_2(t)\quad \textit{for all}\quad t\in[0,\infty).$$

Then, for all $t>0, k>0$, $\alpha>0$ and $r\in\mathbb{R}\backslash\{-1\}$, we have

$$\begin{aligned} &R_k^{\alpha,r}\{u\,\varphi_2\}(t)\,R_k^{\alpha,r}\{v\,f\}(t)+R_k^{\alpha,r}\{u\,f\}(t)\,R_k^{\alpha,r}\{v\,\varphi_1\}(t)\\ &\quad\geq R_k^{\alpha,r}\{u\,\varphi_2\}(t)\,R_k^{\alpha,r}\{v\,\varphi_1\}(t)+R_k^{\alpha,r}\{u\,f\}(t)\,IR_k^{\alpha,r}\{v\,f\}(t).\end{aligned}\tag{3.1}$$

Proof. From (H_1), for all $\tau\geq0$, $\rho\geq0$, we have

$$(\varphi_2(\tau)-f(\tau))\,(f(\rho)-\varphi_1(\rho))\geq0.$$

Therefore

$$\varphi_2(\tau)f(\rho)+\varphi_1(\rho)f(\tau)\geq\varphi_1(\rho)\varphi_2(\tau)+f(\tau)f(\rho).\tag{3.2}$$

Multiplying both sides of (3.2) by $\dfrac{(1+r)^{1-\frac{\alpha}{k}}\,\left(t^{r+1}-\tau^{r+1}\right)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}u(\tau)$, $\tau\in(a,t)$ and integrating both sides with respect to τ on $(0,t)$, we obtain

$$\begin{aligned} &R_k^{\alpha,r}\{u\,\varphi_2\}(t)f(\rho)+R_k^{\alpha,r}\{u\,f\}(t)\varphi_1(\rho)\\ &\quad\geq R_k^{\alpha,r}\{u\,\varphi_2\}(t)\varphi_1(\rho)+R_k^{\alpha,r}\{u\,f\}(t)f(\rho).\end{aligned}\tag{3.3}$$

Multiplying both sides of (3.3) by $\dfrac{(1+r)^{1-\frac{\alpha}{k}}\,\left(t^{r+1}-\rho^{r+1}\right)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}v(\rho)$, $\rho\in(a,t)$, and integrating both sides with respect to ρ on $(0,t)$, we get inequality (3.1) as requested. This completes the proof. □

As special cases of Theorem 3.1, we obtain the following results:

Corollary 3.2. *Let f be an integrable function on $[0,\infty)$ satisfying $m\leq f(t)\leq M$ for all $t\in[0,\infty)$, $u,v:[0,\infty)\to[0,\infty)$ be continuous functions and $m,M\in\mathbb{R}$. Then for all $t>0,k>0$, $\alpha>0$ and $r\in\mathbb{R}\backslash\{-1\}$, we have*

$$\begin{aligned} &MR_k^{\alpha,r}\{u\}(t)\,R_k^{\alpha,r}\{v\,f\}(t)+mR_k^{\alpha,r}\{u\,f\}(t)\,R_k^{\alpha,r}\{v\}(t)\\ &\quad\geq mMR_k^{\alpha,r}\{u\}(t)\,R_k^{\alpha,r}\{v\}(t)+R_k^{\alpha,r}\{u\,f\}(t)\,R_k^{\alpha,r}\{v\,f\}(t).\end{aligned}$$

Corollary 3.3. *Let f be an integrable function on $[1,\infty)$ and $u,v:[0,\infty)\to[0,\infty)$ be continuous functions. Assume that there exists an integrable function $\varphi(t)$ on $[0,\infty)$ and a constant $M>0$ such that*

$$\varphi(t)-M\leq f(t)\leq\varphi(t)+M,$$

then for all $t>0,k>0$, $\alpha>0$ and $r\in\mathbb{R}\backslash\{-1\}$, we have

$$\begin{aligned} &R_k^{\alpha,r}\{u\,\varphi\}(t)\,R_k^{\alpha,r}\{v\,f\}(t)+R_k^{\alpha,r}\{u\,f\}(t)\,R_k^{\alpha,r}\{v\,\varphi\}(t)\\ &\quad+MR_k^{\alpha,r}\{u\}(t)\,R_k^{\alpha,r}\{v\,f\}(t)+MR_k^{\alpha,r}\{v\}(t)\,R_k^{\alpha,r}\{u\,\varphi\}(t)\\ &\quad+M^2R_k^{\alpha,r}\{u\}(t)\,R_k^{\alpha,r}\{v\}(t)\\ &\geq R_k^{\alpha,r}\{u\,\varphi\}(t)\,R_k^{\alpha,r}\{v\,\varphi\}(t)+R_k^{\alpha,r}\{u\,f\}(t)\,R_k^{\alpha,r}\{v\,f\}(t)\\ &\quad+MR_k^{\alpha,r}\{u\}(t)\,R_k^{\alpha,r}\{v\,\varphi\}(t)+MR_k^{\alpha,r}\{u\,f\}(t)\,R_k^{\alpha,r}\{v\}(t).\end{aligned}$$

Theorem 3.4. *Let f be an integrable function on $[0,\infty)$, $u,v:[0,\infty)\to[0,\infty)$ be continuous functions and $\theta_1,\theta_2>0$ satisfying $1/\theta_1+1/\theta_2=1$. Suppose that (H_1) holds. Then, for all $t>0, k>0$, $\alpha>0$ and $r\in\mathbb{R}\backslash\{-1\}$, we have*

$$\begin{aligned}\frac{1}{\theta_1}R_k^{\alpha,r}\{v\}(t)\,R_k^{\alpha,r}\left\{u\,(\varphi_2-f)^{\theta_1}\right\}(t)+\frac{1}{\theta_2}R_k^{\alpha,r}\{u\}(t)\,R_k^{\alpha,r}\left\{v\,(f-\varphi_1)^{\theta_2}\right\}(t)\\ +R_k^{\alpha,r}\{u\,\varphi_2\}(t)\,R_k^{\alpha,r}\{v\,\varphi_1\}(t)+R_k^{\alpha,r}\{u\,f\}(t)\,R_k^{\alpha,r}\{v\,f\}(t)\\ \geq R_k^{\alpha,r}\{u\,\varphi_2\}(t)\,R_k^{\alpha,r}\{v\,f\}(t)+R_k^{\alpha,r}\{u\,f\}(t)\,R_k^{\alpha,r}\{v\,\varphi_1\}(t).\end{aligned}\tag{3.4}$$

Proof. According to the well-known Young's inequality [27]

$$\frac{1}{\theta_1}x^{\theta_1}+\frac{1}{\theta_2}y^{\theta_2}\geq xy\qquad\forall x,y\geq 0,\quad\theta_1,\theta_2>0,\quad\frac{1}{\theta_1}+\frac{1}{\theta_2}=1,$$

setting $x=\varphi_2(\tau)-f(\tau)$ and $y=f(\rho)-\varphi_1(\rho)$, $\tau,\rho\geq 0$, we have

$$\frac{1}{\theta_1}(\varphi_2(\tau)-f(\tau))^{\theta_1}+\frac{1}{\theta_2}(f(\rho)-\varphi_1(\rho))^{\theta_2}\geq(\varphi_2(\tau)-f(\tau))(f(\rho)-\varphi_1(\rho)).\tag{3.5}$$

Multiplying both sides of (3.5) by

$$\frac{(1+)^{2-2\frac{\alpha}{k}}\,\left(t^{r+1}-\tau^{r+1}\right)^{\frac{\alpha}{k}-1}\left(t^{r+1}-\rho^{r+1}\right)^{\frac{\alpha}{k}-1}}{\left(k\Gamma_k(\alpha)\right)^2}u(\tau)v(\rho)$$

for $\tau,\rho\in(0,t)$, and integrating with respect to τ and ρ from 0 to t, we deduce the desired result in (3.4). □

Corollary 3.5. *Let f be an integrable function on $[0,\infty)$ satisfying $m\leq f(t)\leq M$ for all $t\in[0,\infty)$, $u,v:[0,\infty)\to[0,\infty)$ be continuous functions and $m,M\in\mathbb{R}$. Then for all $t>0,k>0$, $\alpha>0$ and $r\in\mathbb{R}\backslash\{-1\}$, we have*

$$\begin{aligned}(m+M)^2R_k^{\alpha,r}\{u\}(t)\,R_k^{\alpha,r}\{v\}(t)+2R_k^{\alpha,r}\{u\,f\}(t)\,R_k^{\alpha,r}\{v\,f\}(t)\\ +R_k^{\alpha,r}\left\{v\,f^2\right\}(t)\left(R_k^{\alpha,r}\{u\}(t)+R_k^{\alpha,r}\{v\}(t)\right)\\ \geq 2(m+M)\left(R_k^{\alpha,r}\{u\,f\}(t)\,R_k^{\alpha,r}\{v\}(t)+R_k^{\alpha,r}\{u\}(t)\,R_k^{\alpha,r}\{v\,f\}(t)\right).\end{aligned}$$

Theorem 3.6. *Let f be an integrable function on $[0,\infty)$, $u,v:[0,\infty)\to[0,\infty)$ be continuous functions and $\theta_1,\theta_2>0$ satisfying $\theta_1+\theta_2=1$. In addition, suppose that (H_1) holds. Then, for all $t>0,k>0$, $\alpha>0$ and $r\in\mathbb{R}\backslash\{-1\}$, we have*

$$\begin{aligned}\theta_1R_k^{\alpha,r}\{u\,\varphi_2\}(t)\,R_k^{\alpha,r}\{v\}(t)+\theta_2R_k^{\alpha,r}\{u\}(t)\,R_k^{\alpha,r}\{v\,f\}(t)\\ \geq\theta_1R_k^{\alpha,r}\{u\,f\}(t)\,R_k^{\alpha,r}\{v\}(t)+\theta_2R_k^{\alpha,r}\{u\}(t)\,R_k^{\alpha,r}\{v\,\varphi_1\}(t)\\ +R_k^{\alpha,r}\left\{u\,(\varphi_2-f)^{\theta_1}\right\}(t)\,R_k^{\alpha,r}\left\{v\,(f-\varphi_1)^{\theta_2}\right\}(t).\end{aligned}\tag{3.6}$$

Proof. From the well-known Weighted AM-GM inequality [27]

$$\theta_1x+\theta_2y\geq x^{\theta_1}y^{\theta_2}\qquad\forall x,y\geq 0,\quad\theta_1,\theta_2>0,\quad\theta_1+\theta_2=1,$$

by setting $x=\varphi_2(\tau)-f(\tau)$ and $y=f(\rho)-\varphi_1(\rho)$, $\tau,\rho>1$, we have

$$\theta_1(\varphi_2(\tau)-f(\tau))+\theta_2(f(\rho)-\varphi_1(\rho))\geq(\varphi_2(\tau)-f(\tau))^{\theta_1}(f(\rho)-\varphi_1(\rho))^{\theta_2}.\tag{3.7}$$

Multiplying both sides of (3.7) by

$$\frac{(1+)^{2-2\frac{\alpha}{k}}\,\left(t^{r+1}-\tau^{r+1}\right)^{\frac{\alpha}{k}-1}\left(t^{r+1}-\rho^{r+1}\right)^{\frac{\alpha}{k}-1}}{\left(k\Gamma_k(\alpha)\right)^2}u(\tau)v(\rho)$$

for $\tau,\rho\in(0,t)$, and integrating with respect to τ and ρ from 0 to t, we deduce inequality (3.6). □

Corollary 3.7. *Let f be an integrable function on $[0,\infty)$ satisfying $m \le f(t) \le M$ for all $t \in [0,\infty)$, $u, v : [0,\infty) \to [0,\infty)$ be continuous functions and $m, M \in \mathbb{R}$. Then for all $t > 0, k > 0$, $\alpha > 0$ and $r \in \mathbb{R}\backslash\{-1\}$, we have*

$$\begin{aligned}(M-m)&R_k^{\alpha,r}\{u\}(t)\,R_k^{\alpha,r}\{v\}(t)+R_k^{\alpha,r}\{u\}(t)\,R_k^{\alpha,r}\{v\,f\}(t)\\ &\ge R_k^{\alpha,r}\{u\,f\}(t)\,R_k^{\alpha,r}\{v\}(t)+2R_k^{\alpha,r}\left\{u\sqrt{M-f}\right\}(t)\,R_k^{\alpha,r}\left\{v\sqrt{f-m}\right\}(t).\end{aligned}$$

Lemma 3.8 ([24])**.** *Assume that $a \ge 0$, $p \ge q \ge 0$ and $p \ne 0$. Then*

$$a^{\frac{q}{p}} \le \left(\frac{q}{p}k^{\frac{q-p}{p}}a+\frac{p-q}{p}k^{\frac{q}{p}}\right) \ \textit{for any } k>0.$$

Theorem 3.9. *Let f be an integrable function on $[0,\infty)$, $u : [0,\infty) \to [0,\infty)$ be a continuous function and constants $p \ge q \ge 0$, $p \ne 0$. In addition, assume that (H_1) holds. Then for all $t > 0, k > 0$, $\alpha > 0$ and $r \in \mathbb{R}\backslash\{-1\}$, the following two inequalities hold:*

$$\begin{aligned}(i)\quad & R_k^{\alpha,r}\left\{u\,(\varphi_2-f)^{\frac{q}{p}}\right\}(t)+\frac{q}{p}k^{\frac{q-p}{p}}R_k^{\alpha,r}\{u\,f\}(t)\\ &\qquad \le \frac{q}{p}k^{\frac{q-p}{p}}R_k^{\alpha,r}\{u\,\varphi_2\}(t)+\frac{p-q}{p}k^{\frac{q}{p}}R_k^{\alpha,r}\{u\}(t),\\ (ii)\quad & R_k^{\alpha,r}\left\{u\,(f-\varphi_1)^{\frac{q}{p}}\right\}(t)+\frac{q}{p}k^{\frac{q-p}{p}}R_k^{\alpha,r}\{u\,\varphi_1\}(t)\\ &\qquad \le \frac{q}{p}k^{\frac{q-p}{p}}R_k^{\alpha,r}\{u\,f\}(t)+\frac{p-q}{p}k^{\frac{q}{p}}R_k^{\alpha,r}\{u\}(t).\end{aligned} \tag{3.8}$$

Proof. By condition (H_1) and Lemma 3.8, for $p \ge q \ge 0$, $p \ne 0$, it follows that

$$(\varphi_2(\tau)-f(\tau))^{\frac{q}{p}} \le \frac{q}{p}k^{\frac{q-p}{p}}(\varphi_2(\tau)-f(\tau))+\frac{p-q}{p}k^{\frac{q}{p}}, \tag{3.9}$$

for any $k > 0$. Multiplying both sides of (3.9) by $\frac{(1+)^{1-\frac{\alpha}{k}}\ (t^{r+1}-\tau^{r+1})^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}u(\tau)$, $\tau \in (0,t)$, and integrating the resulting identity with respect to τ from 0 to t, one has inequality (i). Inequality (ii) is proved by setting $a = f(\tau) - \varphi_1(\tau)$ in Lemma 3.8. □

Corollary 3.10. *Let f be an integrable function on $[0,\infty)$ satisfying $m \le f(t) \le M$, for all $t \in [0,\infty)$, $u, v : [0,\infty) \to [0,\infty)$ be continuous functions and $m, M \in \mathbb{R}$. Then for all $t > 0, k > 0$, $\alpha > 0$ and $r \in \mathbb{R}\backslash\{-1\}$, we have*

$$(i)\quad 2R_k^{\alpha,r}\left\{u\sqrt{M-f}\right\}(t)+R_k^{\alpha,r}\{u\,f\}(t) \le (M+1)R_k^{\alpha,r}\{u\}(t),$$

$$(ii)\quad 2R_k^{\alpha,r}\left\{u\sqrt{f-m}\right\}(t)+(m-1)R_k^{\alpha,r}\{u\}(t) \le R_k^{\alpha,r}\{u\,f\}(t).$$

Theorem 3.11. *Let f and g be two integrable functions on $[0,\infty)$ and $u, v : [0,\infty) \to [0,\infty)$ be continuous functions. Suppose that (H_1) holds and moreover we assume that:*

(H_2) *There exist ψ_1 and ψ_2 integrable functions on $[0,\infty)$ such that*

$$\psi_1(t) \le g(t) \le \psi_2(t) \quad \textit{for all} \quad t \in [0,\infty).$$

Then, for all $t > 0, k > 0$, $a \ge 0, \alpha > 0$ and $r \in \mathbb{R}\backslash\{-1\}$, the following inequalities hold:

$$\begin{aligned}(i)\quad & R_k^{\alpha,r}\{u\,\varphi_2\}(t)\,R_k^{\alpha,r}\{v\,g\}(t)+R_k^{\alpha,r}\{u\,f\}(t)\,R_k^{\alpha,r}\{v\,\psi_1\}(t)\\ &\qquad \ge R_k^{\alpha,r}\{u\,\varphi_2\}(t)\,R_k^{\alpha,r}\{v\,\psi_1\}(t)+R_k^{\alpha,r}\{u\,f\}(t)\,R_k^{\alpha,r}\{v\,g\}(t),\end{aligned}$$

$$(ii)\quad R_k^{\alpha,r}\left\{u\,\psi_2\right\}(t)\,R_k^{\alpha,r}\left\{v\,f\right\}(t)+R_k^{\alpha,r}\left\{u\,g\right\}(t)\,R_k^{\alpha,r}\left\{v\,\varphi_1\right\}(t)$$
$$\geq R_k^{\alpha,r}\left\{u\,\psi_2\right\}(t)\,R_k^{\alpha,r}\left\{v\,\varphi_1\right\}(t)+R_k^{\alpha,r}\left\{u\,g\right\}(t)\,R_k^{\alpha,r}\left\{v\,f\right\}(t),$$
$$(iii)\quad R_k^{\alpha,r}\left\{u\,\varphi_2\right\}(t)\,R_k^{\alpha,r}\left\{v\,\psi_2\right\}(t)+R_k^{\alpha,r}\left\{u\,f\right\}(t)\,R_k^{\alpha,r}\left\{v\,g\right\}(t)$$
$$\geq R_k^{\alpha,r}\left\{u\,\varphi_2\right\}(t)\,R_k^{\alpha,r}\left\{v\,g\right\}(t)+R_k^{\alpha,r}\left\{u\,f\right\}(t)\,R_k^{\alpha,r}\left\{v\,\psi_2\right\}(t),$$
$$(iv)\quad R_k^{\alpha,r}\left\{u\,\varphi_1\right\}(t)\,R_k^{\alpha,r}\left\{v\,\psi_1\right\}(t)+R_k^{\alpha,r}\left\{u\,f\right\}(t)\,R_k^{\alpha,r}\left\{v\,g\right\}(t)$$
$$\geq R_k^{\alpha,r}\left\{u\,\varphi_1\right\}(t)\,R_k^{\alpha,r}\left\{v\,g\right\}(t)+R_k^{\alpha,r}\left\{u\,f\right\}(t)\,R_k^{\alpha,r}\left\{v\,\psi_1\right\}(t).$$

Proof. To prove (i), from (H_1) and (H_2), we have for $t\in[0,\infty)$ that

$$(\varphi_2(\tau)-f(\tau))\,(g(\rho)-\psi_1(\rho))\geq 0.$$

Therefore

$$\varphi_2(\tau)g(\rho)+\psi_1(\rho)f(\tau)\geq\psi_1(\rho)\varphi_2(\tau)+f(\tau)g(\rho). \tag{3.10}$$

Multiplying both sides of (3.10) by $\dfrac{(1+)^{1-\frac{\alpha}{k}}\,(t^{r+1}-\tau^{r+1})^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}u(\tau)$, $\tau\in(0,t)$ and integrating both sides with respect to τ on $(0,t)$, we obtain

$$g(\rho)R_k^{\alpha,r}\left\{u\,\varphi_2\right\}(t)+\psi_1(\rho)R_k^{\alpha,r}\left\{u\,f\right\}(t)\geq\psi_1(\rho)R_k^{\alpha,r}\left\{u\,\varphi_2\right\}(t)+g(\rho)R_k^{\alpha,r}\left\{u\,f\right\}(t). \tag{3.11}$$

Multiplying both sides of (3.11) by $\dfrac{(1+)^{1-\frac{\alpha}{k}}\,(t^{r+1}-\rho^{r+1})^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}v(\rho)$, $\rho\in(0,t)$, and integrating both sides with respect to ρ on $(0,t)$, we get the desired inequality (i).

To prove (ii)-(iv), we use the following inequalities
(ii) $(\psi_2(\tau)-g(\tau))\,(f(\rho)-\varphi_1(\rho))\geq 0$,
(iii) $(\varphi_2(\tau)-f(\tau))\,(g(\rho)-\psi_2(\rho))\leq 0$,
(iv) $(\varphi_1(\tau)-f(\tau))\,(g(\rho)-\psi_1(\rho))\leq 0$. □

Theorem 3.12. *Let f and g be two integrable functions on $[0,\infty)$, $u,v:[0,\infty)\to[0,\infty)$ be continuous functions and $\theta_1,\theta_2>0$ satisfying $1/\theta_1+1/\theta_2=1$. Suppose that (H_1) and (H_2) hold. Then, for all $t>0,k>0$, $\alpha>0$ and $r\in\mathbb{R}\backslash\{-1\}$, the following inequalities hold:*

$$(i)\frac{1}{\theta_1}R_k^{\alpha,r}\left\{u\,(\varphi_2-f)^{\theta_1}\right\}(t)\,R_k^{\alpha,r}\left\{v\right\}(t)+\frac{1}{\theta_2}R_k^{\alpha,r}\left\{v\,(\psi_2-g)^{\theta_2}\right\}(t)\,R_k^{\alpha,r}\left\{u\right\}(t)$$
$$+R_k^{\alpha,r}\left\{u\,\varphi_2\right\}(t)\,R_k^{\alpha,r}\left\{v\,g\right\}(t)+R_k^{\alpha,r}\left\{u\,f\right\}(t)\,R_k^{\alpha,r}\left\{v\,\psi_2\right\}(t)$$
$$\geq R_k^{\alpha,r}\left\{u\,\varphi_2\right\}(t)\,R_k^{\alpha,r}\left\{v\,\psi_2\right\}(t)+R_k^{\alpha,r}\left\{u\,f\right\}(t)\,R_k^{\alpha,r}\left\{v\,g\right\}(t),$$

$$(ii)\frac{1}{\theta_1}R_k^{\alpha,r}\left\{u\,(\varphi_2-f)^{\theta_1}\right\}(t)\,R_k^{\alpha,r}\left\{v\,(\psi_2-g)^{\theta_1}\right\}(t)$$
$$+\frac{1}{\theta_2}R_k^{\alpha,r}\left\{u\,(\psi_2-g)^{\theta_2}\right\}(t)\,R_k^{\alpha,r}\left\{v\,(\varphi_2-f)^{\theta_2}\right\}(t)$$
$$\geq R_k^{\alpha,r}\left\{u\,(\varphi_2-f)\,(\psi_2-g)\right\}(t)\,R_k^{\alpha,r}\left\{v\,(\psi_2-g)\,(\varphi_2-f)\right\}(t),$$

$$(iii)\frac{1}{\theta_1}R_k^{\alpha,r}\left\{u\,(f-\varphi_1)^{\theta_1}\right\}(t)\,R_k^{\alpha,r}\left\{v\right\}(t)+\frac{1}{\theta_2}R_k^{\alpha,r}\left\{v\,(g-\psi_1)^{\theta_2}\right\}(t)\,R_k^{\alpha,r}\left\{u\right\}(t)$$
$$+R_k^{\alpha,r}\left\{u\,f\right\}(t)\,R_k^{\alpha,r}\left\{v\,\psi_1\right\}(t)+R_k^{\alpha,r}\left\{u\,\varphi_1\right\}(t)\,R_k^{\alpha,r}\left\{v\,g\right\}(t)$$
$$\geq R_k^{\alpha,r}\left\{u\,f\right\}(t)\,R_k^{\alpha,r}\left\{v\,g\right\}(t)+R_k^{\alpha,r}\left\{u\,\varphi_1\right\}(t)\,R_k^{\alpha,r}\left\{v\,\psi_1\right\}(t),$$

$$(iv)\frac{1}{\theta_1}R_k^{\alpha,r}\left\{u\,(f-\varphi_1)^{\theta_1}\right\}(t)\,R_k^{\alpha,r}\left\{v\,(g-\psi_1)^{\theta_1}\right\}(t)$$

$$
\begin{aligned}
&+\frac{1}{\theta_2}R_k^{\alpha,r}\left\{u\left(g-\psi_1\right)^{\theta_2}\right\}(t)\,R_k^{\alpha,r}\left\{v\left(f-\varphi_1\right)^{\theta_2}\right\}(t)\\
&\geq R_k^{\alpha,r}\left\{u\left(f-\varphi_1\right)\left(g-\psi_1\right)\right\}(t)\,R_k^{\alpha,r}\left\{v\left(g-\psi_1\right)\left(f-\varphi_1\right)\right\}(t).
\end{aligned}
$$

Proof. The inequalities (i)-(iv) can be proved by choosing the parameters in the Young inequality [27]:

(i) $x=\varphi_2(\tau)-f(\tau),\quad y=\psi_2(\rho)-g(\rho),$

(ii) $x=(\varphi_2(\tau)-f(\tau))(\psi_2(\rho)-g(\rho)),\quad y=(\psi_2(\tau)-g(\tau))(\varphi_2(\rho)-f(\rho)),$

(iii) $x=f(\tau)-\varphi_1(\tau),\quad y=g(\rho)-\psi_1(\rho),$

(iv) $x=(f(\tau)-\varphi_1(\tau))(g(\rho)-\psi_1(\rho)),\quad y=(g(\tau)-\psi_1(\tau))(f(\rho)-\varphi_1(\rho)).$

□

Theorem 3.13. *Let f and g be two integrable functions on $[0,\infty)$, $u,v:[0,\infty)\to[0,\infty)$ be continuous functions and $\theta_1,\theta_2>0$ satisfying $\theta_1+\theta_2=1$. Suppose that (H_1) and (H_2) hold. Then, for all $t>0,k>0$, $\alpha>0$ and $r\in\mathbb{R}\backslash\{-1\}$, the following inequalities hold:*

(i)
$$
\begin{aligned}
&\theta_1R_k^{\alpha,r}\{u\,\varphi_2\}(t)\,R_k^{\alpha,r}\{v\}(t)+\theta_2R_k^{\alpha,r}\{v\,\psi_2\}(t)\,R_k^{\alpha,r}\{u\}(t)\\
&\geq\theta_1R_k^{\alpha,r}\{u\,f\}(t)\,R_k^{\alpha,r}\{v\}(t)+\theta_2R_k^{\alpha,r}\{v\,g\}(t)\,R_k^{\alpha,r}\{u\}(t)\\
&\quad+R_k^{\alpha,r}\left\{u\left(\varphi_2-f\right)^{\theta_1}\right\}(t)\,R_k^{\alpha,r}\left\{v\left(\psi_2-g\right)^{\theta_2}\right\}(t),
\end{aligned}
$$

(ii)
$$
\begin{aligned}
&\theta_1R_k^{\alpha,r}\{u\,\varphi_2\}(t)\,R_k^{\alpha,r}\{v\,\psi_2\}(t)+\theta_1R_k^{\alpha,r}\{u\,f\}(t)\,R_k^{\alpha,r}\{v\,g\}(t)\\
&\quad+\theta_2R_k^{\alpha,r}\{u\,\psi_2\}(t)\,R_k^{\alpha,r}\{v\,\varphi_2\}(t)+\theta_2R_k^{\alpha,r}\{u\,g\}(t)\,R_k^{\alpha,r}\{v\,f\}(t)\\
&\geq\theta_1R_k^{\alpha,r}\{u\,\varphi_2\}(t)\,R_k^{\alpha,r}\{v\,g\}(t)+\theta_1R_k^{\alpha,r}\{u\,f\}(t)\,R_k^{\alpha,r}\{v\,\psi_2\}(t)\\
&\quad+\theta_2R_k^{\alpha,r}\{u\,\psi_2\}(t)\,R_k^{\alpha,r}\{v\,f\}(t)+\theta_2R_k^{\alpha,r}\{u\,g\}(t)\,R_k^{\alpha,r}\{v\,\varphi_2\}(t)\\
&\quad+R_k^{\alpha,r}\left\{u\left(\varphi_2-f\right)^{\theta_1}\left(\psi_2-g\right)^{\theta2}\right\}(t)\,R_k^{\alpha,r}\left\{v\left(\psi_2-g\right)^{\theta_1}\left(\varphi_2-f\right)^{\theta_2}\right\}(t),
\end{aligned}
$$

(iii)
$$
\begin{aligned}
&\theta_1R_k^{\alpha,r}\{u\,f\}(t)\,R_k^{\alpha,r}\{v\}(t)+\theta_2R_k^{\alpha,r}\{v\,g\}(t)\,R_k^{\alpha,r}\{u\}(t)\\
&\geq\theta_1R_k^{\alpha,r}\{u\,\varphi_1\}(t)\,R_k^{\alpha,r}\{v\}(t)+\theta_2R_k^{\alpha,r}\{v\,\psi_1\}(t)\,R_k^{\alpha,r}\{u\}(t)\\
&\quad+R_k^{\alpha,r}\left\{u\left(f-\varphi_1\right)^{\theta_1}\right\}(t)\,R_k^{\alpha,r}\left\{v\left(g-\psi_1\right)^{\theta_2}\right\}(t),
\end{aligned}
$$

(iv)
$$
\begin{aligned}
&\theta_1R_k^{\alpha,r}\{u\,f\}(t)\,R_k^{\alpha,r}\{v\,g\}(t)+\theta_1R_k^{\alpha,r}\{u\,\varphi_1\}(t)\,R_k^{\alpha,r}\{v\,\psi_1\}(t)\\
&\quad+\theta_2R_k^{\alpha,r}\{u\,g\}(t)\,R_k^{\alpha,r}\{v\,f\}(t)+\theta_2R_k^{\alpha,r}\{u\,\psi_1\}(t)\,R_k^{\alpha,r}\{v\,\varphi_1\}(t)\\
&\geq\theta_1R_k^{\alpha,r}\{u\,f\}(t)\,R_k^{\alpha,r}\{v\,\psi_1\}(t)+\theta_1R_k^{\alpha,r}\{u\,\varphi_1\}(t)\,R_k^{\alpha,r}\{v\,g\}(t)\\
&\quad+\theta_2R_k^{\alpha,r}\{u\,g\}(t)\,R_k^{\alpha,r}\{v\,\varphi_1\}(t)+\theta_2R_k^{\alpha,r}\{u\,\psi_1\}(t)\,R_k^{\alpha,r}\{v\,f\}(t)\\
&\quad+R_k^{\alpha,r}\left\{u\left(f-\varphi_1\right)^{\theta_1}\left(g-\psi_1\right)^{\theta2}\right\}(t)\,R_k^{\alpha,r}\left\{v\left(g-\psi_1\right)^{\theta_1}\left(f-\varphi_1\right)^{\theta_2}\right\}(t).
\end{aligned}
$$

Proof. The inequalities (i)-(iv) can be proved by choosing the parameters in the Weighted AM-GM [27]:

(i) $x=\varphi_2(\tau)-f(\tau),\quad y=\psi_2(\rho)-g(\rho),$

(ii) $x=(\varphi_2(\tau)-f(\tau))(\psi_2(\rho)-g(\rho)),\quad y=(\psi_2(\tau)-g(\tau))(\varphi_2(\rho)-f(\rho)),$

(iii) $x=f(\tau)-\varphi_1(\tau),\quad y=g(\rho)-\psi_1(\rho),$

(iv) $x=(f(\tau)-\varphi_1(\tau))(g(\rho)-\psi_1(\rho)),\quad y=(g(\tau)-\psi_1(\tau))(f(\rho)-\varphi_1(\rho)).$

□

Theorem 3.14. *Let f and g be two integrable functions on $[0,\infty)$, $u,v:[0,\infty)\to[0,\infty)$ be continuous functions and constants $p\geq q\geq 0$, $p\neq 0$. Assume that (H_1) and (H_2) hold. Then, for all $t>0, k>0$, $\alpha>0$ and $r\in\mathbb{R}\backslash\{-1\}$, the following inequalities hold:*

$$(i)\ R_k^{\alpha,r}\left\{u\,(\varphi_2-f)^{\frac{q}{p}}\,(\psi_2-g)^{\frac{q}{p}}\right\}(t)+\frac{q}{p}k^{\frac{q-p}{p}}R_k^{\alpha,r}\{u\,\varphi_2\,g\}(t)+\frac{q}{p}k^{\frac{q-p}{p}}R_k^{\alpha,r}\{u\,f\,\psi_2\}(t)$$
$$\leq\frac{q}{p}k^{\frac{q-p}{p}}R_k^{\alpha,r}\{u\,\varphi_2\,\psi_2\}(t)+\frac{q}{p}k^{\frac{q-p}{p}}R_k^{\alpha,r}\{u\,f\,g\}(t)+\frac{p-q}{p}k^{\frac{q}{p}}R_k^{\alpha,r}\{u\}(t),$$

$$(ii)\ R_k^{\alpha,r}\left\{u\,(\varphi_2-f)^{\frac{q}{p}}\right\}(t)\,R_k^{\alpha,r}\left\{v\,(\psi_2-g)^{\frac{q}{p}}\right\}(t)$$
$$+\frac{q}{p}k^{\frac{q-p}{p}}R_k^{\alpha,r}\{u\,\varphi_2\}(t)\,R_k^{\alpha,r}\{v\,g\}(t)+\frac{q}{p}k^{\frac{q-p}{p}}R_k^{\alpha,r}\{u\,f\}(t)R_k^{\alpha,r}\{v\,\psi_2\}(t)$$
$$\leq\frac{q}{p}k^{\frac{q-p}{p}}R_k^{\alpha,r}\{u\,\varphi_2\}(t)\,R_k^{\alpha,r}\{v\,\psi_2\}(t)+\frac{q}{p}k^{\frac{q-p}{p}}R_k^{\alpha,r}\{u\,f\}(t)\,R_k^{\alpha,r}\{v\,g\}(t)g(t)$$
$$+\frac{p-q}{p}k^{\frac{q}{p}}R_k^{\alpha,r}\{u\}(t)\,R_k^{\alpha,r}\{v\}(t),$$

$$(iii)\ R_k^{\alpha,r}\left\{u\,(f-\varphi_1)^{\frac{q}{p}}\,(g-\psi_1)^{\frac{q}{p}}\right\}(t)+\frac{q}{p}k^{\frac{q-p}{p}}R_k^{\alpha,r}\{u\,\psi_1\,f\}(t)+\frac{q}{p}k^{\frac{q-p}{p}}R_k^{\alpha,r}\{u\,\varphi_1\,g\}(t)$$
$$\leq\frac{q}{p}k^{\frac{q-p}{p}}I_q^{\eta,\mu,\beta}\{u\,f\,g\}(t)+\frac{q}{p}k^{\frac{q-p}{p}}R_k^{\alpha,r}\{u\,\varphi_1\,\psi_1\}(t)+\frac{p-q}{p}k^{\frac{q}{p}}R_k^{\alpha,r}\{u\}(t),$$

$$(iv)\ R_k^{\alpha,r}\left\{u\,(f-\varphi_1)^{\frac{q}{p}}\right\}(t)\,R_k^{\alpha,r}\left\{v\,(g-\psi_1)^{\frac{q}{p}}\right\}(t)$$
$$+\frac{q}{p}k^{\frac{q-p}{p}}R_k^{\alpha,r}\{u\,f\}(t)\,R_k^{\alpha,r}\{v\,\psi_1\}(t)+\frac{q}{p}k^{\frac{q-p}{p}}R_k^{\alpha,r}\{u\,\varphi_1\}(t)R_k^{\alpha,r}\{v\,g\}(t)$$
$$\leq\frac{q}{p}k^{\frac{q-p}{p}}R_k^{\alpha,r}\{u\,f\}(t)\,R_k^{\alpha,r}\{v\,g\}(t)+\frac{q}{p}k^{\frac{q-p}{p}}R_k^{\alpha,r}\{u\,\varphi_1\}(t)\,R_k^{\alpha,r}\{v\,\psi_1\}(t)$$
$$+\frac{p-q}{p}k^{\frac{q}{p}}R_k^{\alpha,r}\{u\}(t)\,R_k^{\alpha,r}\{v\}(t).$$

Proof. The inequalities (i)-(iv) can be proved by choosing the parameters in the Lemma 3.8:
(i) $a=(\varphi_2(\tau)-f(\tau))(\psi_2(\tau)-g(\tau))$,
(ii) $a=(\varphi_2(\tau)-f(\tau))(\psi_2(\rho)-g(\rho))$,
(iii) $a=(f(\tau)-\varphi_1(\tau))(g(\tau)-\psi_1(\tau))$,
(iv) $a=(f(\tau)-\varphi_1(\tau))(g(\rho)-\psi_1(\rho))$. □

Remark 3.15. It may be noted that the inequalities (2.6) and (2.11) in Theorems 2.3 and 2.4, respectively, are reversed if the functions are asynchronous on $[0,\infty)$. The special case of (2.11) in Theorem 2.4 when $\beta=\delta$, $\eta=\zeta$ and $\mu=\nu$ is easily seen to yield the inequality (2.6) in Theorem 2.3.

Acknowledgement

This project was supported by the Deanship of Scientific Research at Prince Sattam Bin Abdulaziz University under the research project 2015/01/3883. We also thankful to Editor and Referee for their comments.

References

[1] P. Agarwal, J. Choi, *Certain fractional integral inequalities associated with Pathway fractional integral operators*, Bull. Korean Math. Soc., **53** (2016), 181–193. 1

[2] P. Agarwal, S. S. Dragomir, J. Park, S. Jain, *q-Integral inequalities associated with some fractional q-integral operators*, J. Inequal. Appl., **2015** (2015), 13 pages. 1

[3] P. Agarwal, J. Tariboon, S. K. Ntouyas, *Some generalized Riemann-Liouville k-fractional integral inequalities*, J. Inequal. Appl., **2016** (2016), 13 pages. 1

[4] A. Alsaedi, D. Baleanu, S. Etemad, S. Rezapour, *On coupled systems of time-fractional differential problems by using a new fractional derivative*, J. Funct. Spaces, **2016** (2016), 8 pages. 1

[5] G. A. Anastassiou, *Advances on Fractional Inequalities*, Springer Briefs in Mathematics, Springer, New York, (2011). 1

[6] A. Atangana, *On the new fractional derivative and application to nonlinear Fisher's reaction-diffusion equation*, Appl. Math. Comput., **273** (2016), 948–956. 1

[7] A. Atangana, D. Baleanu, *CaputoFabrizio derivative applied to groundwater flow with in a confined aquifer*, J. Eng. Mech., **2016** (2016). 1

[8] A. Atangana, D. Baleanu, *New fractional derivatives with non local and non-singular kernel:theory and application to heat transfer model*, arXiv preprint arXiv:1602.03408, (2016). 1

[9] S. Belarbi, Z. Dahmani, *On some new fractional integral inequalities*, JIPAM. J. Inequal. Pure Appl. Math., **10** (2009), 5 pages. 1

[10] M. Caputo, M. Fabrizio, *A new definition of fractional derivative with-out singular kernel*, Progr. Fract. Differ. Appl., **1** (2015), 1–13. 1

[11] M. Caputo, M. Fabrizio, *Applications of new time and spatial fractional derivatives with exponential kernels*, Progr. Fract. Differ. Appl., **2** (2016), 1–11. 1

[12] P. L. Chebyshev, *Sur les expressions approximatives des integrales definies par les autres prises entre les mêmes limites*, Proc. Math. Soc. Charkov., **2** (1882), 93–98. 1, 1

[13] J. Choi, P. Agarwal, *Some new Saigo type fractional integral inequalities and their q-analogues*, Abstr. Appl. Anal., **2014** (2014), 11 pages. 1

[14] J. Choi, P. Agarwal, *Certain fractional integral inequalities involving hypergeometric operators*, East Asian Math. J., **30** (2014), 283–291. 1

[15] J. Choi, P. Agarwal, *Certain new pathway type fractional integral inequalities*, Honam Math. J., **36** (2014), 455–465. 1

[16] J. Choi, P. Agarwal, *Certain integral transform and fractional integral formulas for the generalized Gauss hypergeometric functions*, Abstr. Appl. Anal., **2014** (2014), 7 pages. 1

[17] Z. Dahmani, O. Mechouar, S. Brahami, *Certain inequalities related to the Chebyshev's functional involving a Riemann-Liouville operator*, Bull. Math. Anal. Appl., **3** (2011), 38–44. 1

[18] R. Diaz, E. Pariguan, *On hypergeometric functions and pochhammerk-symbol*, Divulg. Mat., **15** (2007), 179–192. 1.1

[19] E. F. Doungmo Goufo, M. K. Pene, J. N. Mwambakana, *Duplication in a model of rock fracture with fractional derivative without singular kernel*, Open. Math., **13** (2015), 839–846. 1

[20] S. S. Dragomir, *Some integral inequalities of Gruss type*, Indian J. Pure Appl. Math., **31** (2000), 397–415. 1

[21] J. F. Gómez-Aguilar, H. Yépez-Martínez, C. Calderón-Ramón, I. Cruz-Orduña, R. F. Escobar-Jiménez, V. H. Olivares-peregrinoictor, *Modeling of a mass-spring-damper system by fractional derivatives with and without a singular kernel*, Entropy, **17** (2015), 6289–6303. 1

[22] J. Hristov, *Transient heat diffusion with a non-singular fading memory: from the cattaneo constitutive equation with Jeffrey's kernel to the CaputoFabriziotime-fractional derivative*, Therm. Sci., **2016** (2016), 19 pages. 1

[23] S. Jain, P. Agarwal, B. Ahmad, S. K. Q. Al-Omari, *Certain recent fractional integral inequalities associated with the hypergeometric operators*, J. King Saud Univ. Sci., **28** (2016), 82–86. 1

[24] F. Jiang, F. Meng, *Explicit bounds on some new nonlinear integral inequalities with delay*, J. Comput. Appl. Math., **205** (2007), 479–486. 3.8

[25] V. Lakshmikantham, A. S. Vatsala, *Theory of fractional differential inequalities and applications*, Commun. Appl. Anal., **11** (2007), 395–402. 1

[26] J. Losada, J. J. Nieto, *Properties of a new fractional derivative without singular kernel*, Progr. Fract. Differ. Appl., **1** (2015), 87–92. 1

[27] D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Berlin, (1970). 1, 3, 3, 3, 3

[28] S. Mubeen, G. M. Habibullah, *k-fractional integrals and application*, Int. J. Contemp. Math. Sci., **7** (2012), 89–94. 1.3

[29] H. Öğünmez, U. M. Özkan, *Fractional quantum integral inequalities*, J. Inequal. Appl., **2011** (2011), 7 pages. 1

[30] A. M. Ostrowski, *On an integral inequality*, Aequations Math., **4** (1970), 358–373. 1

[31] M. Z. Sarikaya, Z. Dahmani, M. E. Kiris, F. Ahmad,$(k; s)$ *Riemann-Liouville fractional integral and applications*, Hacet. J. Math. Stat., (in press). 1, 1.4

[32] E. Set, M. Tomar, M. Z. Sarikaya, *On generalized Grüss type inequalities for k-fractional integrals*, Appl. Math. Comput., **269** (2015), 29–34. 1, 1.3

[33] W. T. Sulaiman, *Some new fractional integral inequalities*, J. Math. Anal., **2** (2011), 23–28. 1

[34] G. Wang, P. Agarwal, M. Chand, *Certain Grüss type inequalities involving the generalized fractional integral operator*, J. Inequal. Appl., **2014** (2014), 8 pages. 1