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Nonexistence of solutions to a fractional differential boundary value problem

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Abstract

We investigate new results about Lyapunov-type inequality by considering a fractional boundary value problem subject to mixed boundary conditions. We give a necessary condition for nonexistence of solutions for a class of boundary value problems involving Riemann–Liouville fractional order. The order considered here is $3 < \alpha \leq 4$. The investigation is based on a construction of Green's function and on finding its corresponding maximum value. In order to illustrate the result, we provide an application of Lyapunov-type inequality for an eigenvalue problem and we show how the necessary condition of existence can be employed to determine intervals for the real zeros of the Mittag-Leffler function.

Keywords: Lyapunov's inequality, Riemann–Liouville derivative, Green's function, mixed boundary conditions.
2010 MSC: 34A08, 34A40, 26D10, 33E12.

1. Introduction

We present a Lyapunov's inequality for the following boundary value problem:

$$\begin{cases} ({}_aD^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b, \quad 3 < \alpha \leq 4, \\ u(a) = u(b) = u''(a) = u''(b) = 0, \end{cases} \tag{1.1}$$

where a and b are consecutive zeros of the solution u. As $u = 0$ is a trivial solution, only nonnegative solutions are taken into consideration.

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We prove that problem (1.1) has a nontrivial solution for $\alpha \in (3,4]$ provided that the real and continuous function q satisfies

$$\int_a^b |q(t)|\,dt > \frac{\Gamma(\alpha)}{(1-(b-a))\,[(b-a)^{\alpha-1}]}. \tag{1.2}$$

Before we prove this result, let us dwell upon some references.

For the fractional boundary value problem

$$\begin{cases} u''(t)+q(t)u(t)=0, & a<t<b,\\ u(a)=u(b)=0, \end{cases}$$

where a and b are consecutive zeros of u and $q \in C([a,b];\mathbb{R})$. Lyapunov [8] proved a necessary condition of existence of nontrivial solutions, which is formulated by

$$\int_a^b |q(t)|\,dt > \frac{4}{b-a}, \tag{1.3}$$

where the constant 4 in (1.3) is sharp. After this result, similar type inequalities have been obtained for other kind of differential equations and boundary conditions; see [4, 10]. In addition, for positive solutions for a class of nonlinear fractional differential equations one may read [2] and the references therein.

Concerning differential equation with fractional derivatives, in [3], Ferreira derived Lyapunov's inequality for the problem

$$\begin{cases} ({}_aD^\alpha u)(t)+q(t)u(t)=0, & a<t<b,\ 1<\alpha\le 2,\\ u(a)=u(b)=0, \end{cases} \tag{1.4}$$

where $q \in C([a,b],\mathbb{R})$, a and b are consecutive zeros of u, and ${}_aD^\alpha$ is the Riemann–Liouville fractional derivative of order $\alpha>0$ defined for an absolute continuous function on $[a,b]$ by

$$({}_aD^\alpha f)(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^\alpha f(s)\,ds,$$

where $n\in\mathbb{N}$, $n<\alpha\le n+1$ (For more details of fractional derivatives, see [7]). His inequality reads

$$\int_a^b |q(t)|\,dt > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}, \tag{1.5}$$

which, in the particular case $\alpha=2$, corresponds to Lyapunov's classical inequality (1.3). Thus, Ferreira [4] and Jleli and Samet [6] dealt with fractional differential boundary value problems with Caputo's derivative, which is defined for an absolutely continuous function f by

$$({}_a^CD^\alpha f)(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^\alpha f^{(n)}(s)\,ds.$$

For the boundary value problem

$$\begin{cases} ({}_a^CD^\alpha u)(t)+q(t)u(t)=0, & a<t<b,\ 1<\alpha\le 2,\\ u(a)=u(b)=0, \end{cases} \tag{1.6}$$

where $q\in C([a,b];\mathbb{R})$ and a and b are consecutive zeros of u, Ferreira [3] proved that if (1.6) has a nontrivial solution, then the following necessary condition is satisfied

$$\int_a^b |q(t)|\,dt > \frac{\Gamma(\alpha)\alpha^\alpha}{[(\alpha-1)(b-a)]^{\alpha-1}}. \tag{1.7}$$

In [6], Jleli and Samet considered the equation (1.6) subject to either

$$u'(a)=0,\ u(b)=0, \tag{1.8}$$

or

$$u(a) = 0,\ u'(b) = 0. \tag{1.9}$$

They showed that an associated nontrivial solution to (1.8) exists if

$$\int_a^b (b-s)^{\alpha-2}|q(t)|\,dt \ \geq\ \Gamma(\alpha) \tag{1.10}$$

is satisfied.

However, in the case of (1.9), the corresponding nontrivial solution exists if

$$\int_a^b (b-s)^{\alpha-2}|q(t)|\,dt \ \geq\ \frac{\Gamma(\alpha)}{\max\{\alpha-1, 2-\alpha\}\,(b-a)}. \tag{1.11}$$

It was shown in [5] that a nontrivial solution corresponding to equation (1.6) where $q \in C([a,b];\mathbb{R})$, a and b are consecutive zeros of u, subject to the boundary conditions

$$u(a) - u'(a) = u(b) + u'(b) = 0, \tag{1.12}$$

exists if the following necessary condition

$$\int_a^b (b-s)^{\alpha-2}(b-s+\alpha-1)|q(s)|\,ds \geq \frac{(b-a+2)\,\Gamma(\alpha)}{\max\{b-a+1, \frac{2-\alpha}{\alpha-1}\,(b-a)-1\}} \tag{1.13}$$

is satisfied. Rong and Bai [11] established a Lyapunov's inequality for a fractional differential equation (1.6) under the following boundary condition

$$({}_a^C D^{\beta} u)(b) = u(a) = 0, \tag{1.14}$$

where $0 < \beta \leq 1$, $1 < \alpha \leq \beta + 1$. Precisely, they proved the following necessary integral condition for existence of a nontrivial solution:

$$\int_a^b (b-s)^{\alpha-\beta-1}|q(s)|\,ds \geq \frac{(b-a)^{-\beta}}{\max\left\{\frac{1}{\Gamma(\alpha)} - \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}, \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}, \left(\frac{2-\alpha}{\alpha-1}\right)\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}\right\}}. \tag{1.15}$$

For the particular case when $\beta = 1$, the necessary condition of existence is reduced to the Lyapunov-type inequality (1.11). In [9], Donal O'Regan and Bessem Samet were concerned with the following fractional boundary problem:

$$({}_a D^{\alpha} u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 3 < \alpha \leq 4, \tag{1.16}$$

$$u(a) = u'(a) = u''(a) = u''(b) = 0, \tag{1.17}$$

where ${}_a D^{\alpha}$ is the standard Riemann–Liouville fractional derivative of fractional order α and $q : [a,b] \to \mathbb{R}$ is a continuous function. They proved that: if a nontrivial continuous solution to the fractional boundary value problem

$$({}_a D^{\alpha} u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 3 < \alpha \leq 4,$$
$$u(a) = u'(a) = u''(a) = u''(b) = 0,$$

exists, where q is a real and continuous function in $[a,b]$, then

$$\int_a^b (b-s)^{\alpha-3}(s-a)(2b-a-s)|q(s)|\,ds \geq \Gamma(\alpha)\cdot \tag{1.18}$$

Motivated by all above results, we are concerned in this paper with the following fractional boundary value problem

$$({}_aD^{\alpha}u)(t) + q(t)u(t) = 0, \quad a < t < b, \ 3 < \alpha \leq 4, \tag{1.19}$$
$$u(a) = u(b) = u''(a) = u''(b) = 0, \tag{1.20}$$

where a and b satisfy

$$0 \ < \ b - a \ < \ 1, \text{ and } (1 - 3\frac{(s-a)^2}{(b-a)^2}) \ \geq \ \frac{6}{\alpha(\alpha-1)}, \tag{1.21}$$

and s is any value fixed in $[a, b]$.

The aim here is to prove that the corresponding nontrivial solution to boundary value problem (1.19)–(1.20) exists if the following integral inequality is satisfied

$$\int_a^b |q(t)|\, dt \ \geq \ \frac{\Gamma(\alpha)}{(1-(b-a))(b-a)^{\alpha-1}}, \tag{1.22}$$

where q is a real and continuous function in $[a, b]$.

2. Preliminary and lemmas

Definition 2.1. If $g \in C([a, b])$ and $\alpha > 0$, then the Riemann–Liouville fractional integral is defined by $I^{\alpha}_{a^+}\, g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g(s)}{(t-s)^{1-\alpha}} ds.$

Definition 2.2. Let $\alpha \geq 0$, and $n = [\alpha]+1$. If $f \in AC([a, b])$, then the Caputo fractional derivative of order α of f defined by $cD^{\alpha}_{a^+}\, g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}}\, ds$ exist almost everywhere on $[a, b]$ ($[\alpha]$ is the entire part of α).

Lemma 2.3 ([7])**.** *Let $\alpha, \beta > 0$ and $n = [\alpha]+1$; then the following relations hold: $cD^{\alpha}_{0^+}\, t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\, t^{\beta-\alpha-1}$, $\beta > n$ and $cD^{\alpha}_{o^+}\, t^k = 0$. $k = 0, \ldots, n-1$.*

Lemma 2.4 ([7])**.** *For $\alpha > 0$, $g(t) \in C(0, 1)$, the homogeneous fractional differential equation $cD^{\alpha}_{a^+}\, g(t) = 0$ has a solution $g(t) = c_1 + c_2t + c_3t^2 + \cdots + c_nt^{n-1}$, where $c_i \in R$, $i = 0, \ldots, n$, and $n = [\alpha]+1$, (α noninteger).*

The following two lemmas focus on properties of Riemann–Liouville fractional integrals and Caputo fractional derivative that we need in the sequel.

Lemma 2.5 ([1])**.** *Let $p, q \geq 0$, $f \in L_1[a, b]$. Then, $I^p_{0^+}\, I^q_{0^+}\, f(t) = I^{p+q}_{0^+} \ = I^q_{0^+}\, I^p_{0^+}$ and $cD^q_{0^+}\, I^q_{0^+}\, f(t) = f(t)$, for all $t \in [a, b]$.*

Lemma 2.6 ([7])**.** *Let $\alpha > \beta > 0$. Then the formula $cD^{\alpha}_{0^+}\, I^{\beta}_{0^+}\, f(t) = I^{\beta-\alpha}_{0^+}$ holds almost everywhere on $[a, b]$; for $f \in L_1[a, b]$ it is valid at any point $x \in [a, b]$ if $f \in C[a, b]$.*

As an auxiliary result, the Green's function, which is the most crucial function involved in the fractional boundary value problem, is given explicitly in the following theorem.

Theorem 2.7. *Let $3 < q \leq 4$ and $f \in C([a, b])$. Then the unique solution of the fractional boundary value problem*

$$({}_aD^{\alpha}u)(t) + q(t)u(t) = 0, \quad a < t < b, \ 3 < \alpha \leq 4,$$
$$u(a) = u(b) = u''(a) = u''(b) = 0,$$

is given by

$$u(t) = \frac{1}{\Gamma(q)} \int_a^t G(t, s)u(s)q(s)ds + \frac{1}{\Gamma(q)} \int_t^b G(t, s)u(s)q(s)ds,$$

where the Green function $G(t,s)$ is defined by

$$\Gamma(\alpha)G(t,s) = \begin{cases} (t-s)^{\alpha-1} - (t-a)(b-s)^{\alpha-1} + \\ \frac{\alpha(\alpha-1)(b-a)(b-s)^{\alpha-3}}{6}(t-a)[1-\frac{(t-a)^2}{(b-a)^2}], \\ s \le t \le b, \\ -(t-a)(b-s)^{\alpha-1} \\ +\frac{\alpha(\alpha-1)(b-a)(b-s)^{\alpha-3}}{6}(t-a)[1-\frac{(t-a)^2}{(b-a)^2}], \\ t \le s \le b. \end{cases} \tag{2.1}$$

Proof. In view of Lemmas 2.4 and 2.5, fractional differential equation (1.19) takes the form

$$u(t) = I_{0^+}^{\alpha}\, u(t)q(t) + c_1 + c_2(t-a) + c_3(t-a)^2 + c_4(t-a)^3, \tag{2.2}$$

where c_1, c_2, c_3, and c_4 are real constants. Now by the boundary condition $u(a) = 0$, we are conducted to $c_1 = 0$ and by $u(b) = 0$ we are lead to $I_{0^+}^{\alpha}\, u(b)q(b)+c_2(b-a)+c_3(b-a)^2+c_4(b-a)^3 = 0$. Now the use of Lemma 2.6 and the differentiating of (2.2) conduct us to $u''(t) = I_{0^+}^{\alpha-2}\, u(t)q(t)+2c_3+6c_4(t-a)$. In view of $u''(a) = 0$, we get $c_3 = 0$ and from $u''(b) = 0$, we find $c_4 = \frac{-I_{0^+}^{\alpha-2}\, u(b)q(b)}{6(b-a)}$. Thus, $c_2 = \frac{I_{0^+}^{\alpha-2}\, u(b)q(b)}{6}(b-a) - \frac{I_{0^+}^{\alpha}\, u(b)q(b)}{b-a}$. By inserting the values of the constants c_1, c_2, c_3, and c_4 into (2.2), we find

$$u(t) = I_{0^+}^{\alpha} + (t-a)\left(\frac{I_{0^+}^{\alpha-2}\, u(b)q(b)}{6}(b-a) - \frac{I_{0^+}^{\alpha}\, u(b)q(b)}{b-a}\right) - (t-a)^3\left(\frac{I_{0^+}^{\alpha-2}u(b)q(b)}{6(b-a)}\right).$$

Equivalently, we have $u(t) = \int_a^b G(t,s)q(s)u(s)ds$ where

$$\Gamma(\alpha)G(t,s) = \begin{cases} (t-s)^{\alpha-1} - (t-a)(b-s)^{\alpha-1} + \\ \frac{\alpha(\alpha-1)(b-a)(b-s)^{\alpha-3}}{6}(t-a)[1-\frac{(t-a)^2}{(b-a)^2}], \\ s \le t \le b, \\ -(t-a)(b-s)^{\alpha-1} \\ +\frac{\alpha(\alpha-1)(b-a)(b-s)^{\alpha-3}}{6}(t-a)[1-\frac{(t-a)^2}{(b-a)^2}], \\ t \le s \le b, \end{cases} \tag{2.3}$$

which completes the proof. □

The strategy for getting the necessary condition of the existence of nontrivial solutions to the fractional boundary value problem (1.19)–(1.20) is based on a construction of the corresponding Green's function and in finding its maximum value. However, to accomplish this fact, we have to overcome some difficulties. This is due to the considered boundary conditions (1.20). So in order to overcome this type of difficulties, we recourse to the successive differentiation of G with respect to their arguments t and s. Once constructed, we provide our objective which is the maximum of the Green's function G.

We formulate this result in the following theorem.

Theorem 2.8. *The Green function G defined in* (2.1) *satisfies:*

(1) $G(t,s) \ge 0$ *for all* $a \le t,\ s \le b$ *and* $\max_{t\in[a,b]} G(t,s) = G(b,s),\ s \in [a,b]$*,*

(2) $G(b,s)$ *has a unique maximum given by*

$$\max_{s\in[a,b]} G(b,s) = \frac{1}{\Gamma(\alpha)}\left(1-(b-a)\right)(b-a)^{(\alpha-1)}.$$

Proof. The proof, upon the criteria of the Green function, splits into two cases. For this purpose, let us differentiate G with respect to t by fixing an arbitrary $s \in [a, b]$, one time, two times, and three times, respectively, and get

$$\Gamma(\alpha)\, G_t = (\alpha-1)(t-s)^{\alpha-2} - \frac{(b-s)^{\alpha-1}}{b-a} + \frac{\alpha(\alpha-1)(b-a)(b-s)^{\alpha-3}}{6}[1-3\left(\frac{t-a^2}{b-a}\right)], \tag{2.4}$$

$$\Gamma(\alpha)\, G_{tt} = (\alpha-1)(\alpha-2)(t-s)^{\alpha-3} - \alpha(\alpha-1)(b-s)^{\alpha-3}\left(\frac{t-a}{(b-a)^2}\right), \tag{2.5}$$

$$\Gamma(\alpha)\, G_{ttt} = (\alpha-1)(\alpha-2)(\alpha-3)(t-s)^{\alpha-4} - \frac{\alpha(\alpha-1)(b-s)^{\alpha-3}}{(b-a)}. \tag{2.6}$$

Now for $s \le t$, we claim that the function G_{ttt} is a nonincreasing function of t. Indeed, deriving G_{ttt} with respect to t and taking in account the fact that $\alpha \in (3, 4]$, we obtain

$$\Gamma(\alpha)\, G_{tttt} = (\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(t-s)^{\alpha-5} \tag{2.7}$$

for $s \le t$, which is nonpositive for $3 < \alpha \le 4$. Thus, $G_{ttt}(t, s) \le G_{ttt}(s, s)$ since G_{ttt} is a nonincreasing function of t in the interval $[s, t]$ by (2.7). Hence the following inequality is satisfied

$$\Gamma(\alpha)\, G_{ttt}(s, s) = -\frac{\alpha(\alpha-1)}{6(b-a)}(b-s)^{\alpha-3} \le 0,$$

since $\alpha \in (3, 4]$. Hence the function G_{tt} is also nonincreasing, and therefore for $s \le t \le b$ we have

$$G_{tt}(b, s) \le G_{tt}(t, s) \le G_{tt}(s, s),$$

where

$$\Gamma(\alpha)\, G_{tt}(s, s) := -\frac{\alpha(\alpha-1)(b-s)^{\alpha-3}(s-a)}{(b-a)} \le 0$$

and

$$\Gamma(\alpha)\, G_{tt}(b, s) := (\alpha-1)(\alpha-2)(b-s)^{\alpha-3} - \frac{\alpha(\alpha-1)(b-s)^{\alpha-3}}{(b-a)} \ (\le 0).$$

Similarly as above, we obtain G_t is a nonincreasing function of t for $s \le t$. Thus

$$\Gamma(\alpha) G_t(t, s) \le \Gamma(\alpha) G_t(s, s) := -\frac{(b-s)^{\alpha-1}}{b-a} + \frac{\alpha(\alpha-1)(b-a)(b-s)^{\alpha-3}}{6}[1-3(\frac{s-a}{b-a})^2]. \tag{2.8}$$

After a simplification and reduction of the right-hand side of (2.8), we get

$$\Gamma(\alpha)\, G_t(s, s) = -(b-s)^{\alpha-1} + ((b-a)^2 - 3(s-a)^2)\frac{\alpha(\alpha-1)(b-s)^{\alpha-3}}{(b-a)}.$$

As a function of s, we define $G_t(s, s)$ by

$$h(s) := \Gamma(\alpha)\, G_t(s, s) = -(b-s)^{\alpha-1} + ((b-a)^2 - 3(s-a)^2)\frac{\alpha(\alpha-1)(b-s)^{\alpha-3}}{(b-a)}. \tag{2.9}$$

A very well known strategy to get the Lyapunov integral inequality within such a fractional boundary value problem appears to be transforming (1.19)–(1.20) into an equivalent integral form and then finding the maximum value of its Green's function. Thus, the obtained result leads to an application illustrated in the last section. So, our purpose is to obtain the maximum of $h(s)$ for $s \in [a, b]$. For this matter, we may distinguish two cases:

1) If $(b-a)^2 - 3(s-a)^2$ is negative, then we are conducted to $G = 0$ and we get a contradiction since the solution u is assumed to be nontrivial.

We will show this fact by using (2.9) and $(b-a)^2 - 3(s-a)^2 \le 0$; we get that $h(s)$ is nonpositive and consequently, in light of (2.8), we conclude that

$$G_t \le 0 \quad \text{for } s \le t.$$

Thus, the Green function G is non increasing for $s \le t$ and therefore the following inequality is satisfied:

$$G(b,s) \le G(t,s) \le G(s,s). \tag{2.10}$$

From (2.1), one may deduce that $G(b,s)$ takes the form

$$\begin{aligned} \Gamma(\alpha)G(b,s) &= (b-s)^{\alpha-1} - (b-a)(b-s)^{\alpha-1} \\ &\quad + \frac{\alpha(\alpha-1)(b-a)(b-s)^{\alpha-3}}{6}(b-a)\,(1 - \frac{(b-a)^2}{(b-a)^2}) \\ &= (1-(b-a))(b-s)^{\alpha-1}, \end{aligned} \tag{2.11}$$

which is positive since $0 < b-a < 1$ by condition (1.21).

Similarly, $G(s,s)$ takes the form

$$\begin{aligned} \Gamma(\alpha)G(s,s) :=& -(s-a)(b-s)^{\alpha-1} \\ &+ \frac{\alpha(\alpha-1)(b-a)(b-s)^{\alpha-3}}{6}(s-a)[1 - \frac{(s-a)^2}{(b-a)^2}], \end{aligned} \tag{2.12}$$

which is nonpositive since $a \le s$ and $3 < \alpha \le 4$. Hence we conclude that $G = 0$ in light of (2.10), (2.11) and (2.12). We obtained a contradiction since the solution u is assumed to be nontrivial.

The second possibility may be formulated as follows.

2) If $(b-a)^2 - 3(s-a)^2$ is positive, then in view of (1.21), (2.8) and the fact that $a \le s \le t \le b$, we find

$$\begin{aligned} h(s) := G_t(s,s) &= -\frac{(b-s)^{\alpha-1}}{(b-a)} + \frac{\alpha(\alpha-1)(b-a)}{6}(b-s)^{\alpha-3}(1 - 3\frac{(s-a)^2}{(b-a)^2}) \\ &\ge -\frac{(b-s)^{(\alpha-3)}(b-s)^2}{b-a} \\ &\quad + \frac{\alpha(\alpha-1)}{6(b-a)^2}(b-s)^{\alpha-3}(b-a)(1 - 3\frac{(s-a)^2}{(b-a)^2}) \\ &\ge (b-s)^{(\alpha-3)}(b-a)[-1 + (1 - 3\frac{(s-a)^2}{(b-a)^2}) \\ &\ge 0. \end{aligned} \tag{2.13}$$

Since G_t is a nonincreasing function of t, we have

$$G_t(b,s) \le G_t(t,s) \le G_t(s,s),$$

and consequently, it remains to prove that $k(s) := \Gamma(\alpha)\, G_t(b,s)$ is positive and we will conclude that $G_t(t,s)$ is positive for $s \leq t$. Indeed,

$$\begin{aligned} k(s) &= (\alpha-1)(b-s)^{\alpha-2} - \frac{(b-s)^{\alpha-1}}{b-a} \\ &\quad + \frac{\alpha(\alpha-1)}{b-a}(b-s)^{\alpha-3}(1-3\frac{(s-a)^2}{(b-a)^2} \\ &= (\alpha-1)(b-s)^{\alpha-2} - \frac{(b-s)^{\alpha-2}(b-s)}{b-a} \\ &\quad + \frac{\alpha(\alpha-1)(b-a)}{6}(b-s)^{\alpha-3}(1-3\frac{(s-a)^2}{(b-a)^2}) \\ &= (b-s)^{\alpha-2}[(\alpha-1) - \frac{(b-s)}{b-a}] \\ &\quad + \frac{\alpha(\alpha-1)(b-a)}{6}(b-s)^{\alpha-3}(1-3\frac{(s-a)^2}{(b-a)^2}) \\ &\geq (b-s)^{\alpha-2}[(\alpha-1) - \frac{(b-a)}{b-a}] \\ &\quad + \frac{\alpha(\alpha-1)(b-a)}{6}(b-s)^{\alpha-3}(1-3\frac{(s-a)^2}{(b-a)^2}) \\ &\geq (b-s)^{\alpha-2}[(\alpha-2)] \\ &\quad + \frac{\alpha(\alpha-1)(b-a)}{6}(b-s)^{\alpha-3}(1-3\frac{(s-a)^2}{(b-a)^2}), \end{aligned} \tag{2.14}$$

which is positive, by assumption (1.21) for $3 < \alpha \leq 4$.

Hence we conclude that G_t is positive and therefore G is an increasing function of t for $s \leq t$. Thus, the following inequality is satisfied

$$G(s,s) \leq G(t,s) \leq G(b,s).$$

The next step is to show that the Green function G attains its maximum value on $[a,b]$. By (2.1), $G(b,s)$ takes the form

$$\begin{aligned} \Gamma(\alpha)\, G(b,s) &= (b-s)^{\alpha-1}[1-(b-a)] \\ &\quad + \frac{\alpha(\alpha-1)(b-a)^2(b-s)^{\alpha-3}}{6}(1-\frac{(b-a)^2}{(b-a)^2}). \end{aligned} \tag{2.15}$$

Let us define the function $G(b,s)$ as a function of s, by $R(s)$ as follows

$$R(s) := \Gamma(\alpha)\, G(b,s) = (b-s)^{\alpha-1}[1-(b-a)], \tag{2.16}$$

which is positive since $0 < b-a < 1$, by condition (1.21). Taking the first derivative of R with respect to s, we get

$$R'(s) := -(\alpha-1)(b-s)^{\alpha-2}[1-(b-a)], \tag{2.17}$$

which is nonpositive for $3 < \alpha \leq 4$. Hence the function R is nonincreasing in s and for $a \leq s \leq b$, we have

$$R(s) \;\leq\; R(a) := \frac{1}{\Gamma(\alpha)}(b-a)^{\alpha-1}[1-(b-a)]. \tag{2.18}$$

To this end, we conclude that

$$\max_{a\leq s\leq t\leq b} G(t,s) = G(b,s) = \frac{1}{\Gamma(\alpha)}(b-a)^{\alpha-1}[1-(b-a)]. \tag{2.19}$$

It results that $0 = R(b) \le R(s) \le R(a)$ and this yields the positivity of the Green function G. So Theorem 2.8 is proved for $s \le t$.

Now to accomplish the second part of the proof, we consider the Green function defined in (2.1) for $t \le s$ by

$$\begin{aligned}\Gamma(\alpha)G(t,s) \,:=& -(t-a)(b-s)^{\alpha-1}\\ &+ \frac{\alpha(\alpha-1)(b-a)(b-s)^{\alpha-3}}{6}(t-a)[1-\frac{(t-a)^2}{(b-a)^2}].\end{aligned} \tag{2.20}$$

Differentiating (2.20) with respect to t, we obtain

$$\begin{aligned}\Gamma(\alpha)G_t(t,s) \,:=& -(\alpha-1)(b-s)^{\alpha-2}\\ &+ \frac{\alpha(\alpha-1)(b-a)(b-s)^{\alpha-3}}{6}[1-3\frac{(t-a)^2}{(b-a)^2}].\end{aligned} \tag{2.21}$$

In turn, we differentiate (2.21) with respect to t and get

$$\Gamma(\alpha)\, G_{tt}(t,s) = -\frac{(t-a)}{b-a}\alpha(\alpha-1)(b-s)^{\alpha-3}, \tag{2.22}$$

which is nonpositive for $3 < \alpha \le 4$ by condition (1.21). Therefore, one may deduce that G_t is a nonincreasing function in t and this in turn leads to the following inequality

$$G_t(b,s) \le G_t(t,s) \le G_t(a,s). \tag{2.23}$$

Let us denote $G_t(a,s)$ as a function of s, by

$$l(s) := G_t(a,s) = \, -(\alpha-1)(b-s)^{\alpha-2} + \frac{\alpha(\alpha-1)(b-a)(b-s)^{\alpha-3}}{3},$$

and derive l two times with respect to s, to get

$$\begin{aligned}l'(s) &= (\alpha-1)(\alpha-2)(b-s)^{\alpha-3} - \frac{\alpha(\alpha-1)(\alpha-3)}{(b-a)}(b-s)^{\alpha-4},\\ l''(s) &= -(\alpha-1)(\alpha-2)(\alpha-3)(b-s)^{\alpha-4} + \frac{\alpha(\alpha-1)(\alpha-3)(\alpha-4)(b-s)^{\alpha-5}}{3},\end{aligned}$$

which is nonpositive for $3 < \alpha \le 4$. We conclude that the Green function G is decreasing for $t \le s$. Indeed, since $l''(s)$ is nonpositive, $l'(s)$ is decreasing as a function of s. This is due to $l'(b) \le l'(s) \le l'(a)$ where $l'(a)$ is positive and $l'(b) = 0$. Now since l' is a nonincreasing function of s, one may conclude that the function $l(s) := G_t(t,s)$ is a nondecreasing function of s and therefore the following inequality is satisfied

$$l(a) \le l(s) \le l(b) = 0.$$

Equivalently, the function $l(s)$ is nonincreasing in s and, as above, we have $G(b,s) \le G(t,s) \le G(a,s)$ and therefore $G = 0$, which leads the solution u to be zero, the trivial one. Contradiction.

In resume, the maximum value of the Green function associated to the boundary fractional value problem is given by

$$\max_{a\le s\le t\le b} G(t,s) = \frac{1}{\Gamma(\alpha)}(b-a)^{\alpha-1}[1-(b-a)].$$

The proof is complete. □

3. A Lyapunov's Inequality

In this section, we establish the necessary condition of the existence of nontrivial solution of (1.19)–(1.20). Based on a construction of the Green function, we are able to achieve the desired result formulated in the following theorem.

Theorem 3.1. *Assume that* u *is a nontrivial solution to the fractional boundary problem* (1.19)–(1.20) *and that condition* (1.21) *is satisfied. Then the necessary integral condition for existence of a nontrivial solution is*

$$\int_a^b |q(t)|\, dt \geq \frac{\Gamma(\alpha)}{(1-(b-a))(b-a)^{\alpha-1}}. \tag{3.1}$$

For proof , we equip the Banach space $C([a,b])$ with the Chebychev norm $||u|| = \max_{t\in[a,b]} |u(t)|$. As

$$u(t) := \int_a^b G(t,s)q(s)u(s)\, ds,$$

which, for all $t \in [a,b]$, yields

$$|u(t)| \leq \int_a^b |G(t,s)|\, |q(s)|\, |u(s)|\, ds.$$

Thus

$$||u(t)|| \leq \int_a^b \max_{t,s\in[a,b]} |G(t,s)|\, |q(s)|\, ||u(s)||\, ds.$$

Due to the nontriviality of the solution u, one may conclude that $||u(s)|| \neq 0$ and therefore

$$1 \leq \int_a^b G(a,s)\, |q(s)|\, ds.$$

Thus, in view of (2.19), we obtain

$$1 \leq \int_a^b \frac{1}{\Gamma(\alpha)}(b-a)^{\alpha-1}[1-(b-a)]\, |q(s)|\, ds.$$

In other words, we have

$$\int_a^b |q(s)|\, ds \geq \frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}[1-(b-a)]}.$$

To this end, the desired inequality (3.1) is achieved.

4. Application

In order to illustrate Theorem 3.1, we give an application of Lyapunov-type inequality (1.22) for the following eigenvalue problem and we get a bound for λ, for which the boundary value problem in consideration has a nontrivial solution. Precisely, we show how the necessary condition of existence can be employed to determine intervals for the real zeros of the Mittag-Leffler function.

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{k\alpha+\alpha}, \quad z \in \mathcal{C}, \quad \text{and } \mathcal{R}(\alpha) \text{ is positive.} \tag{4.1}$$

By setting $a = \frac{1}{2}$ and $b = 1$, and taking into consideration Sturm–Liouville eigenvalue problem, we get

$$\begin{cases} ({}_{\frac{1}{2}}D^\alpha u)(t) + \lambda u(t) = 0, \quad 0 < t < \frac{1}{2}, \quad 3 < \alpha \leq 4, \\ u(1) = u(\frac{1}{2}) = u''(1) = u''(\frac{1}{2}) = 0. \end{cases} \tag{4.2}$$

Theorem 4.1. *If λ is an eigenvalue of fractional boundary value problem* (4.2)*, then the following inequality holds*

$$|\lambda| \geq \frac{\Gamma(\alpha)}{2^{-\alpha}}. \tag{4.3}$$

For a proof of Theorem 4.1, it is sufficient to use the integral inequality (3.1). We assume that λ is an eigenvalue of boundary value problem (4.2), Then there exists only one nontrivial solution depending on λ such that

$$\int_{\frac{1}{2}}^{1} |\lambda|\, dt \geq \frac{\Gamma(\alpha)}{2^{-\alpha}(\frac{1}{2})}, \tag{4.4}$$

or equivalently

$$|\lambda| \geq \frac{\Gamma(\alpha)}{2^{-\alpha}}, \tag{4.5}$$

which leads to completion of the proof.

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