Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 8 (2015), 8–16
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Some common coupled fixed point theorems for generalized contraction in b-metric spaces

Nidhi Malhotra[a], Bindu Bansal[b,*]

[a] *Department of Mathematics, Hindu College, University of Delhi, Delhi, India.*
[b] *Department of Mathematics, Hindu College, University of Delhi, Delhi, India.*

Communicated by R. Saadati

Abstract

The aim of this paper is to prove the existence and uniqueness of a common coupled fixed point for a pair of mappings in a complete b-metric space in view of diverse contractive conditions. In addition, as a bi-product we obtain several new common coupled fixed point theorems.

Keywords: Common fixed point, coupled fixed point, coupled coincidence point, contractive mappings, b-metric spaces.
2010 MSC: 47H10, 54H25.

1. Introduction and preliminaries

The study of fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity. One of the main tools in fixed point theory is the Banach contraction theorem proved by Banach in 1922. This theorem is a very popular and effective tool in solving existence problems in many branches of mathematical analysis and engineering. There are a lot of generalizations of this theorem in the literature. Fixed point theory has many applications in various branches of mathematics and branches of science.

The concept of b-metric space was introduced by Bakhtin [3] and Czerwik [5]. In [5], Czerwik proved the contraction mapping principle in b-metric spaces that generalized the famous Banach contraction principle in metric spaces. Since then several papers have dealt with fixed point theory for single-valued and multi-valued operators in b-metric spaces (see [1], [4], [9], [10], [11] and references therein). A b-metric space was

*Corresponding author
Email addresses: `nidmal25@gmail.com` (Nidhi Malhotra), `bindubansaldu@gmail.com` (Bindu Bansal)

Received 2014-08-06

also called a metric-type space in [7]. The fixed point theory in metric-type spaces was investigated in [7] and [8].

In [6], Bhaskar and Lakshmikantham introduced the concept of coupled fixed points for a given partially ordered set X. The study of common coupled fixed points of mappings satisfying certain contractive conditions has been at the center of vigorous research activity, being the applications of fixed point very important in several areas of mathematics. The purpose of the present paper is to study the notion of common coupled fixed points for a pair of mappings in b-metric spaces and prove the existence and uniqueness of the common coupled fixed point in a complete b-metric space in view of diverse contractive conditions. In addition, as a bi-product we obtain several new common coupled fixed point theorems.

Definition 1.1. [2] Let X be a (nonempty) set and $s \geq 1$ a given real number. A function $d : X \times X \to \Re^+$(nonnegative real numbers) is called a b-metric provided that, for all $x, y, z \in X$, the following conditions are satisfied:

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space with parameter s.

We now give some examples of b-metric spaces.

Example 1.2. [4] The space $l_p(0 < p < 1)$, $l_p = \{(x_n) \in \Re : \sum |x_n|^p < \infty\}$,
together with the function $d : l_p \times l_p \to \Re$

$$d(x, y) = (\sum |x_n - y_n|^p)^{\frac{1}{p}},$$

where $x = (x_n); y = (y_n) \in l_p$ is a b-metric space with $s = 2^{\frac{1}{p}}$.

Example 1.3. [4] The space $L_p(0 < p < 1)$ of all real functions $x(t), t \in [0, 1]$ such that $\int_0^1 |x(t)|^p \, dt < \infty$, is a b-metric space if we take
$d(x, y) = (\int_0^1 |x(t) - y(t)|^p \, dt)^{\frac{1}{p}}$, for each $x, y \in L_p$.

Remark 1.4. We note that a metric space is evidently a b-metric space for $s = 1$. However, in general, a b-metric on X need not be a metric on X as shown in the following example:

Example 1.5. [2] Let $X = \{0, 1, 2\}$ and $d(2, 0) = d(0, 2) = m \geq 2, d(0, 1) = d(1, 2) = d(1, 0) = d(2, 1) = 1$ and $d(0, 0) = d(1, 1) = d(2, 2) = 0$. Then $d(x, y) \leq \frac{m}{2}[d(x, z) + d(z, y)]$ for all $x, y, z \in X$. If $m > 2$, the ordinary triangle inequality does not hold.

Definition 1.6. [4] Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called a Cauchy sequence if for every $\epsilon > 0$, there exists $K(\epsilon) \in \mathbb{N}$, such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq K(\epsilon)$.

Definition 1.7. [4] Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is said to converge to $x \in X$ if for every $\epsilon > 0$, there exists $K(\epsilon) \in \mathbb{N}$, such that $d(x_n, x) < \epsilon$ for all $n \geq K(\epsilon)$. In this case, we write $\lim_{n\to\infty} x_n = x$.

Definition 1.8. [4] The b-metric space (X, d) is complete if every Cauchy sequence in X converges in X.

Remark 1.9. In a b-metric space (X, d) the following assertions hold:

(1) A convergent sequence has a unique limit;
(2) Every convergent sequence is Cauchy.

Definition 1.10. [6] An element $(x, y) \in X \times X$ is called a coupled fixed point of $T : X \times X \to X$ if

$$x = T(x, y) \text{ and } y = T(y, x).$$

Definition 1.11. An element $(x, y) \in X \times X$ is called a coupled coincidence point of $S, T : X \times X \to X$ if

$$S(x, y) = T(x, y) \text{ and } S(y, x) = T(y, x).$$

Example 1.12. Let $X = \Re$ and $S, T : X \times X \to X$ defined as

$$S(x, y) = x^2y^2 \text{ and } T(x, y) = (9/4)(x + y),$$

for all $x, y \in X$. Then $(0, 0)$, $(1, 3)$ and $(3, 1)$ are coupled coincidence points of S and T.

Example 1.13. Let $X = \Re$ and $S, T : X \times X \to X$ defined as

$$S(x, y) = x + y - xy + sin(x + y) \text{ and } T(x, y) = x + y + cos(x + y),$$

for all $x, y \in X$. Then $(0, \pi/4)$, and $(\pi/4, 0)$ are coupled coincidence points of S and T.

Definition 1.14. An element $(x, y) \in X \times X$ is called a common fixed point of $S, T : X \times X \to X$ if

$$x = S(x, y) = T(x, y) \text{ and } y = S(y, x) = T(y, x).$$

Example 1.15. Let $X = \Re$ and $S, T : X \times X \to X$ defined as

$$S(x, y) = xy \text{ and } T(x, y) = x + (y - x)^2,$$

for all $x, y \in X$. Then $(0, 0)$ and $(1, 1)$ are common coupled fixed points of S and T.

2. Main results

Theorem 2.1. *Let (X, d) be a complete b-metric space with parameter $s \geq 1$ and let the mapping $S, T : X \times X \to X$ satisfy*

$$d(S(x, y), T(u, v)) \leq \alpha\frac{d(x, u) + d(y, v)}{2} + \beta\frac{d(x, S(x, y))d(u, T(u, v))}{(1 + d(x, u) + d(y, v))} + \gamma\frac{d(u, S(x, y))d(x, T(u, v))}{(1 + d(x, u) + d(y, v))},$$

for all $x, y, u, v \in X$ and $\alpha, \beta, \gamma \geq 0$ with $s\alpha + \beta < 1$ and $\alpha + \gamma < 1$. Then S and T have a unique common coupled fixed point in X.

Proof. Let x_0 and $y_0 \in X$ be arbitrary points.
Define $x_{2k+1} = S(x_{2k}, y_{2k}), y_{2k+1} = S(y_{2k}, x_{2k})$ and $x_{2k+2} = T(x_{2k+1}, y_{2k+1}), y_{2k+2} = T(y_{2k+1}, x_{2k+1})$ for $k = 0, 1, 2....$
Then

$$\begin{aligned}
&d(x_{2k+1}, x_{2k+2}) = d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1}))\\
&\leq \alpha\frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}{2} + \beta\frac{d(x_{2k}, S(x_{2k}, y_{2k}))d(x_{2k+1}, T(x_{2k+1}, y_{2k+1}))}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})} +\\
&\quad \gamma\frac{d(x_{2k+1}, S(x_{2k}, y_{2k}))d(x_{2k}, T(x_{2k+1}, y_{2k+1}))}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}\\
&= \alpha\frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}{2} + \beta\frac{d(x_{2k}, x_{2k+1})d(x_{2k+1}, x_{2k+2})}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})} + \gamma\frac{d(x_{2k+1}, x_{2k+1})d(x_{2k}, x_{2k+2})}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}\\
&= \alpha\frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}{2} + \beta\frac{d(x_{2k}, x_{2k+1})d(x_{2k+1}, x_{2k+2})}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}\\
&\leq \alpha\frac{d(x_{2k}, x_{2k+1})}{2} + \alpha\frac{d(y_{2k}, y_{2k+1})}{2} + \beta d(x_{2k+1}, x_{2k+2}).\\
&\Rightarrow d(x_{2k+1}, x_{2k+2}) \leq \frac{\alpha}{2(1 - \beta)}d(x_{2k}, x_{2k+1}) + \frac{\alpha}{2(1 - \beta)}d(y_{2k}, y_{2k+1}).
\end{aligned}$$

Similarly

$$d(y_{2k+1}, y_{2k+2}) \leq \frac{\alpha}{2(1-\beta)} d(y_{2k}, y_{2k+1}) + \frac{\alpha}{2(1-\beta)} d(x_{2k}, x_{2k+1}).$$

Adding we get,

$$[d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2})] \leq \frac{\alpha}{1-\beta}[d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})] = h[d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}],$$

where $0 < h = \frac{\alpha}{1-\beta} < 1$.
Also,

$$d(x_{2k+2}, x_{2k+3}) \leq \frac{\alpha}{2(1-\beta)} d(x_{2k+1}, x_{2k+2}) + \frac{\alpha}{2(1-\beta)} d(y_{2k+1}, y_{2k+2})$$

and

$$d(y_{2k+2}, y_{2k+3}) \leq \frac{\alpha}{2(1-\beta)} d(y_{2k+1}, y_{2k+2}) + \frac{\alpha}{2(1-\beta)} d(x_{2k+1}, x_{2k+2}).$$

Adding, we get

$$[d(x_{2k+2}, x_{2k+3}) + d(y_{2k+2}, y_{2k+3})] \leq \frac{\alpha}{1-\beta}[d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2})]$$
$$= h[d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2})].$$

Therefore,

$$(d(x_n, x_{n+1}) + d(y_n, y_{n+1})) \leq h(d(x_{n-1}, x_n) + d(y_{n-1}, y_n)) \leq ... \leq h^n(d(x_0, x_1) + d(y_0, y_1)).$$

Now, if

$$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) = \delta_n, \ then \ \delta_n \leq h\delta_{n-1} \leq ... \leq h^n \delta_0.$$

For $m > n$, we have

$$\begin{aligned}
&(d(x_n, x_m) + d(y_n, y_m)) \leq s(d(x_n, x_{n+1}) + d(y_n, y_{n+1})) + ... + s^{m-n}(d(x_{m-1}, x_m) + d(y_{m-1}, y_m)) \\
&\leq sh^n\delta_0 + s^2h^{n+1}\delta_0 + ... + s^{m-n}h^{m-1}\delta_0 \\
&< sh^n[1 + (sh) + (sh)^2 + ...]\delta_0 \\
&= \frac{sh^n}{1-sh}\delta_0 \to 0 \ as \ n \to \infty.
\end{aligned}$$

This shows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence in X. Since X is a complete b-metric space, there exists $x, y \in X$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.
Now, we show that $x = S(x, y)$ and $y = S(y, x)$.
We suppose on the contrary that $x \neq S(x, y)$ and $y \neq S(y, x)$ so that
$d(x, S(x, y)) = l_1 > 0$ and $d(y, S(y, x)) = l_2 > 0$.
Consider

$$\begin{aligned}
&l_1 = d(x, S(x, y)) \leq s[d(x, x_{2k+2}) + d(x_{2k+2}, S(x, y))] \\
&\leq sd(x, x_{2k+2}) + sd(T(x_{2k+1}, y_{2k+1}), S(x, y)) \\
&\leq sd(x, x_{2k+2}) + s\alpha\frac{d(x_{2k+1}, x) + d(y_{2k+1}, y)}{2} + s\beta\frac{d(x, S(x, y))d(x_{2k+1}, T(x_{2k+1}, y_{2k+1}))}{1 + d(x_{2k+1}, x) + d(y_{2k+1}, y)} \\
&\qquad + s\gamma\frac{d(x_{2k+1}, S(x, y))d(x, T(x_{2k+1}, y_{2k+1}))}{1 + d(x_{2k+1}, x) + d(y_{2k+1}, y)} \\
&\leq sd(x, x_{2k+2}) + s\alpha\frac{d(x_{2k+1}, x) + d(y_{2k+1}, y)}{2} + s\beta\frac{d(x, S(x, y))d(x_{2k+1}, x_{2k+2})}{1 + d(x_{2k+1}, x) + d(y_{2k+1}, y)} \\
&\qquad + s\gamma\frac{d(x_{2k+1}, S(x, y))d(x, x_{2k+2})}{1 + d(x_{2k+1}, x) + d(y_{2k+1}, y)}.
\end{aligned}$$

By taking $k \to \infty$, we get

$l_1 \leq 0$, which is a contradiction.

Therefore, $d(x, S(x, y)) = 0$.

That is, $x = S(x, y)$.

Similarly, one can prove that $y = S(y, x)$.

It follows similarly that $x = T(x, y)$ and $y = T(y, x)$.

So we have proved that (x, y) is a common coupled fixed point of S and T.

We now show that S and T have a unique common coupled fixed point.

Uniqueness: Let $(x^*, y^*) \in X \times X$ be another common coupled fixed point of S and T.

Then,

$$
\begin{aligned}
&d(x, x^*) = d(S(x, y), T(x^*, y^*)) \\
&\leq \alpha \frac{d(x, x^*) + d(y, y^*)}{2} + \beta \frac{d(x, S(x, y))d(x^*, T(x^*, y^*))}{(1 + d(x, x^*) + d(y, y^*))} + \gamma \frac{d(x^*, S(x, y))d(x, T(x^*, y^*))}{(1 + d(x, x^*) + d(y, y^*))} \\
&= \alpha \frac{d(x, x^*) + d(y, y^*)}{2} + \beta \frac{d(x, x)d(x^*, x^*)}{(1 + d(x, x^*) + d(y, y^*))} + \gamma \frac{d(x^*, x)d(x, x^*)}{(1 + d(x, x^*) + d(y, y^*))}. \\
&\Rightarrow d(x^*, x^*) \leq \frac{\alpha}{2} d(x, x^*) + \frac{\alpha}{2} d(y, y^*) + \gamma d(x, x^*). \\
&\Rightarrow d(x, x^*) \leq \frac{\alpha}{2 - \alpha - 2\gamma} d(y, y^*).
\end{aligned}
$$

Similarly, one can easily prove that

$$d(y, y^*) \leq \frac{\alpha}{2 - \alpha - 2\gamma} d(x, x^*).$$

Adding, we get

$$
\begin{aligned}
&d(x, x^*) + d(y, y^*) \leq \frac{\alpha}{2 - \alpha - 2\gamma} [d(x, x^*) + d(y, y^*)]. \\
&\Rightarrow (2 - 2\alpha - 2\gamma)(d(x, x^*) + d(y, y^*)) \leq 0. \\
&\Rightarrow d(x, x^*) + d(y, y^*) = 0. \\
&\Rightarrow x = x^* \textit{ and } y = y^*.
\end{aligned}
$$

□

Corollary 2.2. *Let (X, d) be a complete b-metric space with parameter $s \geq 1$ and let the mapping $T : X \times X \to X$ satisfy*

$$d(T(x, y), T(u, v)) \leq \alpha \frac{d(x, u) + d(y, v)}{2} + \beta \frac{d(x, T(x, y))d(u, T(u, v))}{(1 + d(x, u) + d(y, v))} + \gamma \frac{d(u, T(x, y))d(x, T(u, v))}{(1 + d(x, u) + d(y, v))},$$

for all $x, y, u, v \in X$ and $\alpha, \beta, \gamma \geq 0$ with $s\alpha + \beta < 1$ and $\alpha + \gamma < 1$. Then T has a unique coupled fixed point in X.

Proof. Take $T = S$ in above Theorem. □

Theorem 2.3. *Let (X, d) be a complete b-metric space with parameter $s \geq 1$ and let the mappings $S, T : X \times X \to X$ satisfy*

$$d(S(x, y), T(u, v)) \leq \begin{cases} \alpha \frac{d(x,u)+d(y,v)}{2} + \beta \frac{d(x,S(x,y))d(u,T(u,v))}{s[d(x,T(u,v))+d(u,S(x,y))+d(x,u)+d(y,v)]}, & \textit{if } D \neq 0 \\ 0, & \textit{if } D = 0 \end{cases}$$

for all $x, y, u, v \in X$, where $D = D(x, y, u, v) = s[d(x, T(u, v)) + d(u, S(x, y)) + d(x, u) + d(y, v)]$ and α, β are nonnegative reals with $s(\alpha + \beta) < 1$. Then S and T have a unique common coupled fixed point.

Proof. Let x_0 and $y_0 \in X$ be arbitrary points.
Define $x_{2k+1} = S(x_{2k}, y_{2k}), y_{2k+1} = S(y_{2k}, x_{2k})$ and $x_{2k+2} = T(x_{2k+1}, y_{2k+1}), y_{2k+2} = T(y_{2k+1}, x_{2k+1})$ for $k = 0, 1, 2....$
Now, we assume that
$D_1 = D(x_{2k}, y_{2k}, x_{2k+1}, y_{2k+1}) \neq 0$ and $D_2 = D(y_{2k}, x_{2k}, y_{2k+1}, x_{2k+1}) \neq 0$.
Then,

$$\begin{aligned}
&d(x_{2k+1}, x_{2k+2}) = d(S(x_{2k}, y_{2k}, T(x_{2k+1}, y_{2k+1}))\\
&\leq \alpha \frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}{2} + \beta \frac{d(x_{2k}, S(x_{2k}, y_{2k}))d(x_{2k+1}, T(x_{2k+1}, y_{2k+1}))}{D_1}\\
&= \alpha \frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}{2} + \beta \frac{d(x_{2k}, x_{2k+1})d(x_{2k+1}, x_{2k+2})}{s[d(x_{2k}, x_{2k+2}) + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})]}\\
&\leq \alpha \frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}{2} + \beta d(x_{2k}, x_{2k+1}).\\
&\Rightarrow d(x_{2k+1}, x_{2k+2}) \leq \frac{\alpha + 2\beta}{2} d(x_{2k}, x_{2k+1}) + \frac{\alpha}{2} d(y_{2k}, y_{2k+1}).
\end{aligned}$$

Similarly, one can easily prove that

$$d(y_{2k+1}, y_{2k+2}) \leq \frac{\alpha + 2\beta}{2} d(y_{2k}, y_{2k+1}) + \frac{\alpha}{2} d(x_{2k}, x_{2k+1}).$$

Adding, we get

$$[d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2})] \leq (\alpha + \beta)[d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})].$$

Now, if
$D_3 = D(x_{2k+2}, y_{2k+2}, x_{2k+1}, y_{2k+1}) \neq 0,$
we get

$$\begin{aligned}
&d(x_{2k+2}, x_{2k+3}) = d(T(x_{2k+1}, y_{2k+1}), S(x_{2k+2}, y_{2k+2}))\\
&\leq \alpha \frac{d(x_{2k+2}, x_{2k+1}) + d(y_{2k+2}, y_{2k+1})}{2} + \beta \frac{d(x_{2k+2}, S(x_{2k+2}, y_{2k+2}))d(x_{2k+1}, T(x_{2k+1}, y_{2k+1}))}{D_3}\\
&= \alpha \frac{d(x_{2k+2}, x_{2k+1}) + d(y_{2k+2}, y_{2k+1})}{2} + \beta \frac{d(x_{2k+2}, x_{2k+3})d(x_{2k+1}, x_{2k+2})}{s[d(x_{2k+1}, x_{2k+3}) + d(x_{2k+2}, x_{2k+1}) + d(y_{2k+2}, y_{2k+1})]}\\
&\leq \alpha \frac{d(x_{2k+2}, x_{2k+1}) + d(y_{2k+2}, y_{2k+1})}{2} + \beta d(x_{2k+1}, x_{2k+2}).\\
&\Rightarrow d(x_{2k+2}, x_{2k+3}) \leq \frac{\alpha + 2\beta}{2} d(x_{2k+1}, x_{2k+2}) + \frac{\alpha}{2} d(y_{2k+1}, y_{2k+2}).
\end{aligned}$$

Similarly, if $D_4 = D(y_{2k+2}, x_{2k+2}, y_{2k+1}, x_{2k+1}) \neq 0$, one can easily prove that

$$d(y_{2k+2}, y_{2k+3}) \leq \frac{\alpha + 2\beta}{2} d(y_{2k+1}, y_{2k+2}) + \frac{\alpha}{2} d(x_{2k+1}, x_{2k+2}).$$

Again adding, we get

$$[d(x_{2k+2}, x_{2k+3}) + d(y_{2k+2}, y_{2k+3})] \leq (\alpha + \beta)[d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2})].$$

Therefore,

$$[d(x_n, x_{n+1}) + d(y_n, y_{n+1})] \leq (\alpha + \beta)[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)] = h[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)],$$

where $h = (\alpha + \beta) < 1$.
Now, if $d(x_n, x_{n+1}) + d(y_n, y_{n+1}) = \delta_n$, then

$$\delta_n \le h\delta_{n-1} \le ... \le h^n \delta_0.$$

For $m > n$, we have

$$\begin{aligned}
&(d(x_n, x_m) + d(y_n, y_m)) \le s(d(x_n, x_{n+1}) + d(y_n, y_{n+1})) + ... + s^{m-n}(d(x_{m-1}, x_m) + d(y_{m-1}, y_m))\\
&\le sh^n\delta_0 + s^2h^{n+1}\delta_0 + ... + s^{m-n}h^{m-1}\delta_0\\
&< sh^n[1 + (sh) + (sh)^2 + ...]\delta_0\\
&= \frac{sh^n}{1 - sh}\delta_0 \to 0 \; as \; n \to \infty.
\end{aligned}$$

This shows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence in X. Since X is a complete b-metric space, there exists $x, y \in X$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.
Now we show that $x = S(x, y)$ and $y = S(y, x)$.
We suppose on the contrary that $x \ne S(x, y)$ and $y \ne S(y, x)$ so that
$d(x, S(x, y)) = l_1 > 0$ and $d(y, S(y, x)) = l_2 > 0$.
Consider

$$\begin{aligned}
&l_1 = d(x, S(x, y)) \le s[d(x, x_{2k+2}) + d(x_{2k+2}, S(x, y))]\\
&\le sd(x, x_{2k+2}) + sd(T(x_{2k+1}, y_{2k+1}), S(x, y))\\
&\le sd(x, x_{2k+2}) + s\alpha\frac{d(x_{2k+1}, x) + d(y_{2k+1}, y)}{2} +\\
&\quad \beta\frac{d(x, S(x, y))d(x_{2k+1}, T(x_{2k+1}, y_{2k+1}))}{d(x_{2k+1}, S(x, y)) + d(x, T(x_{2k+1}, y_{2k+1})) + d(x_{2k+1}, x) + d(y_{2k+1}, y)}\\
&= sd(x, x_{2k+2}) + s\alpha\frac{d(x_{2k+1}, x) + d(y_{2k+1}, y)}{2} + \beta\frac{d(x, S(x, y))d(x_{2k+1}, x_{2k+2})}{d(x_{2k+1}, S(x, y)) + d(x, x_{2k+2}) + d(x_{2k+1}, x) + d(y_{2k+1}, y)}.
\end{aligned}$$

By taking $k \to \infty$, we get
$l_1 \le 0$, which is a contradiction.
Therefore, $d(x, S(x, y)) = 0$.
That is, $x = S(x, y)$.
Similarly, one can prove that $y = S(y, x)$.
It follows similarly that $x = T(x, y)$ and $y = T(y, x)$.
So we have proved that (x, y) is a common coupled fixed point of S and T.
We now show that S and T have a unique common coupled fixed point.
Uniqueness: Let $(x^*, y^*) \in X \times X$ be another common coupled fixed point of S and T.
Then,

$$\begin{aligned}
&d(x, x^*) = d(S(x, y), T(x^*, y^*))\\
&\le \alpha\frac{d(x, x^*) + d(y, y^*)}{2} + \beta\frac{d(x, S(x, y))d(x^*, T(x^*, y^*))}{s[(d(x, T(x^*, y^*)) + d(x^*, S(x, y)) + d(x, x^*) + d(y, y^*))]}\\
&= \alpha\frac{d(x, x^*) + d(y, y^*)}{2} + \beta\frac{d(x, x)d(x^*, x^*)}{s[3d(x, x^*) + d(y, y^*)]}\\
&\Rightarrow d(x^*, x^*) \le \frac{\alpha}{2}d(x, x^*) + \frac{\alpha}{2}d(y, y^*).\\
&\Rightarrow d(x, x^*) \le \frac{\alpha}{2 - \alpha}d(y, y^*).
\end{aligned}$$

Similarly, one can easily prove that

$$d(y,y^*) \le \frac{\alpha}{2-\alpha} d(x,x^*).$$

Adding, we get

$$d(x,x^*) + d(y,y^*) \le \frac{\alpha}{2-\alpha}[d(x,x^*) + d(y,y^*)].$$
$$\Rightarrow (2-2\alpha)(d(x,x^*) + d(y,y^*)) \le 0.$$
$$\Rightarrow d(x,x^*) + d(y,y^*) = 0.$$
$$\Rightarrow x = x^* \textit{ and } y = y^*.$$

We have obtained the existence and uniqueness of a common coupled fixed point if
$D_1, D_2, D_3, D_4 \neq 0$ for all $k \in \mathbb{N}$.
Now, assume that $D_1 = 0$ for some $k \in \mathbb{N}$.
That is,

$$s[d(x_{2k}, x_{2k+2}) + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})] = 0.$$
$$\Rightarrow x_{2k} = x_{2k+1} = x_{2k+2} \textit{ and } y_{2k} = y_{2k+1}.$$

If $D_2 \neq 0$, we get

$$d(y_{2k+1}, y_{2k+2}) = d(S(y_{2k}, x_{2k}), T(y_{2k+1}, x_{2k+1}) = 0.$$

That is, $y_{2k+1} = y_{2k+2}$(this equality holds if $D_2 = 0$).
The equalities
$x_{2k} = x_{2k+1} = x_{2k+2}$ and $y_{2k} = y_{2k+1} = y_{2k+2}$ ensures that (x_{2k+1}, y_{2k+1}) is a unique common coupled fixed point of S and T. The same holds if either $D_2 = 0$, $D_3 = 0$, or $D_4 = 0$. □

From above theorem, we obtain following corollary by taking $S = T$.

Corollary 2.4. *Let (X, d) be a complete b-metric space with parameter $s \ge 1$ and let the mappings $T : X \times X \to X$ satisfy*

$$d(T(x,y), T(u,v)) \le \begin{cases} \alpha \frac{d(x,u)+d(y,v)}{2} + \beta \frac{d(x,T(x,y))d(u,T(u,v))}{s[d(x,T(u,v))+d(u,T(x,y))+d(x,u)+d(y,v)]}, & \textit{if } D \neq 0 \\ 0, & \textit{if } D = 0 \end{cases}$$

for all $x, y, u, v \in X$, where $D = D(x,y,u,v) = s[d(x,T(u,v)) + d(u,T(x,y)) + d(x,u) + d(y,v)]$ and α, β are nonnegative reals with $s(\alpha + \beta) < 1$. Then T has a unique common coupled fixed point.

Now, we furnish a nontrivial example to support the result of Theorem 2.1.

Example 2.5. Let $X = \{0, 1\}$. Consider a b-metric $d : X \times X \to \Re$ defined as $d(x,y) = \frac{2}{3}(x-y)^2$ for all $x, y \in X$. Then (X, d) is a $b-$metric space with parameter $s = 2$. Define $S, T : X \times X \to X$ as follows:

$$S(x,y) = \frac{xy}{4} \textit{ and } T(x,y) = \frac{xy}{3},$$

for all $x, y \in X$. It can be easily verified that the maps S and T satisfy the contractive condition of Theorem 2.1 with $\alpha = \frac{3}{8}, \beta = \frac{1}{5}$ and $\gamma = \frac{2}{5}$. Observe that the point $(0, 0)$ is a unique common coupled fixed point of S and T.

References

[1] M. Akkouchi, *A common fixed point theorem for expansive mappings under strict implicit conditions on b-metric spaces*, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math., **50** (2011), 5–15. 1

[2] H. Aydi, M. F. Bota, E. Karapinar, S. Mitrović, *A fixed point theorem for set-valued quasi-contractions in b-metric spaces*, Fixed Point Theory Appl., **2012** (2012), 8 pp. 1.1, 1.5

[3] I. A. Bakhtin, *The contraction mapping principle in quasimetric spaces*, Functional Analysis, **30** (1989), 26–37. 1

[4] M. Boriceanu, *Fixed point theory for multivalued generalized contraction on a set with two b-metrics*, Studia Univ. Babes-Bolyai Math., **LIV (3)** (2009), 1–14. 1, 1.2, 1.3, 1.6, 1.7, 1.8

[5] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostraviensis, **1** (1993), 5–11. 1

[6] T. Gnana Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Analysis: TMA, **65** (2006), 1379–1393. 1, 1.10

[7] N. Hussain, D. Dorić, Z. Kadelburg, S. Radenović, *Suzuki-type fixed point results in metric type spaces*, Fixed Point Theory Appl., **2012** (2012), 12 pp. 1

[8] M. Jovanović, Z. Kadelburg, S. Radenović, *Common fixed point results in metric-type spaces*, Fixed Point Theory Appl., **2010** (2010), 15 pp. 1

[9] M. Kir, H. Kiziltunc, *On some well known fixed point theorems in b-metric spaces*, Turkish J. Anal. Number Theory, **1** (2013), 13–16. 1

[10] M. O. Olatinwo, C. O. Imoru, *A generalization of some results on multi-valued weakly Picard mappings in b-metric space*, Fasciculi Mathematici, **40** (2008), 45–56. 1

[11] M. Păcurar, *A fixed point result for ϕ-contractions on b- metric spaces without the boundedness assumption*, Fasc. Math., **43** (2010), 127–137. 1