Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 933–939
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Fixed points and dynamics on generating function of Genocchi numbers

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Communicated by S.-H. Rim

Abstract

Recently, there have been many works related with dynamics of various functions. In this paper, singular values and fixed points of generating function of Genocchi numbers, $g_\lambda(z) = \lambda \frac{2z}{e^z+1}$, $\lambda(\in \mathbb{R}) > 1$, are investigated. It is shown that the function $g_\lambda(z)$ has infinitely many singular values and its critical values lie in the left half plane and one point on the real axis in the right half plane. Further, the real fixed points of $g_\lambda(z)$ and their nature are determined. Finally, we provide numerical evidence of the existence of chaotic phenomena by illustrating bifurcation diagrams of system and by calculating the Lyapunov exponent.

Keywords: Fixed point, Genocchi number, chaos.
2010 MSC: 30D05, 37C25, 37F45, 58K05.

1. Introduction

The singular values and fixed points play a crucial role in the dynamics or iteration of functions. The theory of dynamics or iteration of functions can be seen in [1, 9]. In recent decades, there have been many works related with dynamics of various functions. Prasad investigated the dynamics of entire functions $\lambda \frac{\sinh(z)}{z}$, $\lambda > 0$ with infinitely many bounded singular values in [6]. And the dynamics of function $\lambda \frac{e^z-1}{z}$, $\lambda > 0$ is investigated by Kapoor and Prasad [3]. Further, the dynamics of certain transcendental meromorphic functions with unbounded singular values was explored by Nayak and Prasad [5]. Especially in [8], Sajid and Alsuwaiyan studied chaotic behavior in the real dynamics of a one parameter family of non linear function $\lambda \frac{xe^x}{x-1}$, $\lambda > 0$, $x \in \mathbb{R}\ \{1\}$. Sajid also discussed singular values and fixed points of generating function of Bernoulli's numbers [7].

In this paper, we consider one parameter family of function $\lambda \frac{2z}{e^z+1}$ and determine its singular values and fixed points. The motivation of the present work comes from the fact that the function $\frac{2z}{e^z+1}$ is a generating

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Received 2015-10-09

funcion of the Genocchi numbers. These Genocchi numbers $G_0 = 1$, $G_1 = -1$, $G_2 = 0$, $G_3 = -3$, $G_4 = 0$, $G_5 = 17$, $\cdots$ are coefficients in the series expansion

$$\frac{2z}{e^z+1} = \sum_{k=0}^{\infty} G_k \frac{z^k}{k!}, \quad |z| < \pi.$$

Recently, many authors have investigated works related with these numbers [2, 4].

Let

$$\mathcal{G} = \left\{ g_\lambda(z) = \lambda \frac{2z}{e^z+1} \;\middle|\; 1 < \lambda (\in \mathbb{R}), z \in \mathbb{C} \right\}$$

be one parameter family of transcendental functions. The function is a transcendental function. Moreover, the function is neither even nor odd and not periodic.

Before going into the main topics, we introduce the basic definitions which are needed in the sequel: A point z^* is said to be a critical point of $f(z)$ if $f'(z^*) = 0$. The value $f(z^*)$ corresponding to a critical point z^* is called a critical value of $f(z)$. A point $w \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is said to be an asymptotic value for $f(z)$, if there exists a continuous curve $\gamma : [0, \infty) \to \hat{\mathbb{C}}$ satisfying $\lim_{t\to\infty} \gamma(t) = \infty$ and $\lim_{t\to\infty} f(\gamma(t)) = w$. A singular value of f is defined to be either a critical value or an asymptotic value of f. A function f is called critically bounded or functions of bounded type if the set of all singular values of f is bounded, otherwise unbounded-type. A point z is said to be a fixed point of function $f(z)$ if $f(z) = z$. A fixed point z_0 is called an attracting, neutral (indifferent) or repelling if $|f'(z_0)| < 1$, $|f'(z_0)| = 1$ or $|f'(z_0)| > 1$, respectively. In addition to, the dynamics of the function changes when the parameter value crosses through a certain point. Each of such a change is called a bifurcation. Lyapunov exponent, for kth iterate x_k of the function $f_\lambda(x)$, is defined as:

$$L = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |f'_\lambda(x_k)|. \tag{1.1}$$

The main object of this paper is to investigate the dynamic properties of the function $g_\lambda \in \mathcal{G}$. In this context, the paper is organized as follows. In Section 2, we show that the function $g_\lambda \in \mathcal{G}$ has infinitely many singular values. Further, it is shown that all the critical values of $g_\lambda(z)$ lie in the left half plane and one point on the real axis in the right half plane. Moreover, the real fixed points of $g_\lambda \in \mathcal{G}$ and their nature are examined in Section 3. Finally, in Section 4, we provide numerical evidence of the existence of chaotic phenomena by illustrating bifurcation diagrams of system and by calculating the Lyapunov exponent.

2. Singular Values of $g_\lambda \in \mathcal{G}$

In this section, it is proved that the function $g_\lambda \in \mathcal{G}$ has infinitely many singular values and all the critical values of $g_\lambda(z)$ lie in the left half plane and one point on the real axis in the right half plane. The following theorem gives the function $g_\lambda(z)$ has infinitely many singular values.

Theorem 2.1. *Let $g_\lambda \in \mathcal{G}$. Then, the function $g_\lambda(z)$ possesses infinitely many singular values.*

Proof. For critical points, $g'_\lambda(z) = \lambda \frac{2(1-z)e^z+2}{(e^z+1)^2} = 0$. This gives the equation $(z-1)e^z - 1 = 0$. The real and imaginary parts of this equation are

$$\begin{aligned} &(1-x)\cos y + y \sin y + e^{-x} = 0, \\ &x = y \cot y - 1. \end{aligned} \tag{2.1}$$

From these equations, we have

$$\frac{y}{\sin y} - 2y \sin y - 2\cos y - e^{1-y\cot y} = 0. \tag{2.2}$$

It is seen that Equation (2.2) has infinitely many solutions (Figure 1). Suppose that the solution of Equation (2.2) are $\{y_n\}_{n=-\infty,n\neq 0}^{n=\infty}$. Now, from Equation (2.1), $x_n = y_n \cot y_n - 1$ for $n = \pm 1, \pm 2, \pm 3, \cdots$. Consequently, It gives that $g'_\lambda(z_n) = 0$ so $z_n = x_n + iy_n$ are critical points for $g_\lambda(z)$. The critical values of $g_\lambda(z)$ are given by $g_\lambda(z_n)$. It is observed that $g_\lambda(z_n)$ are distinct for different n. It shows that the function $g_\lambda(z)$ has infinitely many critical values.

Since $g_\lambda(z)$ tends to 0 as z tends to infinity along positive real axis, it gives that the finite asymptotic value of $g_\lambda(z)$ is 0.

Thus, it follows that the function $g_\lambda \in \mathcal{G}$ possesses infinitely many singular values. $\square$

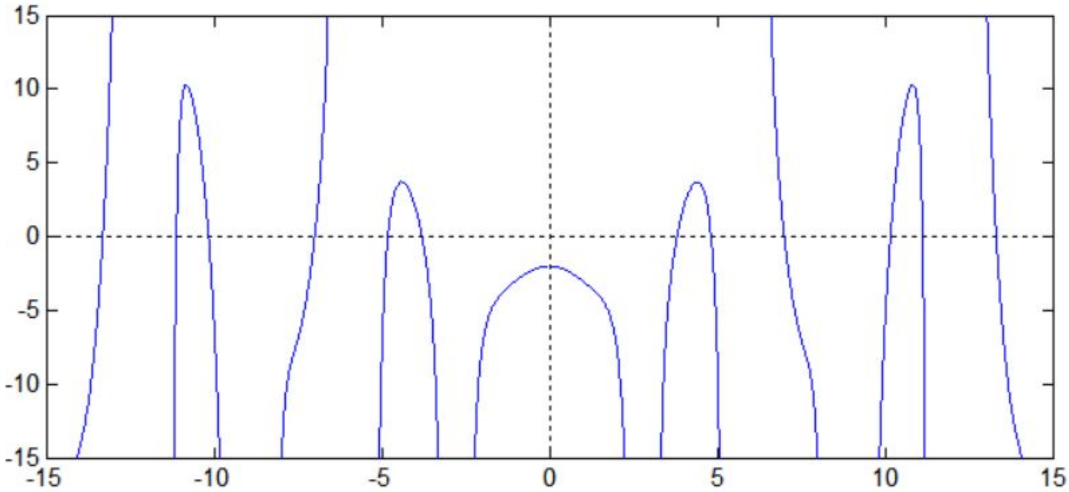


Figure 1: Graph of $\frac{y}{\sin y} - 2y\sin y - 2\cos y - e^{1-y\cot y}$.

Theorem 2.2. *Let $g_\lambda \in \mathcal{G}$. Then, the function $g'_\lambda(z)$ has no zeros in the right half plane $H^+ = \{z \in \hat{\mathbb{C}} \mid Re(z) > 0\}$ except one point on the real axis.*

Proof. Suppose $Re(z) > 0$, and $g'_\lambda(z) = \lambda \frac{2(1-z)e^z+2}{(e^z+1)^2} = 0$ which implies that $e^{-z} = z - 1$. Then,

$$\frac{\cos y - i\sin y}{e^x} = x + iy - 1. \tag{2.3}$$

When $y \neq 0$, then by imaginary part of Equation (2.3) and the fact of $|\sin y| < |y|$, $\frac{\sin y}{y} = -e^x < -1$. This is not true for $y > 0$ and for $y < 0$ because $\frac{\sin y}{y}$ is an even function.

Meanwhile, when $y = 0$, then $z = x > 0$ and, by real part of Equation (2.3), $e^x = \frac{1}{x-1}$. However, there exists unique solution x such that $e^x = \frac{1}{x-1}$.

Consequently, the function $g_\lambda(z)$ has no zeros in the right half plane H^+ except one point on the real axis. $\square$

In the following theorem, it is shown that the function $g_\lambda(z)$ maps the left half plane.

Theorem 2.3. *Let $g_\lambda \in \mathcal{G}$. Then, the function $g_\lambda(z)$ maps the left half plane $H^- = \{z \in \hat{\mathbb{C}} \mid Re(z) > 0\}$ as follows:*

$$\begin{cases} \textit{Inside the open disk centered at origin and having radius } \lambda & , \quad |z| < 1. \\ \textit{Between the closed disk centered at origin and having radius } \lambda & \\ \quad \textit{and the open disk centered at origin and having radius } 2\lambda & , \quad 1 \leq |z| < 2. \\ \textit{Outside the closed disk centered at origin and having radius } 2\lambda & , \quad 2 \leq |z|. \end{cases}$$

Proof. Consider the line segment γ is defined by $\gamma(t) = tz$, $t \in [0,1]$ and the function $h(z) = e^z$ for an arbitrary fixed $z \in H^-$. Then

$$\int_\gamma h(z)dz = \int_0^1 h(\gamma(t))\gamma'(t)dt = z\int_0^1 e^{tz}dz = e^z - 1.$$

Since $m \equiv \min_{t\in[0,1]}|h(\gamma(t))| = \min_{t\in[0,1]}|e^{tz}| > 0$ for $z \in H^-$ and

$$|e^z + 1| = \left|\int_\gamma h(z)dz + 2\right| \geq m|z| + 2 > 2 > |z|,$$

$|\frac{2z}{e^z+1}| < 2$ for $|z| < 2$. Therefore, it follows that $|g_\lambda(z)| = |\lambda\frac{2z}{e^z+1}| < |2\lambda|$ for $|z| < 2$. If $0 < |z| < 1$, then $2 > 2|z|$ and $|g_\lambda(z)| = |\lambda\frac{2z}{e^z+1}| < |\lambda|$. It proves that the function $g_\lambda(z)$ maps the left half plane H^- inside the open disk centered at origin and having radius 2λ for $|z| < 1$ and between the closed disk centered at origin and having radius λ and the open disk centered at origin and having radius 2λ for $1 \leq |z| < 2$.

Meanwhile, since $M \equiv \max_{t\in[0,1]}|h(\gamma(t))| = \max_{t\in[0,1]}|e^{tz}| < 1$ for $z \in H^-$ and

$$|e^z + 1| = \left|\int_\gamma h(z)dz + 2\right| \leq M|z| + 2 < |z| + 2 < |2z|,$$

$|\frac{2z}{e^z+1}| > 1$ for $2 \leq |z|$. Thus we have $|g_\lambda(z)| = |\lambda\frac{2z}{e^z+1}| > |\lambda|$ for $2 \leq |z|$. This shows that the function $g_\lambda(z)$ maps the left half plane H^- outside the closed disk centered at origin and having radius 2λ for $2 \leq |z|$. □

Theorem 2.4. *Let $g_\lambda \in \mathcal{G}$. Then, all the critical values of $g_\lambda(z)$ lie in the left half plane H^- and one point on the real axis in the right half plane H^+.*

Proof. By Theorem 2.2 and Theorem 2.3, it proves that all the critical values of $g_\lambda(z)$ lie in the left half plane H^- and one point on the real axis in the right half plane H^+. □

3. Nature of Real Fixed Points of $g_\lambda \in \mathcal{G}$

The existence of real fixed points and periodic points of the functions $g_\lambda(x) = \lambda g(x)$ where $g(x) = \frac{2x}{e^x+1}$ and their nature are investigated in the present section. The nature of real fixed points of the function $g_\lambda(x)$ is described here for different value of the parameter λ. Since $g_\lambda(0) = 0$ for $\lambda > 1$, the point is a fixed point of all functions $g_\lambda(x)$ for $\lambda > 1$. The non-zero real fixed points of the function $g_\lambda(x)$ are the solutions of the equation $g_\lambda(x) = x$.

The following theorem shows that the function $g_\lambda(x)$ has a unique real fixed point.

Theorem 3.1. *Let $g_\lambda \in \mathcal{G}$. Then, the function $g_\lambda(x)$ has a unique real fixed point x_λ.*

Proof. Since $g_\lambda(x) > 0$ for all $x \in \mathbb{R}$, each real fixed point of $g_\lambda \in \mathcal{G}$ is positive. The function $g'_\lambda(x) = \lambda\frac{2(1-x)e^x+2}{(e^x+1)^2}$, and hence $g'_\lambda(0) = \lambda > 1$ and $g'_\lambda(x) \to -\infty$ as $x \to \infty$. By continuity of $g'_\lambda(x)$, there is a unique $\hat{x} > 0$ such that $g'_\lambda(x) > 0$ for $0 \leq x < \hat{x}$, $g'_\lambda(\hat{x}) = 0$ and $g'_\lambda(x) < 0$ for $x > \hat{x}$. Thus $g_\lambda(x)$ increases in $[0, \hat{x})$, attains its maximum at $\hat{x}$ and decreases thereafter.

Let $h_\lambda(x) = g_\lambda(x) - x$ for $x > 0$ and $h'_\lambda(x) = g'_\lambda(x) - 1$, then we can observe that the function h_λ is increasing firstly and then decreasing in according to the value of x on $\mathbb{R}^+$. Now, $h_\lambda(0) = \lambda - 1 > 0$, $h_\lambda(x) \to -\infty$ as $x \to \infty$ and $h_\lambda(x)$ is continuous on $\mathbb{R}^+$. By the intermediate value theorem, there exists a unique positive x_λ such that $h_\lambda(x_\lambda) = 0$. It proves that $g_\lambda(x)$ has a unique positive fixed point x_λ. □

In the following theorem, the nature of fixed points of $g_\lambda(x)$ are determined:

Theorem 3.2. *Let $g_\lambda(z) \in \mathcal{G}$. Then, the fixed points of $g_\lambda(x)$ is*

(i) attracting for $0 < \lambda < \lambda^*$;
(ii) rationally indifferent for $\lambda = \lambda^*$;
(iii) repelling for $\lambda > \lambda^*$.

Proof. For sake of convenience, we consider the function $\phi(x) = xg'(x) + g(x)$ for $x \geq 0$. It means that

$$\phi(x) = x\frac{2(1-x)e^x + 2}{(e^x+1)^2} + \frac{2x}{e^x+1} = \frac{2x}{(e^x+1)^2}[(2-x)e^x + 2].$$

If $q(x) = (2-x)e^x + 2$, then $q'(x) = (1-x)e^x$ and $q''(x) = -xe^x$. It is seen that $q''(x) < 0$ for $x \in \mathbb{R}^+$. Therefore, the function $q'(x)$ is decreasing on $\mathbb{R}^+$. Since $q'(0) = 1$ and $q'(x) \to -\infty$ as $x \to \infty$, by continuity of $q'(x)$, it follows that there is a unique $\hat{x} > 0$ such that $q'(x) > 0$ for $0 \leq x < \hat{x}$, $q'(\hat{x}) = 0$ and $q'(x) < 0$ for $x > \hat{x}$. Thus $q(x)$ increases in $[0, \hat{x})$, attains its maximum at $\hat{x}$ and decreases thereafter. It ensures from the facts $q(0) = 4$ and $q(x) \to -\infty$ as $x \to \infty$ that there is a unique positive $x^* > \hat{x}$. Since $\frac{2x}{(e^x+1)^2}$ for all $x > 0$. We have

$$\phi(x) = \frac{2x}{(e^x+1)^2}q(x)\begin{cases} > 0 & \text{for} \quad 0 < x < x^*, \\ = 0 & \text{for} \quad x = x^*, \\ < 0 & \text{for} \quad x > x^*. \end{cases} \tag{3.1}$$

Since the derivative of $\frac{x}{g(x)}$ is positive for $x > 0$, the function $\frac{x}{g(x)}$ is increasing on $\mathbb{R}^+$. Using this fact, we prove the following cases for different cases for different values of parameter λ.

(i) For $0 < \lambda < \lambda^*$, since the function $\frac{x}{g(x)}$ is increasing on $\mathbb{R}^+$ and $\lambda = \frac{x_\lambda}{g(x_\lambda)}$, we have $\frac{x_\lambda}{g(x_\lambda)} < \frac{x^*}{g(x^*)}$. It means that $e^{x_\lambda} < e^{x^*}$. Hence $x_\lambda < x^*$. By (3.1), $\phi(x_\lambda) > 0$. Since $g'_\lambda(x_\lambda) = \frac{\phi(x_\lambda)}{g_\lambda} - 1$, it follows that $g'_\lambda(x_\lambda) + 1 = \frac{\phi(x_\lambda)}{g_\lambda} > 0$. Since $g'_\lambda(X)$ is negative on $\mathbb{R}^+$, it shows that $-1 < g'_\lambda(x_\lambda) < 0$ and consequently, the fixed point x_λ of $g_\lambda(x)$ is an attracting for $0 < \lambda < \lambda^*$.

(ii) For $\lambda = \lambda^*$, it is easy to prove $x_\lambda = x^*$. Now, by (3.1), it follows that $g'_\lambda(x_\lambda) + 1 = \frac{\phi(x_\lambda)}{g_\lambda} = 0$ which implies $g'_{\lambda^*}(x_\lambda) = -1$. Therefore, the fixed point x^* of $g_\lambda(x)$ is rationally indifferent for $\lambda = \lambda^*$.

(iii) For $\lambda > \lambda^*$, by similar arguments used in (i), it follows that $x_\lambda > x^*$. By (3.1) and the fact $x_\lambda > x^*$, we have $\phi(x_\lambda) < 0$. It gives that $g'_\lambda(x_\lambda) + 1 = \frac{\phi(x_\lambda)}{g_\lambda} < 0$ and hence $g'_\lambda(x_\lambda) < -1$. Therefore, x_λ is repelling fixed point of $g(x_\lambda)$ for $\lambda > \lambda^*$.

□

It is observed from Theorem 3.2 that the nature of the fixed point changes whenever parameter λ crosses parameter value λ^*. Actually for $\lambda > \lambda^*$, there may exist periodic points of period greater than or equal to 2.

4. Bifurcation Analysis via Numerical simulations

In this section, we will investigate various dynamic behaviors of $g_\lambda(x)$ by using numerical simulations. As seen in the Chapter 3, the dynamics of $g_\lambda(x)$ changes when the parameter value crosses through a certain point. In order to research various dynamical behavior, we need to draw a bifurcation diagram with respective to parameter λ. It is interesting to note that g_λ reproduces classical dynamics behaviors like quadratic maps dynamics including periodic doubling, periodic windows, chaotic region and so on, as we can see in the bifurcation diagram in Figure 2, which are drawn by using the famous software MATLAB.

In fact, if $\lambda < 5$, every initial condition is attracted, under g_λ, to a fixed point. If $\lambda = 9.3385\cdots$, we have Feigenbaum point, the beginning of "chaos". Figure 3(b) illustrates a chaotic motion.

In order to provide a quantitative measure of the degree of chaotic motion, the Lyapunov exponent is considered, which is the average loss of information during successive iterations of points near x. In fact, a trajectory with the positive Lyapunov exponent is chaotic provided that it is not asymptotic to an unstable periodic solution. Or if the Lyapunov exponent of a trajectory is negative, then it is stable. Reviewing the bifurcation diagrams in Figures 2, the corresponding Lyapunov exponents λ from 7 to 20 for $g_\lambda(x)$ are calculated in Figure 4 by using the method in Equation (1.1). Doing that yields,

$$L = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} \ln\left[\lambda \frac{|2(1-x_k)e^{x_k} + 2|}{(e^{x_k}+1)^2}\right].$$

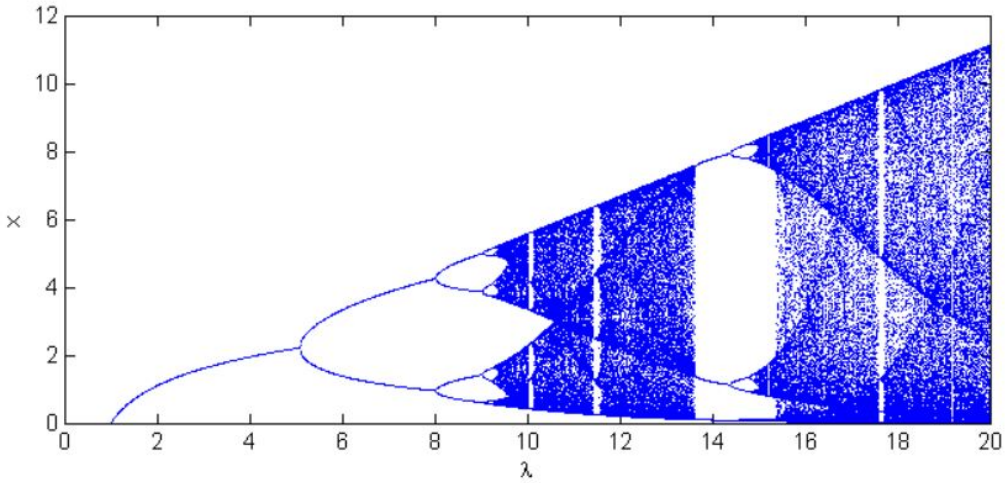
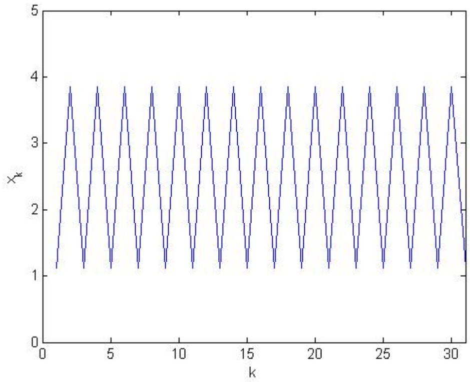
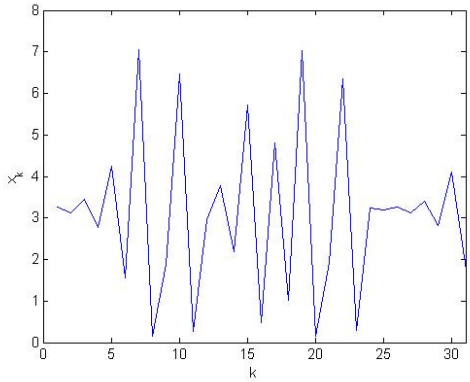


Figure 2: Bifurcation diagram of g_λ.



(a) $\lambda = 7$



(b) $\lambda = 13$

Figure 3: Evolution of the transition x_k of g_λ.

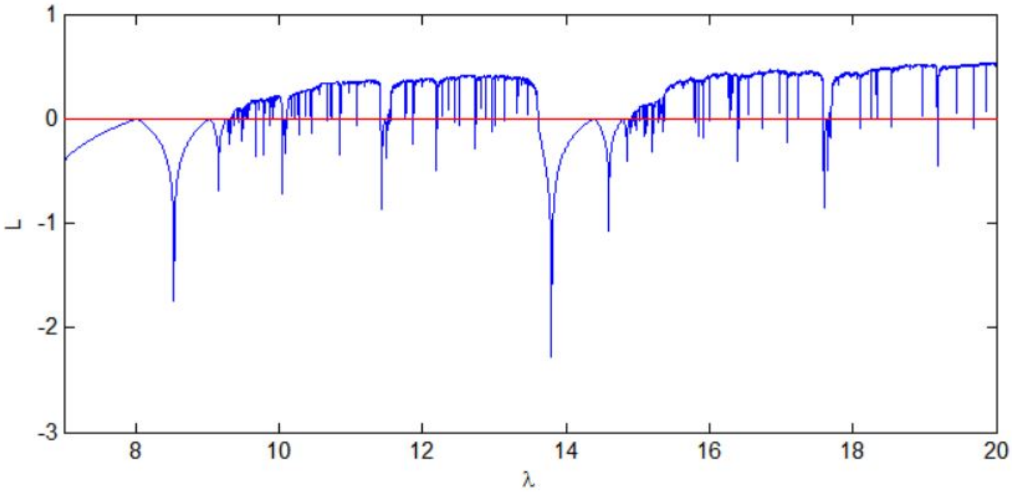


Figure 4: Lyapunov exponent of g_λ.

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