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On certain Euler difference sequence spaces of fractional order and related dual properties

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Abstract

The purpose of this paper is to generalize the Euler sequences of nonabsolute type by introducing a generalized Euler mean difference operator $E^r(\Delta^{(\tilde{\alpha})})$ of order α. We investigate their topological structures as well as some interesting results concerning the operator $E^r(\Delta^{(\tilde{\alpha})})$ for a proper fraction $\tilde{\alpha}$. Also we obtain the α-, β- and γ-duals of these sets.

Keywords: Euler sequence spaces of nonabsolute type, linear operator, matrix transformations, α-, β- and γ-duals.
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1. Introduction

By $\Gamma(\tilde{\alpha})$, we denote the *Euler gamma function* of a real number $\tilde{\alpha}$. Using the definition, $\Gamma(\tilde{\alpha})$ with $\tilde{\alpha} \notin \{0, -1, -2, -3 \dots\}$ can be expressed as an improper integral as follows:

$$\Gamma(\tilde{\alpha}) = \int_0^\infty e^{-t} t^{\tilde{\alpha}-1} dt. \tag{1.1}$$

Also, the Euler gamma function is known as the generalized factorial function. Let w be the set of all sequences of real numbers and ℓ_∞, c and c_0 respectively be the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $\|x\| = \sup_k |x_k|$. By bs, cs, ℓ_1 and ℓ_p we denote the spaces of all bounded, convergent, absolutely and $p-$absolutely convergent series, respectively.

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For a positive proper fraction $\tilde{\alpha}$, Baliarsingh and Dutta [6, 7] (also, see [8, 13]) have defined the generalized fractional difference operator $\Delta^{(\tilde{\alpha})}$ as

$$\Delta^{(\tilde{\alpha})}(x_k) = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\tilde{\alpha}+1)}{i!\Gamma(\tilde{\alpha}-i+1)} x_{k-i}. \tag{1.2}$$

Throughout the text it is assumed that the series defined in (1.2) is convergent for $x \in w$. More specifically, it is convenient to express the difference operator $\Delta^{(\tilde{\alpha})}$ as a triangle i.e.,

$$(\Delta^{(\tilde{\alpha})})_{nk} = \begin{cases} (-1)^{(n-k)} \frac{\Gamma(\tilde{\alpha}+1)}{(n-k)!\Gamma(\tilde{\alpha}-n+k+1)}, & (0 \leq k \leq n) \\ 0, & (k > n) \end{cases}$$

In fact, this difference matrix includes several difference matrices introduced by Ahmad and Mursaleen [1], Malkowsky et al. [18] and many others (see[5, 10, 14, 15, 16, 17, 19, 20, 23]).

The well known Euler mean matrix $E^r = (e^r_{nk})$ of order $r, (0 < r < 1)$ is defined by the matrix

$$e^r_{nk} = \begin{cases} \binom{n}{k}(1-r)^{n-k} r^k, & (k \leq n) \\ 0, & (k > n). \end{cases}$$

Equivalently, we may write

$$E^r = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1-r & r & 0 & 0 & \dots \\ (1-r)^2 & 2(1-r)r & r^2 & 0 & \dots \\ (1-r)^3 & 3(1-r)^2 r & 3(1-r)r^2 & r^3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Combining the Euler mean matrix of order r and the difference matrix of order $\tilde{\alpha}$, we define the product matrix $E^r(\Delta^{(\tilde{\alpha})})$ as

$$(E^r(\Delta^{(\tilde{\alpha})}))_{nk} = \begin{cases} \sum_{i=k}^{n}(-1)^{i-k}\binom{n}{n-i}\frac{\Gamma(\tilde{\alpha}+1)}{(i-k)!\Gamma(\tilde{\alpha}-i+k+1)} r^i(1-r)^{n-i}, & (0 \leq k \leq n) \\ 0, & (k > n) \end{cases}$$

Moreover, $(E^r(\Delta^{(\tilde{\alpha})}))_{nk}$ can be written as follows:

$$(E^r(\Delta^{(\tilde{\alpha})}))_{nk} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ (1-r)-\tilde{\alpha}r & r & 0 & 0 & \cdots \\ (1-r)^2 - 2\tilde{\alpha}(1-r)r + \frac{\tilde{\alpha}(\tilde{\alpha}-1)}{2!}r^2 & 2(1-r)r - \tilde{\alpha}r^2 & r^2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}_0$, the set of all natural numbers including zero. For the sequence spaces X and Y, we write a matrix mapping $A : X \to Y$ defined by

$$(Ax)_n = \sum_k a_{nk}x_k, \quad (n \in \mathbb{N}_0). \tag{1.3}$$

For every $x = (x_k) \in X$, we call Ax the $A-$transform of x if the series $\sum_k a_{nk}x_k$ converges for each $n \in \mathbb{N}_0$. By (X, Y), we denote the class of all infinite matrices A such that $A : X \to Y$. Thus, $A \in (X, Y)$ if and only if the series in (1.3) converges for each $n \in \mathbb{N}_0$.

The matrix domain λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A = \{x = (x_k) \in w : Ax \in \lambda\},$$

In fact, for most cases the new sequence space λ_A generated by the limitation matrix A can be expressed as either an expansion or contraction of the original space λ. In some cases they may overlap each others.

It is known that the duality is an important concept in the theory of sequence spaces, especially for studying of topological structures of sequence spaces. Let $X, Y \subset w$ and define the set $S(X, Y)$ by

$$S(X, Y) = \{z = (z_k) : xz = (x_k z_k) \in Y \text{ for all } x \in X\}. \tag{1.4}$$

With the notation of (1.4), we redefine the $\alpha-, \beta-$ and $\gamma-$duals of a sequence space X respectively as follows:

$$X^\alpha = S(X, \ell_1), \quad X^\beta = S(X, cs) \text{ and } X^\gamma = S(X, bs).$$

Now, we give the following results involving the inverse of the matrices $\Delta^{(\tilde{\alpha})}, E^r$ and $E^r(\Delta^{(\tilde{\alpha})})$

Lemma 1.1 ([9, 12])**.** *The inverse of the difference matrix $\Delta^{(\tilde{\alpha})}$ is given by the triangle*

$$(\Delta^{(-\tilde{\alpha})})_{nk} = \begin{cases} (-1)^{(n-k)} \frac{\Gamma(-\tilde{\alpha}+1)}{(n-k)!\Gamma(-\tilde{\alpha}-n+k+1)}, & (0 \le k \le n), \\ 0, & (k > n). \end{cases}$$

Lemma 1.2. *The inverse of the Euler mean matrix E^r is given by the triangle*

$$(E^{1/r})_{nk} = \begin{cases} (-1)^{(n-k)} \binom{n}{k} (1-r)^{n-k} r^{-n}, & (0 \le k \le n), \\ 0, & (k > n). \end{cases}$$

Proof. Proof is straightforward (see [12]). □

Lemma 1.3. *The inverse of the Euler mean difference matrix $E^r(\Delta^{(\tilde{\alpha})})$ is given by a triangle (b_{nk}), where*

$$b_{nk} = \begin{cases} \sum_{j=k}^{n} (-1)^{(n-k)} \binom{j}{k} \frac{\Gamma(-\tilde{\alpha}+1)(1-r)^{j-k} r^{-j}}{(n-j)!\Gamma(-\tilde{\alpha}-n+j+1)}, & (0 < k \le n), \\ \frac{1}{r^n}, & (k = n), \\ 0, & (k > n). \end{cases}$$

Proof. Proof follows from Lemma 1.1 and Lemma 1.2. □

2. New Euler difference sequence spaces

In this section, we define certain sequence spaces of non absolute type $e_p^r(\Delta^{(\tilde{\alpha})})$, $e_0^r(\Delta^{(\tilde{\alpha})})$,$e_c^r(\Delta^{(\tilde{\alpha})})$ and $e_\infty^r(\Delta^{(\tilde{\alpha})})$ by combining the Euler mean operator E^r and the fractional difference operator $\Delta^{(\tilde{\alpha})}$. We also investigate their certain topological properties. In addition, the $\alpha-, \beta-$ and $\gamma-$duals of these spaces have been determined.

For a positive real number $\tilde{\alpha}$ and $0 < r < 1$, we define certain classes of Euler fractional difference sequence spaces as

$$e_p^r(\Delta^{(\tilde{\alpha})}) = \left\{ x = (x_k) : \sum_n \left| \sum_{j=0}^{n} \sum_{i=j}^{n} (-1)^{i-j} \binom{n}{n-i} \frac{\Gamma(\tilde{\alpha}+1)}{(i-j)!\Gamma(\tilde{\alpha}-i+j+1)} r^i (1-r)^{n-i} x_j \right|^p < \infty \right\}$$

$$e_0^r(\Delta^{(\tilde{\alpha})}) = \left\{ x = (x_k) : \lim_{n\to\infty} \sum_{j=0}^{n} \sum_{i=j}^{n} (-1)^{i-j} \binom{n}{n-i} \frac{\Gamma(\tilde{\alpha}+1)}{(i-j)!\Gamma(\tilde{\alpha}-i+j+1)} r^i (1-r)^{n-i} x_j = 0 \right\}$$

$$e_c^r(\Delta^{(\tilde{\alpha})}) = \left\{ x = (x_k) : \lim_{n\to\infty} \sum_{j=0}^{n} \sum_{i=j}^{n} (-1)^{i-j} \binom{n}{n-i} \frac{\Gamma(\tilde{\alpha}+1)}{(i-j)!\Gamma(\tilde{\alpha}-i+j+1)} r^i (1-r)^{n-i} x_j \text{ exists} \right\}$$

$$e_\infty^r(\Delta^{(\tilde{\alpha})}) = \left\{ x = (x_k) : \sup_n \left| \sum_{j=0}^{n} \sum_{i=j}^{n} (-1)^{i-j} \binom{n}{n-i} \frac{\Gamma(\tilde{\alpha}+1)}{(i-j)!\Gamma(\tilde{\alpha}-i+j+1)} r^i (1-r)^{n-i} x_j \right| < \infty \right\}$$

It is noted that the spaces $e_p^r(\Delta^{(\tilde{\alpha})}), e_0^r(\Delta^{(\tilde{\alpha})}), e_c^r(\Delta^{(\tilde{\alpha})})$ and $e_\infty^r(\Delta^{(\tilde{\alpha})})$ can be derived by taking $E^r(\Delta^{(\tilde{\alpha})})-$ transform of x in the spaces ℓ_p, c_0, c and ℓ_∞, respectively i.e.,

$$e_p^r(\Delta^{(\tilde{\alpha})}) = (\ell_p)_{E^r(\Delta^{(\tilde{\alpha})})}; c_0^r(\Delta^{(\tilde{\alpha})}) = (c_0)_{E^r(\Delta^{(\tilde{\alpha})})}; c^r(\Delta^{(\tilde{\alpha})}) = (c)_{E^r(\Delta^{(\tilde{\alpha})})}; \ell_\infty(\Delta^{(\tilde{\alpha})}) = (\ell_\infty)_{E^r(\Delta^{(\tilde{\alpha})})}$$

Keeping the above new sets in mind, we define the sequence $y = (y_k)$, which is used as the $E^r(\Delta^{(\tilde{\alpha})})-$transform of a sequence $x = (x_k)$ i.e.,

$$y_k = \sum_{j=0}^{k} \sum_{i=j}^{k} (-1)^{i-j} \binom{k}{k-i} \frac{\Gamma(\tilde{\alpha}+1)}{(i-j)!\Gamma(\tilde{\alpha}-i+j+1)} r^i (1-r)^{k-i} x_j, \quad (k \in \mathbb{N}_0). \tag{2.1}$$

In particular, the above new spaces include the following special cases:

(i) For $\tilde{\alpha} = 0$, above classes reduce to the classes defined by Altay and Başar [2, 3].

(ii) For $\tilde{\alpha} = 1$, above classes reduce to the classes defined by Altay and Polat [4].

(iii) For $\tilde{\alpha} = m \in \mathbb{N}$, above classes reduce to the classes defined by Polat and Başar [21].

We will now give some interesting results of these spaces concerning their topological structures, bases and duals.

Theorem 2.1. *For a positive proper fraction $\tilde{\alpha}$, the sequence space $e_p^r(\Delta^{(\tilde{\alpha})})$ is a complete normed linear space with co-ordinate wise addition and scalar multiplication which is a BK-space with the norm*

$$\|x\|_{e_p^r(\Delta^{(\tilde{\alpha})})} = \|E^r(\Delta^{(\tilde{\alpha})})x\|_p \quad (1 \le p < \infty).$$

Also, the sequence spaces $e_0^r(\Delta^{(\tilde{\alpha})}), e_c^r(\Delta^{(\tilde{\alpha})})$ and $e_\infty^r(\Delta^{(\tilde{\alpha})})$ are complete normed linear spaces with co-ordinate wise addition and scalar multiplication, moreover BK-space with the norm

$$\|x\|_{e_c^r(\Delta^{(\tilde{\alpha})})} = \|x\|_{e_\infty^r(\Delta^{(\tilde{\alpha})})} = \|E^r(\Delta^{(\tilde{\alpha})})x\|_\infty.$$

Proof. The proof is straightforward, hence omitted. □

Theorem 2.2. *For a positive proper fraction $\tilde{\alpha}$, the sequence spaces $e_p^r(\Delta^{(\tilde{\alpha})}), e_0^r(\Delta^{(\tilde{\alpha})})$, $e_c^r(\Delta^{(\tilde{\alpha})})$ and $e_\infty^r(\Delta^{(\tilde{\alpha})})$ are linearly isomorphic to the classical spaces ℓ_p, c_0, c and ℓ_∞, respectively.*

Proof. We prove the theorem for the space $e_\infty^r(\Delta^{(\tilde{\alpha})})$. We show that there exists a linear bijection between the spaces $e_\infty^r(\Delta^{(\tilde{\alpha})})$ and ℓ_∞. With the notation (2.1) we define a mapping $T : e_\infty^r(\Delta^{(\tilde{\alpha})}) \to \ell_\infty$ by $x \mapsto y = Tx$. Clearly, T is linear. If $Tx = \theta = (0, 0, 0, \dots)$, then $x = \theta$; therefore, T is injective. Let $y \in \ell_\infty$ and using Lemma 1.3, define a sequence $x = (x_k)$ via y_k as

$$x_k = \sum_{i=0}^{k} \sum_{j=i}^{k} (-1)^{(k-i)} \binom{j}{i} \frac{\Gamma(-\tilde{\alpha}+1)(1-r)^{j-i} r^{-j}}{(k-j)!\Gamma(-\tilde{\alpha}-k+j+1)} y_i, \quad (k \in \mathbb{N}_0). \tag{2.2}$$

Then, we have

$$\sup_n \left| \sum_{j=0}^{n} \sum_{i=j}^{n} (-1)^{i-j} \binom{n}{n-i} \frac{\Gamma(\tilde{\alpha}+1)}{(i-j)!\Gamma(\tilde{\alpha}-i+j+1)} r^i (1-r)^{n-i} x_j \right| = \sup_n |y_n| = \|y\|_\infty < \infty.$$

Thus, we obtain that $x \in e_\infty^r(\Delta^{(\tilde{\alpha})})$ and consequently T is surjective. This completes the proof. □

Theorem 2.3. *Let* $\lambda_k = (E^r(\Delta^{(\tilde{\alpha})})x)_k$ *for all* $k \in \mathbb{N}_0$. *Now for fixed* $k \in \mathbb{N}_0$ *define the sequence* $b_n^{(k)} = \{b_n^{(k)}\}_{n\in\mathbb{N}_0}$ *by*

$$b_n^{(k)} = \begin{cases} \frac{1}{r^n}, & (k=n), \\ \sum_{j=k}^{n} (-1)^{(n-k)} \binom{j}{k} \frac{\Gamma(-\tilde{\alpha}+1)(1-r)^{j-k}r^{-j}}{(n-j)!\Gamma(-\tilde{\alpha}-n+j+1)}, & (0 \le k \le n), \\ 0, & (k>n), \end{cases}$$

for all $n, k \in \mathbb{N}_0$. *Then*

(i) *The sequence* $\{b_n^{(k)}\}_{n\in\mathbb{N}_0}$ *is a basis for the space* $e_0^r(\Delta^{(\tilde{\alpha})})$ *and any* $x \in e_0^r(\Delta^{(\tilde{\alpha})})$ *has a unique representation in the form*

$$x = \sum_k \lambda_k b^{(k)}.$$

(ii) *The set* $\{z, \beta^{(k)}\}$ *is a basis for the space* $e_c^r(\Delta^{(\tilde{\alpha})})$ *and any* $x \in e_c^r(\Delta^{(\tilde{\alpha})})$ *has a unique representation in the form*

$$x = lz + \sum_k (\lambda_k - l) b^{(k)},$$

where $l = \lim_{k\to\infty} \lambda_k$ *and* $z = (z_k)$, *defined by*

$$z_k = \sum_{i=0}^{k} \sum_{j=i}^{k} (-1)^{(k-i)} \binom{j}{i} \frac{\Gamma(-\tilde{\alpha}+1)(1-r)^{j-i}r^{-j}}{(k-j)!\Gamma(-\tilde{\alpha}-k+j+1)}.$$

3. Dual properties

In this section, we formulate and prove theorems determining the α-, β-, and γ-duals of the Euler sequence spaces of nonabsolute type.

It is well known that $\{\ell_\infty\}^\beta = \ell_1$ and $\{\ell_p\}^\beta = \ell_q$ where $1 \le p < \infty$ and $p^{-1} + q^{-1} = 1$. We shall throughout denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$. We begin with quoting lemmas, due to Stieglitz and Tietz [22], that are needed in proving the next theorems.

Lemma 3.1. **(i)** $A = (a_{nk}) \in (c_0 : \ell_1) = (c : \ell_1)$ *if and only if*

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} a_{nk} \right| < \infty. \tag{3.1}$$

(ii) $A = (a_{nk}) \in (c_0 : c)$ *if and only if*

$$\lim_{n\to\infty} a_{nk} = \ell_k \text{ for all } k, \quad \text{and} \tag{3.2}$$

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty. \tag{3.3}$$

(iii) $A = (a_{nk}) \in (c_0 : \ell_\infty)$ *if and only if* (3.3) *holds.*

Theorem 3.2. *Define the sets $h^{\tilde{\alpha}}_{\infty}(r), h^{\tilde{\alpha}}(r)$ and $h^{\tilde{\alpha}}_{p}(r)$ as follows:*

$$h^{\tilde{\alpha}}_{\infty}(r) := \left\{ x=(x_k)\in\omega : \sup_{k\in\mathbb{N}} \sum_n \left| \sum_{j=k}^{n} \frac{(-1)^{n-k}\Gamma(1-\tilde{\alpha})}{(n-j)!\ \Gamma(1-\tilde{\alpha}-n+j)} \binom{j}{k}(1-r)^{j-k}r^{-j}x_j \right| < \infty \right\},$$

$$h^{\tilde{\alpha}}(r) := \left\{ x=(x_k)\in\omega : \sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K}\sum_{j=k}^{n} \frac{(-1)^{n-k}\Gamma(1-\tilde{\alpha})}{(n-j)!\ \Gamma(1-\tilde{\alpha}-n+j)} \binom{j}{k}(1-r)^{j-k}r^{-j}x_j \right| < \infty \right\},$$

$$h^{\tilde{\alpha}}_{p}(r) := \left\{ x=(x_k)\in\omega : \sup_{K\in\mathcal{F}} \sum_k \left| \sum_{n\in K}\sum_{j=k}^{n} \frac{(-1)^{n-k}\Gamma(1-\tilde{\alpha})}{(n-j)!\ \Gamma(1-\tilde{\alpha}-n+j)} \binom{j}{k}(1-r)^{j-k}r^{-j}x_j \right|^p < \infty \right\}.$$

Then $\{e^r_1(\Delta^{(\tilde{\alpha})})\}^{\alpha} = h^{\tilde{\alpha}}_{\infty}(r)$, $\{e^r_0(\Delta^{(\tilde{\alpha})})\}^{\alpha} = \{e^r_c(\Delta^{(\tilde{\alpha})})\}^{\alpha} = h^{\tilde{\alpha}}(r)$ and $\{e^r_p(\Delta^{(\tilde{\alpha})})\}^{\alpha} = h^{\tilde{\alpha}}_{p}(r)$.

Proof. Since the proof for the spaces $e^r_1(\Delta^{(\tilde{\alpha})})$ and $e^r_p(\Delta^{(\tilde{\alpha})})$ is obtained by analogy, we consider only the spaces $e^r_0(\Delta^{(\tilde{\alpha})})$ and $e^r_c(\Delta^{(\tilde{\alpha})})$.

Let $x=(x_n)\in\omega$ consider the matrix $B^{\tilde{\alpha}}_r = (b^{\tilde{\alpha}}_{nk}(r))$ defined by

$$b^{\tilde{\alpha}}_{nk}(r) = \begin{cases} \sum_{j=k}^{n} \frac{(-1)^{n-k}\Gamma(1-\tilde{\alpha})}{(n-j)!\ \Gamma(1-\tilde{\alpha}-n+j)} \binom{j}{k}(1-r)^{j-k}r^{-j} & , \ (k<n), \\ 1/r^n & , \ (k=n), \\ 0 & , \ (k>n). \end{cases} \tag{3.4}$$

Bearing in mind the relation (2.2), we easily obtain that

$$x_n w_n = \sum_{k=0}^{n} \left[\sum_{j=k}^{n} \frac{(-1)^{n-k}\Gamma(1-\tilde{\alpha})}{(n-j)!\ \Gamma(1-\tilde{\alpha}-n+j)} \binom{j}{k}(1-r)^{j-k}r^{-j}x_j \right] y_k = (B^{\tilde{\alpha}}_r y)_n,\ n\in\mathbb{N}. \tag{3.5}$$

We, therefore, observe by (3.5) that $xw=(x_nw_n)\in\ell_1$ whenever $w\in e^r_0(\Delta^{(\tilde{\alpha})})$ or $e^r_c(\Delta^{(\tilde{\alpha})})$ if and only if $B^{\tilde{\alpha}}_r y\in\ell_1$ whenever $y\in c_0$ or c. Then using Lemma 3.1(i) we derive that

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K}\sum_{j=k}^{n} \frac{(-1)^{n-k}\Gamma(1-\tilde{\alpha})}{(n-j)!\ \Gamma(1-\tilde{\alpha}-n+j)} \binom{j}{k}(1-r)^{j-k}r^{-j}x_j \right| < \infty.$$

This yields that $\{e^r_0(\Delta^{(\tilde{\alpha})})\}^{\alpha} = \{e^r_c(\Delta^{(\tilde{\alpha})})\}^{\alpha} = h^{\tilde{\alpha}}(r)$. □

Theorem 3.3. *Define the sets $d^{\tilde{\alpha}}_1(r), d^{\tilde{\alpha}}_2(r)$ and $d^{\tilde{\alpha}}_3(r)$ by*

$$d^{\tilde{\alpha}}_1(r) := \left\{ (x_k)\in\omega : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \frac{(-1)^{n-k}\Gamma(1-\tilde{\alpha})}{(n-j)!\ \Gamma(1-\tilde{\alpha}-n+j)} \binom{j}{k}(1-r)^{j-k}r^{-j}x_j \right| < \infty \right\},$$

$$d^{\tilde{\alpha}}_2(r) := \left\{ (x_k)\in\omega : \lim_{n\to\infty} \sum_{j=k}^{n} \frac{(-1)^{n-k}\Gamma(1-\tilde{\alpha})}{(n-j)!\ \Gamma(1-\tilde{\alpha}-n+j)} \binom{j}{k}(1-r)^{j-k}r^{-j}x_j \ \textit{exists} \right\},$$

$$d^{\tilde{\alpha}}_3(r) := \left\{ (x_k)\in\omega : \lim_{n\to\infty} \sum_{k=0}^{n}\sum_{j=k}^{n} \frac{(-1)^{n-k}\Gamma(1-\tilde{\alpha})}{(n-j)!\ \Gamma(1-\tilde{\alpha}-n+j)} \binom{j}{k}(1-r)^{j-k}r^{-j}x_j \ \textit{exists} \right\}.$$

Then $\{e^r_0(\Delta^{(\tilde{\alpha})})\}^{\beta} = d^{\tilde{\alpha}}_1(r)\cap d^{\tilde{\alpha}}_2(r)$ and $\{e^r_c(\Delta^{(\tilde{\alpha})})\}^{\beta} = d^{\tilde{\alpha}}_1(r)\cap d^{\tilde{\alpha}}_2(r)\cap d^{\tilde{\alpha}}_3(r)$.

Proof. Since the proof can also be obtained for the space $e_c^r(\Delta^{(\tilde{\alpha})})$ in a similar way, we omit it and give only the proof for the space $e_0^r(\Delta^{(\tilde{\alpha})})$. Consider the equality

$$\begin{aligned}\sum_{k=0}^{n} x_k w_k &= \sum_{k=0}^{n}\left[\sum_{j=0}^{k}\frac{(-1)^{n-k}\Gamma(1-\tilde{\alpha})}{(n-j)!\,\Gamma(1-\tilde{\alpha}-n+j)}\binom{k}{j}(r-1)^{k-j}r^{-k}y_j\right]x_k\\ &= \sum_{k=0}^{n}\left[\sum_{j=k}^{n}\frac{(-1)^{n-k}\Gamma(1-\tilde{\alpha})}{(n-j)!\,\Gamma(1-\tilde{\alpha}-n+j)}\binom{j}{k}(1-r)^{j-k}r^{-j}x_j\right]y_k\\ &= (B_r^{\tilde{\alpha}}y)_n \end{aligned}\tag{3.6}$$

where $B_r^{\tilde{\alpha}} = (b_{nk}^{\tilde{\alpha}}(r))$ is defined in (3.4). Thus, we deduce from Lemma 3.1(ii) with (3.6) that $(x_k w_k) \in cs$ whenever $w \in e_0^r(\Delta^{(\tilde{\alpha})})$ if and only if $B_r^{\tilde{\alpha}}y \in c$ whenever $y = (y_k) \in c_0$. Therefore, we derive from (3.2) and (3.3) that

$$\lim_{n\to\infty} b_{nk}^{\tilde{\alpha}}(r) \text{ exists for each } k \in \mathbb{N} \text{ and } \sup_{n\in\mathbb{N}}\sum_{k=0}^{n}|b_{nk}^{\tilde{\alpha}}(r)| < \infty,$$

which shows that $\{e_0^r(\Delta^{(\tilde{\alpha})})\}^\beta = d_1^{\tilde{\alpha}}(r) \cap d_2^{\tilde{\alpha}}(r)$. □

Theorem 3.4. *The γ-duals of the spaces $e_0^r(\Delta^{(\tilde{\alpha})})$, $e_c^r(\Delta^{(\tilde{\alpha})})$ and $e_\infty^r(\Delta^{(\tilde{\alpha})})$ is $d_1^{\tilde{\alpha}}(r)$.*

Proof. It is natural that the present theorem may be proved by the technique used in the proofs of Theorem 3.2 and 3.3. But, we prefer to follow the classical way and give only the proof for the space $e_0^{\tilde{\alpha}}(r)$.

Let $x = (x_k) \in d_1^{\tilde{\alpha}}(r)$ and $w = (w_k) \in e_0^r(\Delta^{(\tilde{\alpha})})$. Consider the equality

$$\begin{aligned}\left|\sum_{k=0}^{n} x_k w_k\right| &= \left|\sum_{k=0}^{n}\left[\sum_{j=0}^{k}\frac{(-1)^{n-k}\Gamma(1-\tilde{\alpha})}{(n-j)!\,\Gamma(1-\tilde{\alpha}-n+j)}\binom{k}{j}(r-1)^{k-j}r^{-k}y_j\right]x_k\right|\\ &= \left|\sum_{k=0}^{n} b_{nk}^{\tilde{\alpha}}(r)y_k\right|\\ &\le \sum_{k=0}^{n}|b_{nk}^{\tilde{\alpha}}(r)|\,|y_k|\end{aligned}$$

which gives us by taking supremum over $n \in \mathbb{N}$ that

$$\sup_{n\in\mathbb{N}}\left|\sum_{k=0}^{n} x_k w_k\right| \le \sup_{n\in\mathbb{N}}\sum_{k=0}^{n}|b_{nk}^{\tilde{\alpha}}(r)|\,|y_k| \le \|y\|_\infty \sup_{n\in\mathbb{N}}\sum_{k=0}^{n}|b_{nk}^{\tilde{\alpha}}(r)| \le \infty.$$

This means that $x = (x_k) \in \{e_0^r(\Delta^{(\tilde{\alpha})})\}^\gamma$. Hence, we have

$$d_1^{\tilde{\alpha}}(r) \subset \{e_0^r(\Delta^{(\tilde{\alpha})})\}^\gamma. \tag{3.7}$$

Conversely, let $x = (x_k) \in \{e_0^r(\Delta^{(\tilde{\alpha})})\}^\gamma$ and $w \in e_0^r(\Delta^{(\tilde{\alpha})})$. Then, one can conclude that the sequence $\left(\sum_{k=0}^{n} b_{nk}^{\tilde{\alpha}}(r)y_k\right)_{n\in\mathbb{N}} \in \ell_\infty$ whenever $(x_k w_k) \in bs$. This implies that the triangle matrix $B_r^{\tilde{\alpha}} = (b_{nk}^{\tilde{\alpha}}(r))$ is in the class $(c_0 : \ell_\infty)$. Hence, the condition

$$\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}|b_{nk}^{\tilde{\alpha}}(r)| < \infty$$

is satisfied, which implies that $x = (x_k) \in d_1^{\tilde{\alpha}}(r)$. In other words,

$$\{e_0^r(\Delta^{(\tilde{\alpha})})\}^\gamma \subset d_1^{\tilde{\alpha}}(r). \tag{3.8}$$

Therefore, by combining inclusions (3.7) and (3.8), we establish that the γ-dual of the space $e_0^r(\Delta^{(\tilde{\alpha})})$ is $d_1^{\tilde{\alpha}}(r)$, which completes the proof. □

Conclusion

In this article, certain results on some Euler spaces of difference sequences of order m, ($m \in \mathbb{N}$), have been extended to the sequence spaces of positive fractional order $\tilde{\alpha}$. The results presented in this article not only generalize the earlier works done by several authors [2, 3, 4, 21] but also give a new perspective concerning the development of the Euler spaces of difference sequences. As future work we will study certain matrix transformations of Euler spaces of fractional order and Riesz mean difference sequence of fractional order.

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References

[1] Z. U. Ahmad, M. Mursaleen, *Köthe-Toeplitz duals of some new sequence spaces and their matrix maps*, Publ. Inst. Math. (Beograd), **42** (1987), 57–61. 1

[2] B. Altay, F. Başar, *On some Euler sequence spaces of non-absolute type*, Ukrainian Math. J., **57** (2005), 1–17. 2, 3

[3] B. Altay, F. Başar, M. Mursaleen, *On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞*, I. Inform. Sci., **76** (2006), 1450–1462. 2, 3

[4] B. Altay, H. Polat, *On some new Euler difference sequence spaces*, Southeast Asian Bull. Math., **30** (2006), 209–220. 2, 3

[5] C. Aydın, F. Başar, *Some new difference sequence spaces*, Appl. Math. Comput., **157** (3) (2004), 677–693. 1

[6] P. Baliarsingh, S. Dutta, *A unifying approach to the difference operators and their applications*, Bol. Soc. Paran. Mat., **33** (2015), 49–57. 1

[7] P. Baliarsingh, S. Dutta, *On the classes of fractional order difference sequence spaces and their matrix transformations*, Appl. Math. Comput., **250** (2015), 665–674. 1

[8] P. Baliarsingh, *Some new difference sequence spaces of fractional order and their dual spaces*, Appl. Math. Comput., **219** (2013), 9737–9742. 1

[9] P. Baliarsingh, S. Dutta, *On an explicit formula for inverse of triangular matrices*, J. Egypt. Math. Soc., **23** (2015), 297–302. 1.1

[10] F. Başar, *Summability Theory and Its Applications*, Bentham Science Publishers, Istanbul, (2012). 1

[11] Ç. Bektas, M. Et, R. Çolak, *Generalized difference sequence spaces and their dual spaces*, J. Math. Anal. Appl., **292** (2004), 423–432.

[12] S. Dutta, P. Baliarsingh, *On some Toeplitz matrices and their inversion*, J. Egypt. Math. Soc., **22** (2014), 420–423. 1.1, 1

[13] S. Dutta, P. Baliarsingh, *A note on paranormed difference sequence spaces of fractional order and their matrix transformations*, J. Egypt. Math. Soc., **22** (2014), 249–253. 1

[14] M. Et, M. Basarir, *On some new generalized difference sequence spaces*, Periodica Math. Hungar., **35** (1997), 169–175. 1

[15] M. Et, R. Çolak, *On some generalized difference sequence spaces*, Soochow J. Math., **21** (1995), 377–386. 1

[16] U. Kadak, *Generalized lacunary statistical difference sequence spaces of fractional order*, Internat. J. Math. Math. Sci., (2015), in press. 1

[17] H. Kızmaz, *On Certain Sequence spaces*, Canad. Math. Bull. **24** (1981), 169–176. 1

[18] E. Malkowsky, M. Mursaleen, S. Suantai, *The dual spaces of sets of difference sequences of order m and matrix transformations*, Acta Math. Sin. (English Series), **23** (2007), 521–532. 1

[19] M. Mursaleen, *Generalized spaces of difference sequences*, J. Math. Anal. Appl., **203** (1996), 738–745. 1

[20] M. Mursaleen, A. K. Noman, *On some new difference sequence spaces of non-absolute type*, Math. Comput. Modelling, **52** (2010), 603–617. 1

[21] H. Polat, F. Başar, *Some Euler spaces of difference sequences of order m*, Acta Math. Sci. Ser. B., Engl. Ed., **27** (2007), 254–266. 2, 3

[22] M. Stieglitz, H. Tietz, *Matrix transformationen von Folgenraumen Eine Ergebnisubersict*, Math. Z., **154** (1977), 1–16. 3

[23] B. C. Tripathy, Y. Altin, M. Et, *Generalized difference sequence spaces defined by Orlicz functions*, Math Slovaca, **58** (2008), 315–324.