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Additive ρ-functional inequalities

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Abstract

In this paper, we solve the additive ρ-functional inequalities

$$\|f(x+y)-f(x)-f(y)\| \leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\|, \tag{1}$$

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\| \leq \|\rho\left(f(x+y)-f(x)-f(y)\right)\|, \tag{2}$$

where ρ is a fixed non-Archimedean number with $|\rho| < 1$ or ρ is a fixed complex number with $|\rho| < 1$. Using the direct method, we prove the Hyers-Ulam stability of the additive ρ-functional inequalities (1) and (2) in non-Archimedean Banach spaces and in complex Banach spaces, and prove the Hyers-Ulam stability of additive ρ-functional equations associated with the additive ρ-functional inequalities (1) and (2) in non-Archimedean Banach spaces and in complex Banach spaces.

Keywords: Hyers-Ulam stability, additive ρ-functional equation, additive ρ-functional inequality, non-Archimedean normed space, Banach space.
2010 MSC: 46S10, 39B62, 39B52, 47S10, 12J25.

1. Introduction and preliminaries

A *valuation* is a function $|\cdot|$ from a field K into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s| \leq |r| + |s|, \qquad \forall r, s \in K.$$

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A field K is called a *valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \le \max\{|r|, |s|\}, \qquad \forall r, s \in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1| = |-1| = 1$ and $|n| \le 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

Definition 1.1. ([11]) Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0, \infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:
(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|rx\| = |r|\|x\| \qquad (r \in K, x \in X)$;
(iii) the strong triangle inequality

$$\|x+y\| \le \max\{\|x\|, \|y\|\}, \qquad \forall x, y \in X$$

holds. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

Definition 1.2. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer N such that

$$\|x_n - x_m\| \le \varepsilon$$

for all $n, m \ge N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\|x_n - x\| \le \varepsilon$$

for all $n \ge N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space*.

The stability problem of functional equations originated from a question of Ulam [16] concerning the stability of group homomorphisms.

The functional equation

$$f(x+y) = f(x) + f(y)$$

is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [10] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [14] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation

$$f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$$

is called the *Jensen equation*. See [2, 3, 4, 13] for more information on functional equations.

In [8], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \le \|f(xy)\| \tag{3}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

See also [6, 15]. Gilányi [9] and Fechner [5] proved the Hyers-Ulam stability of the functional inequality (3). Park, Cho and Han [12] proved the Hyers-Ulam stability of additive functional inequalities.

In Section 2, we solve the additive functional inequality (1) and prove the Hyers-Ulam stability of the additive functional inequality (1) in non-Archimedean Banach spaces. We moreover prove the Hyers-Ulam stability of an additive functional equation associated with the functional inequality (1) in non-Archimedean Banach spaces.

In Section 3, we solve the additive functional inequality (2) and prove the Hyers-Ulam stability of the additive functional inequality (2) in non-Archimedean Banach spaces. We moreover prove the Hyers-Ulam stability of an additive functional equation associated with the functional inequality (2) in non-Archimedean Banach spaces.

In Section 4, we solve the additive functional inequality (1) and prove the Hyers-Ulam stability of the additive functional inequality (1) in complex Banach spaces. We moreover prove the Hyers-Ulam stability of an additive functional equation associated with the functional inequality (1) in complex Banach spaces.

In Section 5, we solve the additive functional inequality (2) and prove the Hyers-Ulam stability of the additive functional inequality (2) in complex Banach spaces. We moreover prove the Hyers-Ulam stability of an additive functional equation associated with the functional inequality (2) in complex Banach spaces.

2. Additive ρ-functional inequality (1) in non-Archimedean Banach spaces

Throughout Sections 2 and 3, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|2| \neq 1$ and let ρ be a non-Archimedean number with $|\rho| < 1$.

We solve and investigate the additive ρ-functional inequality (1) in non-Archimedean normed spaces.

Lemma 2.1. *A mapping $f : X \to Y$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \le \left\| \rho \left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right) \right\| \tag{4}$$

for all $x, y \in X$ if and only if $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (4).

Letting $x = y = 0$ in (4), we get

$$\|f(0)\| \le 0.$$

So $f(0) = 0$.

Letting $y = x$ in (4), we get

$$\|f(2x) - 2f(x)\| \le 0$$

and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{5}$$

for all $x \in X$.

It follows from (4) and (5) that

$$\|f(x+y) - f(x) - f(y)\| \le \left\| \rho \left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right) \right\| = |\rho| \|f(x+y) - f(x) - f(y)\|$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

The converse is obviously true.

Corollary 2.2. *A mapping $f : X \to Y$ satisfies*

$$f(x+y) - f(x) - f(y) = \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) \tag{6}$$

for all $x, y \in X$ if and only if $f : X \to Y$ is additive.

We prove the Hyers-Ulam stability of the additive ρ-functional inequality (4) in non-Archimedean Banach spaces.

Theorem 2.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping such that*

$$\Psi(x,y) := \sum_{j=1}^{\infty} |2|^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty, \tag{7}$$

$$\|f(x+y) - f(x) - f(y)\| \le \left\|\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| + \varphi(x,y) \tag{8}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{|2|}\Psi(x,x) \tag{9}$$

for all $x \in X$.

Proof. Letting $y = x$ in (8), we get

$$\|f(2x) - 2f(x)\| \le \varphi(x,x) \tag{10}$$

for all $x \in X$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x \in X$. Hence

$$\begin{aligned}
&\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \\
&\le \max\left\{\left\|2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|\right\} \\
&= \max\left\{|2|^l \left\|f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |2|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right)\right\|\right\} \\
&\le \sum_{j=l}^{\infty} |2|^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right)
\end{aligned} \tag{11}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (11) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a non-Archimedean Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (11), we get (9).

Now, let $T : X \to Y$ be another additive mapping satisfying (9). Then we have

$$\begin{aligned}
\|A(x) - T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\
&\le \max\left\{\left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\|, \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\|\right\} \\
&\le |2|^{q-1}\Psi\left(\frac{x}{2^q}, \frac{x}{2^q}\right),
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A.

It follows from (7) and (8) that

$$\begin{aligned} \|A(x+y) - A(x) - A(y)\| &= \lim_{n\to\infty} \left\| 2^n \left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\ &\le \lim_{n\to\infty} \left\| 2^n \rho \left(2f\left(\frac{x+y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\ &+ \lim_{n\to\infty} |2|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\ &= \left\| \rho \left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\| \end{aligned}$$

for all $x, y \in X$. So

$$\|A(x+y) - A(x) - A(y)\| \le \left\| \rho \left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\|$$

for all $x, y \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive.

Corollary 2.4. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|f(x+y) - f(x) - f(y)\| \le \left\| \rho \left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| + \theta(\|x\|^r + \|y\|^r) \tag{12}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{|2|^r - |2|}\|x\|^r \tag{13}$$

for all $x \in X$.

Theorem 2.5. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying (8) and*

$$\Psi(x, y) := \sum_{j=0}^{\infty} \frac{1}{|2|^j}\varphi(2^j x, 2^j y) < \infty \tag{14}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{|2|}\Psi(x, x) \tag{15}$$

for all $x \in X$.

Proof. It follows from (10) that

$$\left\| f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{|2|}\varphi(x, x)$$

for all $x \in X$. Hence

$$\begin{aligned} &\left\| \frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| \\ &\le \max\left\{ \left\| \frac{1}{2^l}f\left(2^l x\right) - \frac{1}{2^{l+1}}f\left(2^{l+1}x\right)\right\|, \cdots, \left\| \frac{1}{2^{m-1}}f\left(2^{m-1}x\right) - \frac{1}{2^m}f\left(2^m x\right)\right\| \right\} \\ &= \max\left\{ \frac{1}{|2|^l}\left\| f\left(2^l x\right) - \frac{1}{2}f\left(2^{l+1}x\right)\right\|, \cdots, \frac{1}{|2|^{m-1}}\left\| f\left(2^{m-1}x\right) - \frac{1}{2}f\left(2^m x\right)\right\| \right\} \\ &\le \sum_{j=l}^{\infty} \frac{1}{|2|^{j+1}}\varphi(2^j x, 2^j x) \end{aligned} \tag{16}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (16) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (16), we get (15).

The rest of the proof is similar to the proof of Theorem 2.3.

Corollary 2.6. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying (12). Then there exists a unique additive mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \le \frac{2\theta}{|2| - |2|^r} \|x\|^r \tag{17}$$

for all $x \in X$.

Let $A(x, y) := f(x + y) - f(x) - f(y)$ and $B(x, y) := \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)$ for all $x, y \in X$.

For $x, y \in X$ with $\|A(x, y)\| \le \|B(x, y)\|$,

$$\|A(x, y)\| - \|B(x, y)\| \le \|A(x, y) - B(x, y)\|.$$

For $x, y \in X$ with $\|A(x, y)\| > \|B(x, y)\|$,

$$\begin{aligned}
\|A(x, y)\| &= \|A(x, y) - B(x, y) + B(x, y)\| \\
&\le \max\{\|A(x, y) - B(x, y)\|, \|B(x, y)\|\} \\
&= \|A(x, y) - B(x, y)\| \\
&\le \|A(x, y) - B(x, y)\| + \|B(x, y)\|,
\end{aligned}$$

since $\|A(x, y)\| > \|B(x, y)\|$. So we have

$$\begin{aligned}
&\|f(x + y) - f(x) - f(y)\| - \left\|\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| \\
&\le \left\|f(x + y) - f(x) - f(y) - \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\|.
\end{aligned}$$

As corollaries of Theorems 2.3 and 2.5, we obtain the Hyers-Ulam stability results for the additivec ρ-functional equation (6) in non-Archimedean Banach spaces.

Corollary 2.7. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying (7) and*

$$\left\|f(x + y) - f(x) - f(y) - \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| \le \varphi(x, y) \tag{18}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (9).

Corollary 2.8. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\left\|f(x + y) - f(x) - f(y) - \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| \le \theta(\|x\|^r + \|y\|^r) \tag{19}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (13).

Corollary 2.9. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying (14) and (18). Then there exists a unique additive mapping $A : X \to Y$ satisfying (15).*

Corollary 2.10. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying (19). Then there exists a unique additive mapping $A : X \to Y$ satisfying (17).*

3. Additive ρ-functional inequality (2) in non-Archimedean Banach spaces

We solve and investigate the additive ρ-functional inequality (2) in non-Archimedean normed spaces.

Lemma 3.1. *A mapping $f : X \to Y$ satisfis $f(0) = 0$ and*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \quad \leq \quad \|\rho\left(f(x+y) - f(x) - f(y)\right)\| \tag{20}$$

for all $x, y \in X$ if and if $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (20).

Letting $y = 0$ in (20), we get

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \leq 0 \tag{21}$$

and so $f\left(\frac{x}{2}\right) - \frac{1}{2}f(x)$ for all $x \in X$.

It follows from (20) and (21) that

$$\|f(x+y) - f(x) - f(y)\| = \left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \leq |\rho| \|f(x+y) - f(x) - f(y)\|$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

The converse is obviously true.

Corollary 3.2. *A mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = \rho(f(x+y) - f(x) - f(y)) \tag{22}$$

for all $x, y \in X$ if and only if $f : X \to Y$ is additive.

Now, we prove the Hyers-Ulam stability of the additive ρ-functional inequality (20) in non-Archimedean Banach spaces.

Theorem 3.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\Psi(x, y) : \quad = \quad \sum_{j=0}^{\infty} |2|^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty, \tag{23}$$

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \quad \leq \quad \|\rho\left(f(x+y) - f(x) - f(y)\right)\| + \varphi(x, y) \tag{24}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \Psi(x, 0) \tag{25}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (24), we get

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| = \left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \leq \varphi(x, 0) \tag{26}$$

for all $x \in X$. So

$$\begin{aligned}
&\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \qquad (27)\\
&\quad\le \max\left\{\left\|2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\quad= \max\left\{|2|^l \left\|f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |2|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\quad\le \sum_{j=l}^{\infty} |2|^j \varphi\left(\frac{x}{2^j}, 0\right)
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (27) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a non-Archimedean Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (27), we get (25).

The rest of the proof is similar to the proof of Theorem 2.3.

Corollary 3.4. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \le \|\rho(f(x+y) - f(x) - f(y))\| + \theta(\|x\|^r + \|y\|^r) \qquad (28)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{|2|^r\theta}{|2|^r - |2|}\|x\|^r \qquad (29)$$

for all $x \in X$.

Theorem 3.5. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (24) and*

$$\Psi(x,y) := \sum_{j=1}^{\infty} \frac{1}{|2|^j}\varphi(2^j x, 2^j y) < \infty \qquad (30)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \Psi(x, 0) \qquad (31)$$

for all $x \in X$.

Proof. It follows from (26) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{|2|}\varphi(2x, 0)$$

for all $x \in X$. Hence

$$\begin{aligned}
&\left\|\frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x)\right\| \qquad (32)\\
&\quad\le \max\left\{\left\|\frac{1}{2^l} f\left(2^l x\right) - \frac{1}{2^{l+1}} f\left(2^{l+1}x\right)\right\|, \cdots, \left\|\frac{1}{2^{m-1}} f\left(2^{m-1}x\right) - \frac{1}{2^m} f\left(2^m x\right)\right\|\right\}\\
&\quad= \max\left\{\frac{1}{|2|^l}\left\|f\left(2^l x\right) - \frac{1}{2} f\left(2^{l+1}x\right)\right\|, \cdots, \frac{1}{|2|^{m-1}}\left\|f\left(2^{m-1}x\right) - \frac{1}{2} f\left(2^m x\right)\right\|\right\}\\
&\quad\le \sum_{j=l+1}^{\infty} \frac{1}{|2|^j}\varphi(2^j x, 0)
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (32) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (32), we get (31).

The rest of the proof is similar to the proof of Theorem 2.3.

Corollary 3.6. *Let $r > 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (28). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{|2|^r \theta}{|2| - |2|^r} \|x\|^r \tag{33}$$

for all $x \in X$.

Let $A(x, y) := 2f\left(\frac{x+y}{2}\right) - f(x) - f(y)$ and $B(x, y) := \rho\left(f(x + y) - f(x) - f(y)\right)$ for all $x, y \in X$.
For $x, y \in X$ with $\|A(x, y)\| \le \|B(x, y)\|$,

$$\|A(x, y)\| - \|B(x, y)\| \le \|A(x, y) - B(x, y)\|.$$

For $x, y \in X$ with $\|A(x, y)\| > \|B(x, y)\|$,

$$\begin{aligned} \|A(x, y)\| &= \|A(x, y) - B(x, y) + B(x, y)\| \\ &\le \max\{\|A(x, y) - B(x, y)\|, \|B(x, y)\|\} \\ &= \|A(x, y) - B(x, y)\| \\ &\le \|A(x, y) - B(x, y)\| + \|B(x, y)\|, \end{aligned}$$

since $\|A(x, y)\| > \|B(x, y)\|$. So we have

$$\begin{aligned} &\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| - \left\|\rho\left(f(x + y) - f(x) - f(y)\right)\right\| \\ &\le \left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho\left(f(x + y) - f(x) - f(y)\right)\right\|. \end{aligned}$$

As corollaries of Theorems 3.3 and 3.5, we obtain the Hyers-Ulam stability results for the additivec ρ-functional equation (22) in non-Archimedean Banach spaces.

Corollary 3.7. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (23) and*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho\left(f(x + y) - f(x) - f(y)\right)\right\| \le \varphi(x, y) \tag{34}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (25).

Corollary 3.8. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho\left(f(x + y) - f(x) - f(y)\right)\right\| \le \theta(\|x\|^r + \|y\|^r) \tag{35}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (29).

Corollary 3.9. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (30) and (34). Then there exists a unique additive mapping $A : X \to Y$ satisfying (31).*

Corollary 3.10. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (35). Then there exists a unique additive mapping $A : X \to Y$ satisfying (33).*

4. Additive ρ-functional inequality (1) in complex Banach spaces

Throughout Sections 4 and 5, assume that X is a complex normed space and that Y is a complex Banach space. Let ρ be a complex number with $|\rho| < 1$.

We solve and investigate the additive ρ-functional inequality (1) in complex normed spaces.

Lemma 4.1. *A mapping $f : X \to Y$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \le \left\|\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| \tag{36}$$

for all $x, y \in X$ if and only if $f : X \to Y$ is additive.

Proof. The proof is similar to the proof of Lemma 2.1.

Corollary 4.2. *A mapping $f : X \to Y$ satisfies*

$$f(x+y) - f(x) - f(y) = \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) \tag{37}$$

for all $x, y \in X$ if and only if $f : X \to Y$ is additive.

We prove the Hyers-Ulam stability of the additive ρ-functional inequality (36) in complex Banach spaces.

Theorem 4.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping such that*

$$\Psi(x, y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty, \tag{38}$$

$$\|f(x+y) - f(x) - f(y)\| \le \left\|\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| + \varphi(x, y) \tag{39}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2}\Psi(x, x) \tag{40}$$

for all $x \in X$.

Proof. Letting $y = x$ in (39), we get

$$\|f(2x) - 2f(x)\| \le \varphi(x, x) \tag{41}$$

for all $x \in X$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x \in X$. Hence

$$\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \le \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \le \sum_{j=l}^{m-1} 2^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \tag{42}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (42) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (42), we get (40).

Now, let $T : X \to Y$ be another additive mapping satisfying (40). Then we have

$$\begin{aligned}
\|A(x) - T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\
&\le \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| \\
&\le 2^q \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}\right),
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A.

It follows from (38) and (39) that

$$\begin{aligned}
\|A(x+y) - A(x) - A(y)\| &= \lim_{n\to\infty} \left\|2^n \left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\
&\le \lim_{n\to\infty} \left\|2^n \rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\
&+ \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\
&= \left\|\rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\|
\end{aligned}$$

for all $x, y \in X$. So

$$\|A(x+y) - A(x) - A(y)\| \le \left\|\rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\|$$

for all $x, y \in X$. By Lemma 4.1, the mapping $A : X \to Y$ is additive.

Corollary 4.4. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|f(x+y) - f(x) - f(y)\| \le \left\|\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| + \theta(\|x\|^r + \|y\|^r) \tag{43}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{2^r - 2}\|x\|^r \tag{44}$$

for all $x \in X$.

Theorem 4.5. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying (39) and*

$$\Psi(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y) < \infty \tag{45}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2}\Psi(x, x) \tag{46}$$

for all $x \in X$.

Proof. It follows from (41) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{2}\varphi(x, x)$$

for all $x \in X$. Hence

$$\left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f\left(2^j x\right) - \frac{1}{2^{j+1}} f\left(2^{j+1} x\right) \right\| \leq \sum_{j=l}^{m-1} \frac{1}{2^{j+1}} \varphi(2^j x, 2^j x) \tag{47}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (47) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (47), we get (46).

The rest of the proof is similar to the proof of Theorem 4.3.

Corollary 4.6. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying* (43). *Then there exists a unique additive mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \leq \frac{2\theta}{2 - 2^r} \|x\|^r \tag{48}$$

for all $x \in X$.

By the triangle inequality, we have

$$\begin{aligned} &\|f(x+y) - f(x) - f(y)\| - \left\| \rho \left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right) \right\| \\ &\leq \left\| f(x+y) - f(x) - f(y) - \rho \left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right) \right\|. \end{aligned}$$

As corollaries of Theorems 4.3 and 4.5, we obtain the Hyers-Ulam stability results for the additivec ρ-functional equation (37) in complex Banach spaces.

Corollary 4.7. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying* (38) *and*

$$\left\| f(x+y) - f(x) - f(y) - \rho \left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right) \right\| \leq \varphi(x, y) \tag{49}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (40).

Corollary 4.8. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\left\| f(x+y) - f(x) - f(y) - \rho \left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right) \right\| \leq \theta(\|x\|^r + \|y\|^r) \tag{50}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (44).

Corollary 4.9. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying* (45) *and* (49). *Then there exists a unique additive mapping $A : X \to Y$ satisfying* (46).

Corollary 4.10. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying* (50). *Then there exists a unique additive mapping $A : X \to Y$ satisfying* (48).

Remark 4.11. If ρ is a real number such that $-1 < \rho < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

5. Additive ρ-functional inequality (2) in complex Banach spaces

We solve and investigate the additive ρ-functional inequality (2) in complex normed spaces.

Lemma 5.1. *A mapping $f : X \to Y$ satisfis $f(0) = 0$ and*

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \leq \| \rho \left(f(x+y) - f(x) - f(y) \right) \| \tag{51}$$

for all $x, y \in X$ if and if $f : X \to Y$ is additive.

Proof. The proof is similar to the proof of Lemma 3.1.

Corollary 5.2. *A mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = \rho(f(x+y) - f(x) - f(y)) \tag{52}$$

for all $x, y \in X$ if and only if $f : X \to Y$ is additive.

Now, we prove the Hyers-Ulam stability of the additive ρ-functional inequality (51) in complex Banach spaces.

Theorem 5.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\Psi(x, y) : \;=\; \sum_{j=0}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty, \tag{53}$$

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \leq \| \rho \left(f(x+y) - f(x) - f(y) \right) \| + \varphi(x, y) \tag{54}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \Psi(x, 0) \quad \forall x \in X. \tag{55}$$

Proof. Letting $y = 0$ in (54), we get

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| = \left\| 2f\left(\frac{x}{2}\right) - f(x) \right\| \leq \varphi(x, 0) \tag{56}$$

for all $x \in X$. So

$$\left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| \leq \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \leq \sum_{j=l}^{m-1} 2^j \varphi\left(\frac{x}{2^j}, 0\right) \tag{57}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (57) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (57), we get (55).

The rest of the proof is similar to the proof of Theorem 4.3.

Corollary 5.4. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \leq \|\rho(f(x+y) - f(x) - f(y))\| + \theta(\|x\|^r + \|y\|^r) \tag{58}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2^r\theta}{2^r - 2}\|x\|^r \tag{59}$$

for all $x \in X$.

Theorem 5.5. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$,* (54) *and*

$$\Psi(x, y) := \sum_{j=1}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y) < \infty \tag{60}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \Psi(x, 0) \tag{61}$$

for all $x \in X$.

Proof. It follows from (56) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \leq \frac{1}{2}\varphi(2x, 0)$$

for all $x \in X$. Hence

$$\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| \leq \sum_{j=l+1}^{m}\left\|\frac{1}{2^j}f\left(2^j x\right) - \frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\| \leq \sum_{j=l+1}^{m}\frac{1}{2^j}\varphi(2^j x, 0) \tag{62}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (62) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (62), we get (61).

The rest of the proof is similar to the proof of Theorem 4.3.

Corollary 5.6. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and* (58)*. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2^r\theta}{2 - 2^r}\|x\|^r \tag{63}$$

for all $x \in X$.

By the triangle inequality, we have

$$\begin{aligned}&\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\| - \|\rho\left(f(x+y) - f(x) - f(y)\right)\| \\ &\leq \left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho\left(f(x+y) - f(x) - f(y)\right)\right\|.\end{aligned}$$

As corollaries of Theorems 5.3 and 5.5, we obtain the Hyers-Ulam stability results for the additivec ρ-functional equation (52) in complex Banach spaces.

Corollary 5.7. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$,* (53) *and*

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho\left(f(x+y) - f(x) - f(y)\right) \right\| \leq \varphi(x, y) \tag{64}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (55)*.*

Corollary 5.8. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho\left(f(x+y) - f(x) - f(y)\right) \right\| \leq \theta(\|x\|^r + \|y\|^r) \tag{65}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (59)*.*

Corollary 5.9. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$,* (60) *and* (64)*. Then there exists a unique additive mapping $A : X \to Y$ satisfying* (61)*.*

Corollary 5.10. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and* (65)*. Then there exists a unique additive mapping $A : X \to Y$ satisfying* (63)*.*

Remark 5.11. If ρ is a real number such that $-1 < \rho < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

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