Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 8 (2015), 64–85
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Stability of an ACQ-functional equation in various matrix normed spaces

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Communicated by Choonkil Park

Abstract

Using the direct method and the fixed point method, we prove the Hyers-Ulam stability of the following additive-cubic-quartic (ACQ) functional equation

$$\begin{aligned} &11[f(x+2y)+f(x-2y)] \\ &\quad = 44[f(x+y)+f(x-y)]+12f(3y)-48f(2y)+60f(y)-66f(x) \end{aligned}$$

in matrix Banach spaces. Furthermore, using the fixed point method, we also prove the Hyers-Ulam stability of the above functional equation in matrix fuzzy normed spaces.

Keywords: Fixed point method, Hyers-Ulam stability, matrix Banach space, matrix fuzzy normed space, additive-cubic-quartic functional equation.
2010 MSC: 47L25, 39B82, 39B72, 46L07.

1. Introduction

In 1940, Ulam [28] posed the first stability problem concerning group homomorphisms. In the next year, Hyers [8] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' result was generalized by Aoki [1] for additive mappings and by Rassias [24] for linear mappings. Găvruta [5] obtained generalized Rassias' result which allows the Cauchy difference to be controlled by a general unbounded function in the spirit of Rassias' approach. Since then, the stability of several functional equations has been extensively investigated by several mathematicians (see [9, 10, 11, 25, 26] and references therein); as well

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Received 2014-11-1

as various fuzzy stability results concerning Cauchy, Jensen, quadratic and cubic functional equations (cf. [16, 17, 18, 19]). Furthermore some stability results of functional equations and inequalities were investigated [13, 14, 20, 21, 22] in matrix normed spaces, matrix paranormed spaces and matrix fuzzy normed spaces.

In this paper, we consider the following functional equation derived from additive, cubic and quaritc mappings:

$$\begin{aligned} &11[f(x+2y)+f(x-2y)] \\ &\quad = 44[f(x+y)+f(x-y)]+12f(3y)-48f(2y)+60f(y)-66f(x). \end{aligned} \tag{1.1}$$

It is easy to see that the function $f(x) = ax + bx^3 + cx^4$ satisfies the functional equation equation (1.1), where a, b, c are arbitrary constants. In [6], the authors established the general solution and proved the generalized Hyers-Ulam stability of the functional equation (1.1) in Banach spaces. And using the fixed point method, the Hyers-Ulam stability results for the functional equation (1.1) in fuzzy Banach spaces and multi-Banach spaces were established in [12, 27], respectively.

The main purpose of this paper is to apply the direct method and fixed point method to investigate the Hyers-Ulam stability of functional equation (1.1) in matrix Banach spaces. We also prove the Hyers-Ulam stability of the functional equation (1.1) in matrix fuzzy normed spaces by using the fixed point method.

2. Preliminaries

In this section, some definitions and preliminary results are given which will be used in this paper. Following [2, 16, 17], we give the following notion of a fuzzy norm.

Definition 2.1. Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$:
(N1) $N(x, c) = 0$ for $c \le 0$;
(N2) $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;
(N3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \ne 0$;
(N4) $N(x + y, s + t) \ge \min\{N(x, s), N(y, t)\}$;
(N5) $N(x, \cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
(N6) for $x \ne 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.
In this case (X, N) is called a fuzzy normed vector space.

Definition 2.2 ([2, 16, 17])**.** Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ $(t > 0)$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote by $N - \lim_{n\to\infty} x_n = x$.

Definition 2.3 ([2, 16, 17])**.** Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in N$ such that $N(x_m - x_n, t) > 1 - \varepsilon$ $(m, n \ge n_0)$. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

We will also use the following notations. The set of all $m\times n$-matrices in X will be denoted by $M_{m,n}(X)$. When $m = n$, the matrix $M_{m,n}(X)$ will be written as $M_n(X)$. The symbols $e_j \in M_{1,n}(\mathbb{C})$ will denote the row vector whose jth component is 1 and the other components are 0. Similarly, $E_{ij} \in M_n(\mathbb{C})$ will denote the $n \times n$ matrix whose (i, j)-component is 1 and the other components are 0. The $n \times n$ matrix whose (i, j)-component is x and the other components are 0 will be denoted by $E_{ij} \otimes x \in M_n(X)$.

Let $(X, \|\cdot\|)$ be a normed space. Note that $(X, \{\|\cdot\|_n\})$ is a matrix normed space if and only if $(M_n(X), \|\cdot\|_n)$ is a normed space for each positive integer n and $\|AxB\|_k \le \|A\|\|B\|\|x\|_n$ holds for $A \in M_{k,n}$, $x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}$, and that $(X, \{\|\cdot\|_n\})$ is a matrix Banach space if and only if X is a Banach space and $(X, \{\|\cdot\|_n\})$ is a matrix normed space.

Let E, F be vector spaces. For a given mapping $h : E \to F$ and a given positive integer n, define $h_n : M_n(E) \to M_n(F)$ by

$$h_n([x_{ij}]) = [h(x_{ij})]$$

for all $[x_{ij}] \in M_n(E)$.

We introduce the concept of a matrix fuzzy normed space.

Definition 2.4 ([22])**.** Let (X, N) be a fuzzy normed space.
(1) $(X, \{N_n\})$ is called a matrix fuzzy normed space if for each positive integer n, $(M_n(X), N_n)$ is a fuzzy normed space and $N_k(AxB, t) \geq N_n(x, \frac{t}{\|A\| \cdot \|B\|})$ for all $t > 0$, $A \in M_{k,n}(\mathbb{R})$, $x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}(\mathbb{R})$ with $\|A\| \cdot \|B\| \neq 0$.
(2) $(X, \{N_n\})$ is called a matrix fuzzy Banach space if (X, N) is a fuzzy Banach space and $(X, \{N_n\})$ is a matrix fuzzy normed space.

Example 2.5. Let $(X, \{\| \cdot \|_n\})$ be a matrix normed space and $\alpha, \beta > 0$. Define

$$N_n(x,t) = \begin{cases} \frac{\alpha t}{\alpha t + \beta \|x\|_n}, & t > 0, x = [x_{ij}] \in M_n(X), \\ 0, & t \leq 0, x = [x_{ij}] \in M_n(X). \end{cases}$$

Then $(X, \{N_n\})$ is a matrix fuzzy normed space.

3. Stability of the functional equation (1.1) in matrix Banach spaces: Direct method

Throughout this section, let $(X, \{\| \cdot \|_n\})$ be a matrix normed space, $(Y, \{\| \cdot \|_n\})$ be a matrix Banach space and let n be a fixed positive integer. In this section, we prove the Hyers-Ulam stability of the ACQ -functional equation (1.1) in matrix Banach spaces by using the direct method. We need the following Lemmas:

Lemma 3.1 ([6])**.** *Let V and W be real vector spaces. If an odd mapping $f : V \to W$ satisfies (1.1), then f is cubic-additive.*

Lemma 3.2 ([6])**.** *Let V and W be real vector spaces. If an even mapping $f : V \to W$ satisfies (1.1), then f is quartic.*

Lemma 3.3 ([13, 14, 20, 21])**.** *Let $(X, \{\| \cdot \|_n\})$ be a matrix normed space. Then*
(1) $\|E_{kl} \otimes x\|_n = \|x\|$ *for* $x \in X$;
(2) $\|x_{kl}\| \leq \|[x_{ij}]\|_n \leq \sum_{i,j=1}^{n} \|x_{ij}\|$ *for* $[x_{ij}] \in M_n(X)$;
(3) $\lim_{n\to\infty} x_n = x$ *if and only if* $\lim_{n\to\infty} x_{ijn} = x_{ij}$ *for* $x_n = [x_{ijn}], x = [x_{ij}] \in M_k(X)$.

For a mapping $f : X \to Y$, define $Df : X^2 \to Y$ and

$$Df_n : M_n(X^2) \to M_n(Y)$$

by

$$\begin{aligned} Df(a,b) :=\ & 11[f(a+2b) + f(a-2b)] - 44[f(a+b) + f(a-b)] \\ & - 12f(3b) + 48f(2b) - 60f(b) + 66f(a), \\ Df_n([x_{ij}],[y_{ij}]) :=\ & 11[f_n([x_{ij}] + 2[y_{ij}]) + f_n([x_{ij}] - 2[y_{ij}])] \\ & - 44[f_n([x_{ij}] + [y_{ij}]) + f_n([x_{ij}] - [y_{ij}])] \\ & - 12f_n(3[y_{ij}]) + 48f_n(2[y_{ij}]) - 60f_n([y_{ij}]) + 66f_n([x_{ij}]) \end{aligned}$$

for all $a, b \in X$ and all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\sum_{l=0}^{\infty} \frac{1}{8^{l+1}} \varphi(2^{l+1}a, 2^l a) + \sum_{l=0}^{\infty} \frac{1}{8^{l+1}} \varphi(0, 2^l a) < +\infty, \tag{3.1}$$

$$\lim_{k\to\infty} \frac{1}{8^k} \varphi(2^k a, 2^k b) = 0 \tag{3.2}$$

for all $a, b \in X$. Suppose that $f : X \to Y$ is an odd mapping satisfying

$$\|Df_n([x_{ij}], [y_{ij}])\|_n \leq \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij}) \tag{3.3}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\begin{aligned} &\|f_n(2[x_{ij}]) - 2f_n([x_{ij}]) - C_n([x_{ij}])\|_n \\ &\leq \sum_{i,j=1}^{n} \left(\frac{1}{11} \sum_{l=0}^{\infty} \frac{\varphi(2^{l+1}x_{ij}, 2^l x_{ij})}{8^{l+1}} + \frac{14}{33} \sum_{l=0}^{\infty} \frac{\varphi(0, 2^l x_{ij})}{8^{l+1}} \right) \end{aligned} \tag{3.4}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. When $n = 1$, (3.3) is equivalent to

$$\|Df(a, b)\| \leq \varphi(a, b) \tag{3.5}$$

for all $a, b \in X$. Letting $a = 0$ in (3.5), we get

$$\|12f(3b) - 48f(2b) + 60f(b)\| \leq \varphi(0, b) \tag{3.6}$$

for all $b \in X$. Replacing a by $2b$ in (3.5), we get

$$\|11f(4b) - 56f(3b) + 114f(2b) - 104f(b)\| \leq \varphi(2b, b) \tag{3.7}$$

for all $b \in X$. It follows from (3.6) and (3.7) that

$$\|f(4b) - 10f(2b) + 16f(b)\| \leq \frac{1}{11}\varphi(2b, b) + \frac{14}{33}\varphi(0, b) \tag{3.8}$$

for all $b \in X$. Replacing b by a and $g(a) := f(2a) - 2f(a)$ in (3.8), we get

$$\|g(2a) - 8g(a)\| \leq \frac{1}{11}\varphi(2a, a) + \frac{14}{33}\varphi(0, a) \tag{3.9}$$

for all $a \in X$. Replacing a by $2^l a$ and dividing both sides by 8^{l+1} in (3.9), we have

$$\|\frac{g(2^{l+1}a)}{8^{l+1}} - \frac{g(2^l a)}{8^l}\| \leq \frac{1}{11} \frac{\varphi(2^{l+1}a, 2^l a)}{8^{l+1}} + \frac{14}{33} \frac{\varphi(0, 2^l a)}{8^{l+1}} \tag{3.10}$$

for all $a \in X$. Hence

$$\begin{aligned} \|\frac{g(2^q a)}{8^q} - \frac{g(2^p a)}{8^p}\| &\leq \sum_{l=p}^{q-1} \|\frac{g(2^l a)}{8^l} - \frac{g(2^{l+1}a)}{8^{l+1}}\| \\ &\leq \frac{1}{11} \sum_{l=p}^{q-1} \frac{\varphi(2^{l+1}a, 2^l a)}{8^{l+1}} + \frac{14}{33} \sum_{l=p}^{m-1} \frac{\varphi(0, 2^l a)}{8^{l+1}} \end{aligned} \tag{3.11}$$

for all nonnegative integers p, q with $p < q$ and all $a \in X$. It follows from (3.1) and (3.11) that the sequence $\{\frac{g(2^k a)}{8^k}\}$ is a Cauchy sequence in Y for all $a \in X$. Since Y is complete, the sequence $\{\frac{g(2^k a)}{8^k}\}$ converges. So one can define the mapping $C : X \to Y$ by

$$C(a) = \lim_{k\to\infty} \frac{1}{8^k} g(2^k a) \tag{3.12}$$

for all $a \in X$. Moreover, letting $p = 0$ and passing the limit $q \to \infty$ in (3.11), we get

$$\|g(a) - C(a)\| \le \frac{1}{11}\sum_{l=0}^{\infty}\frac{\varphi(2^{l+1}a, 2^l a)}{8^{l+1}} + \frac{14}{33}\sum_{l=0}^{\infty}\frac{\varphi(0, 2^l a)}{8^{l+1}} \tag{3.13}$$

for all $a \in X$.

Now, we show that the mapping C is cubic. By (3.1), (3.9) and (3.12),

$$\begin{aligned}\|C(2a) - 8C(a)\| &= \lim_{n\to\infty}\|\frac{1}{8^n}g(2^{n+1}a) - \frac{1}{8^{n-1}}g(2^n a)\| \\ &= \lim_{n\to\infty} 8\|\frac{1}{8^{n+1}}g(2^{n+1}a) - \frac{1}{8^n}g(2^n a)\| = 0 \end{aligned} \tag{3.14}$$

for all $a \in X$. Therefore, we obtain

$$C(2a) = 8C(a) \tag{3.15}$$

for all $a \in X$. On the other hand it follows from (3.2), (3.5) and (3.12) that

$$\begin{aligned}\|DC(a,b)\| &= \lim_{k\to\infty}\|\frac{1}{8^k}Dg(2^k a, 2^k b)\| \\ &= \lim_{k\to\infty}\frac{1}{8^k}\|Df(2^{k+1}a, 2^{k+1}b) - 2Df(2^k a, 2^k b)\| \\ &\le \lim_{k\to\infty}\frac{1}{8^k}(\varphi(2^{k+1}a, 2^{k+1}b) + 2\varphi(2^k a, 2^k b)) = 0 \end{aligned} \tag{3.16}$$

for all $a, b \in X$. Hence the mapping C satisfies (1.1). So by Lemma 3.1, the mapping

$$a \mapsto C(2a) - 2C(a)$$

is cubic. Hence (3.15) implies that the mapping C is cubic.

To prove the uniqueness of C, let $C' : X \to Y$ be another cubic mapping satisfying (3.13). Let $n = 1$. Then we get

$$\begin{aligned}\|C(a) - C'(a)\| &= \|\frac{1}{8^q}C(2^q a) - \frac{1}{8^q}C'(2^q a)\| \\ &\le \|\frac{1}{8^q}C(2^q a) - \frac{1}{8^q}g(2^q a)\| + \|\frac{1}{8^q}C'(2^q a) - \frac{1}{8^q}g(2^q a)\| \\ &\le 2(\frac{1}{11}\sum_{l=0}^{\infty}\frac{\varphi(2^{l+q+1}a, 2^{l+q}a)}{8^{l+q+1}} + \frac{14}{33}\sum_{l=0}^{\infty}\frac{\varphi(0, 2^{l+q}a)}{8^{l+q+1}}) \\ &= 2(\frac{1}{11}\sum_{l=q}^{\infty}\frac{\varphi(2^{l+1}a, 2^l a)}{8^{l+1}} + \frac{14}{33}\sum_{l=q}^{\infty}\frac{\varphi(0, 2^l a)}{8^{l+1}}) \end{aligned}$$

for all $a \in X$. Letting $q \to \infty$ in the above inequality, we get $C(a) = C'(a)$ for all $a \in X$, which gives the conclusion. Thus the mapping $C : X \to Y$ is a unique cubic mapping.

By Lemma 3.3 and (3.13), we get

$$\begin{aligned}\|f_n(2[x_{ij}]) - 2f_n([x_{ij}]) - C_n([x_{ij}])\|_n &\le \sum_{i,j=1}^{n}\|f(2x_{ij}) - 2f(x_{ij}) - C(x_{ij})\| \\ &\le \sum_{i,j=1}^{n}\left(\frac{1}{11}\sum_{l=0}^{\infty}\frac{\varphi(2^{l+1}x_{ij}, 2^l x_{ij})}{8^{l+1}} + \frac{14}{33}\sum_{l=0}^{\infty}\frac{\varphi(0, 2^l x_{ij})}{8^{l+1}}\right)\end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$. Thus $C : X \to Y$ is a unique cubic mapping satisfying (3.4), as desired. This completes the proof of the theorem. □

Corollary 3.5. *Let r, θ be positive real numbers with $r < 3$. Suppose that $f : X \to Y$ is an odd mapping satisfying*

$$\|Df_n([x_{ij}],[y_{ij}])\|_n \le \sum_{i,j=1}^{n} \theta(\|x_{ij}\|^r + \|y_{ij}\|^r) \tag{3.17}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\|f_n(2[x_{ij}]) - 2f_n([x_{ij}]) - C_n([x_{ij}])\|_n \le \frac{1}{33}\sum_{i,j=1}^{n} \frac{17 + 3\cdot 2^r}{8 - 2^r}\theta\|x_{ij}\|^r \tag{3.18}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. The proof follows immediately by taking $\varphi(a,b) = \theta(\|a\|^r + \|b\|^b)$ for all $a, b \in X$ in Theorem 3.4. □

Theorem 3.6. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$\sum_{l=1}^{\infty} 8^{l-1}\varphi(\frac{a}{2^{l-1}}, \frac{a}{2^l}) + \sum_{l=1}^{\infty} 8^{l-1}\varphi(0, \frac{a}{2^l}) < +\infty, \tag{3.19}$$

$$\lim_{k\to\infty} 8^k \varphi(\frac{a}{2^k}, \frac{b}{2^k}) = 0 \tag{3.20}$$

for all $a, b \in X$. Suppose that $f : X \to Y$ is an odd mapping satisfying (3.3) for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\begin{aligned}&\|f_n(2[x_{ij}]) - 2f_n([x_{ij}]) - C_n([x_{ij}])\|_n \\ &\qquad \le \sum_{i,j=1}^{n}\left(\frac{1}{11}\sum_{l=1}^{\infty} 8^{l-1}\varphi(\frac{x_{ij}}{2^{l-1}}, \frac{x_{ij}}{2^l}) + \frac{14}{33}\sum_{l=1}^{\infty} 8^{l-1}\varphi(0, \frac{x_{ij}}{2^l})\right)\end{aligned} \tag{3.21}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. The proof of the theorem is similar to the proof of Theorem 3.4 and thus it is omitted. □

Corollary 3.7. *Let r, θ be positive real numbers with $r > 3$. Suppose that $f : X \to Y$ is an odd mapping satisfying (3.17) for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique cubic mapping $C : X \to Y$ such that*

$$\|f_n(2[x_{ij}]) - 2f_n([x_{ij}]) - C_n([x_{ij}])\|_n \le \frac{1}{33}\sum_{i,j=1}^{n} \frac{3\cdot 2^r + 17}{2^r - 8}\theta\|x_{ij}\|^r \tag{3.22}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. The asserted result in Corollary 3.7 can be easily derived by considering $\varphi(a,b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$ in Theorem 3.6. □

Theorem 3.8. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$\sum_{l=0}^{\infty} \frac{1}{2^{l+1}}\varphi(2^{l+1}a, 2^l a) + \sum_{l=0}^{\infty} \frac{1}{2^{l+1}}\varphi(0, 2^l a) < +\infty, \tag{3.23}$$

$$\lim_{k\to\infty} \frac{1}{2^k}\varphi(2^k a, 2^k b) = 0 \tag{3.24}$$

for all $a, b \in X$. Suppose that $f : X \to Y$ is an odd mapping satisfying (3.3) for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f_n(2[x_{ij}]) - 8f_n([x_{ij}]) - A_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \left(\frac{1}{11} \sum_{l=0}^{\infty} \frac{\varphi(2^{l+1}x_{ij}, 2^l x_{ij})}{2^{l+1}} + \frac{14}{33} \sum_{l=0}^{\infty} \frac{\varphi(0, 2^l x_{ij})}{2^{l+1}} \right) \tag{3.25}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. As in the proof of Theorem 3.4, we have

$$\|f(4b) - 10f(2b) + 16f(b)\| \leq \frac{1}{11}\varphi(2b, b) + \frac{14}{33}\varphi(0, b) \tag{3.26}$$

for all $b \in X$. Replacing b by a and $h(a) := f(2a) - 8f(a)$ in (3.26), we get

$$\|h(2a) - 2h(a)\| \leq \frac{1}{11}\varphi(2a, a) + \frac{14}{33}\varphi(0, a) \tag{3.27}$$

for all $a \in X$. The rest of the proof is similar to the proof of Theorem 3.4. □

Corollary 3.9. *Let r, θ be positive real numbers with $r < 1$. Suppose that $f : X \to Y$ is an odd mapping satisfying (3.17) for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f_n(2[x_{ij}]) - 8f_n([x_{ij}]) - A_n([x_{ij}])\|_n \leq \frac{1}{33} \sum_{i,j=1}^{n} \frac{17 + 3 \cdot 2^r}{2 - 2^r} \theta \|x_{ij}\|^r \tag{3.28}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. The asserted result in Corollary 3.9 can be easily derived by considering $\varphi(a, b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$ in Theorem 3.8. □

Theorem 3.10. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\sum_{l=1}^{\infty} 2^{l-1} \varphi(\frac{a}{2^{l-1}}, \frac{a}{2^l}) + \sum_{l=1}^{\infty} 2^{l-1} \varphi(0, \frac{a}{2^l}) < +\infty, \tag{3.29}$$

$$\lim_{k \to \infty} 2^k \varphi(\frac{a}{2^k}, \frac{b}{2^k}) = 0 \tag{3.30}$$

for all $a, b \in X$. Suppose that $f : X \to Y$ is an odd mapping satisfying (3.3) for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f_n(2[x_{ij}]) - 8f_n([x_{ij}]) - A_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \left(\frac{1}{11} \sum_{l=1}^{\infty} 2^{l-1} \varphi(\frac{x_{ij}}{2^{l-1}}, \frac{x_{ij}}{2^l}) + \frac{14}{33} \sum_{l=1}^{\infty} 2^{l-1} \varphi(0, \frac{x_{ij}}{2^l}) \right) \tag{3.31}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. The proof of the theorem is similar to the proof of Theorem 3.6 and 3.8 thus it is omitted. □

Corollary 3.11. *Let r, θ be positive real numbers with $r > 1$. Suppose that $f : X \to Y$ is an odd mapping satisfying (3.17) for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f_n(2[x_{ij}]) - 8f_n([x_{ij}]) - A_n([x_{ij}])\|_n \leq \frac{1}{33} \sum_{i,j=1}^{n} \frac{3 \cdot 2^r + 17}{2^r - 2} \theta \|x_{ij}\|^r \tag{3.32}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. Letting $\varphi(a,b)=\theta(\|a\|^r+\|b\|^b)$ in Theorem 3.10, we obtain the result. $\square$

Theorem 3.12. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$\sum_{l=0}^{\infty}\frac{1}{16^{l+1}}\varphi(2^l a,2^l a)+\sum_{l=0}^{\infty}\frac{1}{16^{l+1}}\varphi(0,2^l a)<+\infty, \tag{3.33}$$

$$\lim_{k\to\infty}\frac{1}{16^k}\varphi(2^k a,2^k b)=0 \tag{3.34}$$

for all $a,b\in X$. Suppose that $f : X\to Y$ is an even mapping satisfying (3.3) and $f(0)=0$ for all $x=[x_{ij}],y=[y_{ij}]\in M_n(X)$. Then there exists a unique quartic mapping $Q : X\to Y$ such that

$$\|f_n([x_{ij}])-Q_n([x_{ij}])\|_n\le\sum_{i,j=1}^{n}\left(\frac{6}{11}\sum_{l=0}^{\infty}\frac{\varphi(2^l x_{ij},2^l x_{ij})}{16^{l+1}}+\frac{1}{22}\sum_{l=0}^{\infty}\frac{\varphi(0,2^l x_{ij})}{16^{l+1}}\right) \tag{3.35}$$

for all $x=[x_{ij}]\in M_n(X)$.

Proof. Putting $a=0$ in (3.5), we get

$$\|-12f(3b)+70f(2b)-148f(b)\|\le\varphi(0,b) \tag{3.36}$$

for all $b\in X$. On the other hand, substituting $a=b$ in (3.5), we obtain the following

$$\|-f(3b)+4f(2b)+17f(b)\|\le\varphi(b,b) \tag{3.37}$$

for all $b\in X$. By (3.36) and (3.37), we have

$$\|f(2b)-16f(b)\|\le\frac{6}{11}\varphi(b,b)+\frac{1}{22}\varphi(0,b) \tag{3.38}$$

for all $b\in X$. Replacing a by $2^l a$ and dividing both sides by 16^{l+1} in (3.38), we have

$$\|\frac{f(2^{l+1}a)}{16^{l+1}}-\frac{f(2^l a)}{16^l}\|\le\frac{6}{11}\frac{\varphi(2^l a,2^l a)}{16^{l+1}}+\frac{1}{22}\frac{\varphi(0,2^l a)}{16^{l+1}} \tag{3.39}$$

for all $a\in X$. Hence

$$\begin{aligned}\|\frac{f(2^q a)}{16^q}-\frac{f(2^p a)}{16^p}\| &\le\sum_{l=p}^{q-1}\|\frac{f(2^l a)}{16^l}-\frac{f(2^{l+1}a)}{16^{l+1}}\|\\ &\le\frac{6}{11}\sum_{l=p}^{q-1}\frac{\varphi(2^l a,2^l a)}{16^{l+1}}+\frac{1}{22}\sum_{l=p}^{m-1}\frac{\varphi(0,2^l a)}{16^{l+1}}\end{aligned} \tag{3.40}$$

for all nonnegative integers p,q with $p<q$ and all $a\in X$. It follows from (3.33) and (3.40) that the sequence $\{\frac{f(2^k a)}{16^k}\}$ is a Cauchy sequence in Y for all $a\in X$. Since Y is complete, the sequence $\{\frac{f(2^k a)}{16^k}\}$ converges. So one can define the mapping $Q : X\to Y$ by

$$Q(a)=\lim_{k\to\infty}\frac{1}{16^k}f(2^k a) \tag{3.41}$$

for all $a\in X$. Moreover, letting $p=0$ and passing the limit $q\to\infty$ in (3.40), we get

$$\|f(a)-Q(a)\|\le\frac{6}{11}\sum_{l=0}^{\infty}\frac{\varphi(2^l a,2^l a)}{16^{l+1}}+\frac{1}{22}\sum_{l=0}^{\infty}\frac{\varphi(0,2^l a)}{16^{l+1}} \tag{3.42}$$

for all $a\in X$. By (3.5), (3.34) and (3.41), we get

$$\|DQ(a,b)\|=\lim_{k\to\infty}\|\frac{1}{16^k}Df(2^k a,2^k b)\|\le\lim_{k\to\infty}\frac{1}{16^k}\varphi(2^k a,2^k b)=0 \tag{3.43}$$

for all $a, b \in X$. Hence by Lemma 3.2, Q is quartic.

Now, Let $Q' : X \to Y$ be another quartic mapping satisfying (3.42). Let $n = 1$. Then we get

$$\begin{aligned} \|Q(a) - Q'(a)\| &= \|\frac{1}{16^q}Q(2^q a) - \frac{1}{16^q}Q'(2^q a)\| \\ &\le \|\frac{1}{16^q}C(2^q a) - \frac{1}{16^q}f(2^q a)\| + \|\frac{1}{8^q}Q'(2^q a) - \frac{1}{8^q}f(2^q a)\|\| \\ &\le 2(\frac{6}{11}\sum_{l=0}^{\infty}\frac{\varphi(2^{l+q}a, 2^{l+q}a)}{16^{l+q+1}} + \frac{1}{22}\sum_{l=0}^{\infty}\frac{\varphi(0, 2^{l+q}a)}{16^{l+q+1}}) \\ &= 2(\frac{6}{11}\sum_{l=q}^{\infty}\frac{\varphi(2^{l}a, 2^{l}a)}{16^{l+1}} + \frac{1}{22}\sum_{l=q}^{\infty}\frac{\varphi(0, 2^{l}a)}{16^{l+1}}) \end{aligned}$$

which tends to zero as $q \to \infty$ for all $a \in X$. So we can conclude that $Q(a) = Q'(a)$ for all $a \in X$. This proves the uniqueness of Q. Thus the mapping $Q : X \to Y$ is a unique quartic mapping.

By Lemma 3.3 and (3.42), we get

$$\begin{aligned} \|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n &\le \sum_{i,j=1}^{n}\|f(2x_{ij}) - Q(x_{ij})\| \\ &\le \sum_{i,j=1}^{n}\left(\frac{6}{11}\sum_{l=0}^{\infty}\frac{\varphi(2^l x_{ij}, 2^l x_{ij})}{16^{l+1}} + \frac{1}{22}\sum_{l=0}^{\infty}\frac{\varphi(0, 2^l x_{ij})}{16^{l+1}}\right) \end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$. Thus $Q : X \to Y$ is a unique quartic mapping satisfying (3.35), as desired. This completes the proof of the theorem. □

Corollary 3.13. *Let r, θ be positive real numbers with $r < 4$. Suppose that $f : X \to Y$ is an even mapping satisfying (3.17) and $f(0) = 0$ for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quartic mapping $Q : X \to Y$ such that*

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \le \frac{25}{22}\sum_{i,j=1}^{n}\frac{\theta}{16 - 2^r}\|x_{ij}\|^r \tag{3.44}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. The proof follows immediately by taking $\varphi(a, b) = \theta(\|a\|^r + \|b\|^b)$ for all $a, b \in X$ in Theorem 3.12. □

Theorem 3.14. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\sum_{l=1}^{\infty}16^{l-1}\varphi(\frac{a}{2^l}, \frac{a}{2^l}) + \sum_{l=1}^{\infty}16^{l-1}\varphi(0, \frac{a}{2^l}) < +\infty, \tag{3.45}$$

$$\lim_{k\to\infty}16^k\varphi(\frac{a}{2^k}, \frac{b}{2^k}) = 0 \tag{3.46}$$

for all $a, b \in X$. Suppose that $f : X \to Y$ is an even mapping satisfying (3.3) and $f(0) = 0$ for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quartic mapping $Q : X \to Y$ such that

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n}\left(\frac{6}{11}\sum_{l=1}^{\infty}16^{l-1}\varphi(\frac{x_{ij}}{2^l}, \frac{x_{ij}}{2^l}) + \frac{1}{22}\sum_{l=1}^{\infty}16^{l-1}\varphi(0, \frac{x_{ij}}{2^l})\right) \tag{3.47}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. The proof of the theorem is similar to the proof of Theorem 3.12 and thus it is omitted. □

Corollary 3.15. *Let r, θ be positive real numbers with $r > 4$. Suppose that $f : X \to Y$ is an even mapping satisfying (3.17) and $f(0) = 0$ for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quartic mapping $Q : X \to Y$ such that*

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \leq \frac{25}{22} \sum_{i,j=1}^{n} \frac{\theta}{2^r - 16} \|x_{ij}\|^r \tag{3.48}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. Letting $\varphi(a, b) = \theta(\|a\|^r + \|b\|^b)$ in Theorem 3.14, we obtain the result. □

4. Stability of the functional equation (1.1) in matrix Banach spaces: Fixed point method

Throughout this section, let $(X, \{\|\cdot\|_n\})$ be a matrix normed space, $(Y, \{\|\cdot\|_n\})$ be a matrix Banach space and let n be a fixed positive integer. In this section, we prove the Hyers-Ulam stability of the ACQ - functional equation (1.1) in matrix Banach spaces by using Fixed point method. We begin with the definition of a generalized metric on a set.

Let E be a set. A function $d : E \times E \to [0, \infty]$ is called a generalized metric on E if d satisfies
(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$, $\forall x, y \in E$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$, $\forall x, y, z \in E$.

Before proceeding to the proof of the main results, we begin with a result due to Diaz and Margolis [4].

Lemma 4.1 ([4] or [23]). *Let (E, d) be a complete generalized metric space and $J : E \to E$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each fixed element $x \in E$, either*

$$d(J^n x, J^{n+1} x) = \infty \quad \forall n \geq 0,$$

or

$$d(J^n x, J^{n+1} x) < \infty \quad \forall n \geq n_0,$$

for some natural number n_0. Moreover, if the second alternative holds then:
(i) The sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;
(ii) y^* is the unique fixed point of J in the set $E^* := \{y \in E \mid d(J^{n_0} x, y) < +\infty\}$ and $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$, $\forall x, y \in E^*$.

Theorem 4.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(a, b) \leq 8\alpha\varphi(\frac{a}{2}, \frac{b}{2}) \tag{4.1}$$

for all $a, b \in X$. Suppose that $f : X \to Y$ is an odd mapping satisfying (3.3) for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\begin{aligned} &\|f_n(2[x_{ij}]) - 2f_n([x_{ij}]) - C_n([x_{ij}])\|_n \\ &\qquad \leq \sum_{i,j=1}^{n} \frac{1}{8(1-\alpha)} \left(\frac{1}{11}\varphi(2x_{ij}, x_{ij}) + \frac{14}{33}\varphi(0, x_{ij}) \right) \end{aligned} \tag{4.2}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. When $n = 1$, similar to the proof of Theorem 3.4, and by (3.9),

$$\|g(a) - \frac{1}{8}g(2a)\| \leq \frac{1}{8}\left(\frac{1}{11}\varphi(2a, a) + \frac{14}{33}\varphi(0, a) \right) \tag{4.3}$$

for all $a \in X$.

Let $S_1 := \{q_1 : X \to Y\}$, and introduce a generalized metric d_1 on S_1 as follows:

$$d_1(q_1, k_1) := \inf \left\{ \lambda \in \mathbb{R}_+ \middle| \|q_1(a) - k_1(a)\| \le \frac{1}{11}\varphi(2a, a) + \frac{14}{33}\varphi(0, a), \forall a \in X \right\}.$$

It is easy to prove that (S_1, d_1) is a complete generalized metric space [3, 7, 15].

Now we consider the mapping $\mathcal{J}_1 : S_1 \to S_1$ defined by

$$\mathcal{J}_1 q_1(a) := \frac{1}{8} q_1(2a), \quad \text{for all } q_1 \in S_1 \text{ and } a \in X. \tag{4.4}$$

Let $q_1, k_1 \in S_1$ and let $\lambda \in \mathbb{R}_+$ be an arbitrary constant with $d_1(q_1, k_1) \le \lambda$. From the definition of d_1, we get

$$\|q_1(a) - k_1(a)\| \le \lambda \left(\frac{1}{11}\varphi(2a, a) + \frac{14}{33}\varphi(0, a) \right)$$

for all $a \in X$. Therefore, using (4.1), we get

$$\begin{aligned} \|\mathcal{J}_1 q_1(a) - \mathcal{J}_1 k_1(a)\| &= \|\frac{1}{8} q_1(2a) - \frac{1}{8} k_1(2a)\| \\ &\le \frac{\lambda}{8} \left(\frac{1}{11}\varphi(2^2 a, 2a) + \frac{14}{33}\varphi(0, 2a) \right) \\ &\le \alpha\lambda \left(\frac{1}{11}\varphi(2a, a) + \frac{14}{33}\varphi(0, a) \right) \end{aligned} \tag{4.5}$$

for some $\alpha < 1$ and for all $a \in X$. Hence, it holds that $d_1(\mathcal{J}_1 q_1, \mathcal{J}_1 k_1) \le \alpha\lambda$, that is, $d_1(\mathcal{J}_1 q_1, \mathcal{J}_1 k_1) \le \alpha d_1(q_1, k_1)$ for all $q_1, k_1 \in S_1$.

It follows from (4.3) that $d_1(g, \mathcal{J}_1 g) \le \frac{1}{8}$. Therefore according to Lemma 4.1, the sequence $\mathcal{J}_1^n g$ converges to a fixed point C of $\mathcal{J}_1$, that is,

$$C : X \to Y, \quad \lim_{n\to\infty} \frac{1}{8^n} g(2^n a) = C(a)$$

for all $a \in X$, and

$$C(2a) = 8C(a) \tag{4.6}$$

for all $a \in X$. Also C is the unique fixed point of $\mathcal{J}_1$ in the set $S_1^* = \{q_1 \in S_1 : d_1(g, q_1) < \infty\}$. This implies that C is a unique mapping satisfying (4.6) such that there exists a $\lambda \in \mathbb{R}_+$ such that

$$\|g(a) - C(a)\| \le \lambda \left(\frac{1}{11}\varphi(2a, a) + \frac{14}{33}\varphi(0, a) \right)$$

for all $a \in X$. Also,

$$d_1(g, C) \le \frac{1}{1-\alpha} d_1(g, \mathcal{J}_1 g) \le \frac{1}{8(1-\alpha)}.$$

So

$$\|g(a) - C(a)\| \le \frac{1}{8(1-\alpha)} \left(\frac{1}{11}\varphi(2a, a) + \frac{14}{33}\varphi(0, a) \right) \tag{4.7}$$

for all $a \in X$.

It follows from (3.5) and (4.1) that

$$\begin{aligned} \|DC(a, b)\| &= \lim_{l\to\infty} \frac{1}{8^l} \|Dg(2^l a, 2^l b)\| \\ &\le \lim_{l\to\infty} \frac{1}{8^l} (\varphi(2^l \cdot 2a, 2^l \cdot 2b) + 2\varphi(2^l a, 2^l b)) \\ &\le \lim_{l\to\infty} \frac{8^l \alpha^l}{8^l} (\varphi(2a, 2b) + 2\varphi(a, b)) = 0 \end{aligned}$$

for all $a, b \in X$. Hence $DC(a, b) = 0$. So by Lemma 3.1, the mapping $x \mapsto C(2a) - 2C(a)$ is cubic. Hence (4.6) implies that the mapping $C : X \to Y$ is cubic.

By Lemma 3.3 and (4.7),

$$\begin{aligned}\|f_n(2[x_{ij}]) - 2f_n([x_{ij}]) - C_n([x_{ij}])\|_n &\leq \sum_{i,j=1}^{n} \|f(2x_{ij}) - 2f(x_{ij}) - C(x_{ij})\| \\ &\leq \sum_{i,j=1}^{n} \frac{1}{8(1-\alpha)}\left(\frac{1}{11}\varphi(2x_{ij}, x_{ij}) + \frac{14}{33}\varphi(0, x_{ij})\right)\end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$. Thus $C : X \to Y$ is a unique cubic mapping satisfying (4.2), as desired. This completes the proof of the theorem. □

Corollary 4.3. *Let r, θ be positive real numbers with $r < 3$. Suppose that $f : X \to Y$ is an odd mapping satisfying (3.17) for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying (3.18) for all $x = [x_{ij}] \in M_n(X)$.*

Proof. The proof follows immediately by taking

$$\varphi(a, b) = \theta(\|a\|^r + \|b\|^r)$$

for all $a, b \in X$ and choosing $\alpha = 2^{r-3}$ in Theorem 4.2. □

Theorem 4.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(a, b) \leq \frac{\alpha}{8}\varphi(2a, 2b) \tag{4.8}$$

for all $a, b \in X$. Suppose that $f : X \to Y$ is an odd mapping satisfying (3.3) *for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique cubic mapping $C : X \to Y$ such that*

$$\begin{aligned}&\|f_n(2[x_{ij}]) - 2f_n([x_{ij}]) - C_n([x_{ij}])\|_n \\ &\qquad \leq \sum_{i,j=1}^{n} \frac{\alpha}{8(1-\alpha)}\left(\frac{1}{11}\varphi(2x_{ij}, x_{ij}) + \frac{14}{33}\varphi(0, x_{ij})\right)\end{aligned} \tag{4.9}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. Let (S_1, d_1) be the generalized metric space defined in the proof of Theorem 4.2.

Now, we consider the mapping $\mathcal{J}_1 : S_1 \to S_1$ defined by

$$\mathcal{J}_1 q_1(a) := 8q_1(\frac{a}{2}), \quad \text{for all } q_1 \in S_1 \text{ and } a \in X. \tag{4.10}$$

It follows from (3.9) that

$$\|g(a) - 8g(\frac{a}{2})\| \leq \frac{\alpha}{8}\left(\frac{1}{11}\varphi(2a, a) + \frac{14}{33}\varphi(0, a)\right) \tag{4.11}$$

for all $a \in X$. Thus $d_1(g, \mathcal{J}_1 g) \leq \frac{\alpha}{8}$. So

$$d_1(g, C) \leq \frac{1}{1-\alpha} d_1(g, \mathcal{J}_1 g) \leq \frac{\alpha}{8(1-\alpha)}.$$

The rest of the proof is similar to the proof of Theorem 4.2. □

Corollary 4.5. *Let r, θ be positive real numbers with $r > 3$. Suppose that $f : X \to Y$ is an odd mapping satisfying (3.17) for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying (3.22) for all $x = [x_{ij}] \in M_n(X)$.*

Proof. The asserted result in Corollary 4.5 can be easily derived by considering

$$\varphi(a,b) = \theta(\|a\|^r + \|b\|^r)$$

for all $a, b \in X$ and $\alpha = 2^{3-r}$ in Theorem 4.4. □

Theorem 4.6. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(a,b) \leq 2\alpha\varphi(\frac{a}{2}, \frac{b}{2}) \tag{4.12}$$

for all $a, b \in X$. Suppose that $f : X \to Y$ is an odd mapping satisfying (3.3) for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\begin{aligned}&\|f_n(2[x_{ij}]) - 8f_n([x_{ij}]) - A_n([x_{ij}])\|_n \\ &\qquad \leq \sum_{i,j=1}^{n} \frac{1}{2(1-\alpha)}\left(\frac{1}{11}\varphi(2x_{ij}, x_{ij}) + \frac{14}{33}\varphi(0, x_{ij})\right)\end{aligned} \tag{4.13}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. When $n = 1$, similar to the proof of Theorem 3.8, and by (3.27),

$$\|h(a) - \frac{1}{2}h(2a)\| \leq \frac{1}{2}\left(\frac{1}{11}\varphi(2a, a) + \frac{14}{33}\varphi(0, a)\right) \tag{4.14}$$

for all $a \in X$. Let (S_1, d_1) be the generalized metric space defined in the proof of Theorems 4.2.

Now we consider the mapping $\mathcal{J}_1 : S_1 \to S_1$ defined by

$$\mathcal{J}_1 q_1(a) := \frac{1}{2}q_1(2a), \quad \text{for all } q_1 \in S_1 \text{ and } a \in X. \tag{4.15}$$

Thus $d_1(h, \mathcal{J}_1 h) \leq \frac{1}{2}$. So

$$d_1(h, A) \leq \frac{1}{1-\alpha} d_1(h, \mathcal{J}_1 h) \leq \frac{1}{2(1-\alpha)}.$$

The rest of the proof is similar to the proof of Theorem 4.2. □

Corollary 4.7. *Let r, θ be positive real numbers with $r < 1$. Suppose that $f : X \to Y$ is an odd mapping satisfying (3.17) for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (3.28) for all $x = [x_{ij}] \in M_n(X)$.*

Proof. The proof follows from Theorem 4.6 by taking asserted $\varphi(a,b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$. Then we can choose $\alpha = 2^{r-1}$ and we get the desired result. □

Theorem 4.8. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(a,b) \leq \frac{\alpha}{2}\varphi(2a, 2b) \tag{4.16}$$

for all $a, b \in X$. Suppose that $f : X \to Y$ is an odd mapping satisfying (3.3) for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\begin{aligned}&\|f_n(2[x_{ij}]) - 8f_n([x_{ij}]) - A_n([x_{ij}])\|_n \\ &\qquad \leq \sum_{i,j=1}^{n} \frac{\alpha}{2(1-\alpha)}\left(\frac{1}{11}\varphi(2x_{ij}, x_{ij}) + \frac{14}{33}\varphi(0, x_{ij})\right)\end{aligned} \tag{4.17}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. Let (S_1, d_1) be the generalized metric space defined in the proof of Theorem 4.2.

Now we consider the mapping $\mathcal{J}_1 : S_1 \to S_1$ defined by

$$\mathcal{J}_1 q_1(a) := 2q_1(\frac{a}{2}), \quad \text{for all } q_1 \in S_1 \text{ and } a \in X. \tag{4.18}$$

It follows from (3.27) that

$$\|h(a) - 2h(\frac{a}{2})\| \le \frac{\alpha}{2}\left(\frac{1}{11}\varphi(2a, a) + \frac{14}{33}\varphi(0, a)\right) \tag{4.19}$$

for all $a \in X$. Thus $d_1(h, \mathcal{J}_1 h) \le \frac{\alpha}{2}$. So

$$d_1(h, A) \le \frac{1}{1-\alpha} d_1(h, \mathcal{J}_1 h) \le \frac{\alpha}{2(1-\alpha)}.$$

The rest of the proof is similar to the proof of Theorem 4.2 and 4.6. □

Corollary 4.9. *Let r, θ be positive real numbers with $r > 1$. Suppose that $f : X \to Y$ is an odd mapping satisfying (3.17) for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (3.32) for all $x = [x_{ij}] \in M_n(X)$.*

Proof. By choosing $\varphi(a, b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$ and $\alpha = 2^{1-r}$ in Theorem 4.8, we obtain the inequality (3.32). □

Theorem 4.10. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(a, b) \le 16\alpha\varphi(\frac{a}{2}, \frac{b}{2}) \tag{4.20}$$

for all $a, b \in X$. Suppose that $f : X \to Y$ is an even mapping satisfying (3.3) and $f(0) = 0$ for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quartic mapping $Q : X \to Y$ such that

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{1}{16(1-\alpha)}\left(\frac{6}{11}\varphi(x_{ij}, x_{ij}) + \frac{1}{22}\varphi(0, x_{ij})\right) \tag{4.21}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. When $n = 1$, similar to the proof of Theorem 3.12, and by (3.38),

$$\|f(a) - \frac{1}{16}f(2a)\| \le \frac{1}{16}\left(\frac{6}{11}\varphi(a, a) + \frac{1}{22}\varphi(0, a)\right) \tag{4.22}$$

for all $a \in X$.

Let $S_2 := \{q_2 : X \to Y\}$, and introduce a generalized metric d_2 on S_2 as follows:

$$d_2(q_2, k_2) := \inf\left\{\mu \in \mathbb{R}_+ \middle| \|q_2(a) - k_2(a)\| \le \frac{6}{11}\varphi(a, a) + \frac{1}{22}\varphi(0, a), \forall a \in X\right\}.$$

It is easy to prove that (S_2, d_2) is a complete generalized metric space [3, 7, 15].

Now we consider the mapping $\mathcal{J}_2 : S_2 \to S_2$ defined by

$$\mathcal{J}_2 q_2(a) := \frac{1}{16} q_2(2a), \quad \text{for all } q_2 \in S_2 \text{ and } a \in X. \tag{4.23}$$

Let $q_2, k_2 \in S_2$ and let $\mu \in \mathbb{R}_+$ be an arbitrary constant with $d_2(q_2, k_2) \le \mu$. From the definition of d_2, we get

$$\|q_2(a) - k_2(a)\| \le \mu\left(\frac{6}{11}\varphi(a, a) + \frac{1}{22}\varphi(0, a)\right)$$

for all $a \in X$. Therefore, using (4.20), we get

$$\begin{aligned}\|\mathcal{J}_2 q_2(a) - \mathcal{J}_2 k_2(a)\| &= \|\frac{1}{16} q_1(2a) - \frac{1}{16} k_1(2a)\| \\ &\le \frac{\mu}{16}\left(\frac{6}{11}\varphi(2a,2a) + \frac{1}{22}\varphi(0,2a)\right) \\ &\le \alpha\mu\left(\frac{6}{11}\varphi(a,a) + \frac{1}{22}\varphi(0,a)\right) \qquad (4.24)\end{aligned}$$

for some $\alpha < 1$ and for all $a \in X$. Hence, it holds that $d_2(\mathcal{J}_2 q_2, \mathcal{J}_2 k_2) \le \alpha\mu$, that is, $d_2(\mathcal{J}_2 q_2, \mathcal{J}_2 k_2) \le \alpha d_2(q_2, k_2)$ for all $q_2, k_2 \in S_2$.

It follows from (4.22) that $d_2(f, \mathcal{J}_2 f) \le \frac{1}{16}$. Therefore according to Lemma 4.1, the sequence $\mathcal{J}_2^n g$ converges to a fixed point Q of $\mathcal{J}_2$, that is,

$$Q : X \to Y, \quad \lim_{n\to\infty} \frac{1}{16^n} f(2^n a) = Q(a)$$

for all $a \in X$, and

$$Q(2a) = 16Q(a) \qquad (4.25)$$

for all $a \in X$. Also Q is the unique fixed point of $\mathcal{J}_2$ in the set $S_2^* = \{q_2 \in S_2 : d_2(f, q_2) < \infty\}$. This implies that Q is a unique mapping satisfying (4.25) such that there exists a $\mu \in \mathbb{R}_+$ such that

$$\|f(a) - Q(a)\| \le \mu\left(\frac{6}{11}\varphi(a,a) + \frac{1}{22}\varphi(0,a)\right)$$

for all $a \in X$. Also,

$$d_2(f, Q) \le \frac{1}{1-\alpha} d_2(f, \mathcal{J}_2 f) \le \frac{1}{16(1-\alpha)}.$$

So

$$\|f(a) - Q(a)\| \le \frac{1}{16(1-\alpha)}\left(\frac{6}{11}\varphi(a,a) + \frac{1}{22}\varphi(0,a)\right) \qquad (4.26)$$

for all $a \in X$.

It follows from (3.5) and (4.20) that

$$\begin{aligned}\|DQ(a,b)\| &= \lim_{l\to\infty} \frac{1}{16^l}\|Df(2^l a, 2^l b)\| \le \lim_{l\to\infty} \frac{1}{16^l}\varphi(2^l a, 2^l b) \\ &\le \lim_{l\to\infty} \frac{16^l \alpha^l}{16^l}\varphi(a,b) = 0\end{aligned}$$

for all $a, b \in X$. Hence $DQ(a,b) = 0$. So by Lemma 3.2, the mapping $Q : X \to Y$ is quartic.

By Lemma 3.3 and (4.26),

$$\begin{aligned}\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n &\le \sum_{i,j=1}^{n} \|f(2x_{ij}) - Q(x_{ij})\| \\ &\le \sum_{i,j=1}^{n} \frac{1}{16(1-\alpha)}\left(\frac{6}{11}\varphi(x_{ij}, x_{ij}) + \frac{1}{22}\varphi(0, x_{ij})\right)\end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$. Thus $Q : X \to Y$ is a unique quartic mapping satisfying (4.21), as desired. This completes the proof of the theorem. □

Corollary 4.11. *Let r, θ be positive real numbers with $r < 4$. Suppose that $f : X \to Y$ is an even mapping satisfying (3.17) and $f(0) = 0$ for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quartic mapping $Q : X \to Y$ satisfying (3.44) for all $x = [x_{ij}] \in M_n(X)$.*

Proof. The proof follows immediately by taking $\varphi(a,b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$ and choosing $\alpha = 2^{r-4}$ in Theorem 4.10. $\square$

Theorem 4.12. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(a,b) \le \frac{\alpha}{16}\varphi(2a,2b) \tag{4.27}$$

for all $a, b \in X$. Suppose that $f : X \to Y$ is an even mapping satisfying (3.3) and $f(0) = 0$ for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quartic mapping $Q : X \to Y$ such that

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{\alpha}{16(1-\alpha)}\left(\frac{6}{11}\varphi(x_{ij}, x_{ij}) + \frac{1}{22}\varphi(0, x_{ij})\right) \tag{4.28}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. Let (S_2, d_2) be the generalized metric space defined in the proof of Theorem 4.10.

Now we consider the mapping $\mathcal{J}_2 : S_2 \to S_2$ defined by

$$\mathcal{J}_2 q_2(a) := 16 q_2(\frac{a}{2}), \quad \text{for all } q_2 \in S_2 \text{ and } a \in X. \tag{4.29}$$

It follows from (3.38) that

$$\|f(a) - 16 f(\frac{a}{2})\| \le \frac{\alpha}{16}\left(\frac{6}{11}\varphi(a,a) + \frac{1}{22}\varphi(0,a)\right) \tag{4.30}$$

for all $a \in X$. Thus $d_2(f, \mathcal{J}_2 f) \le \frac{\alpha}{16}$. So

$$d_2(f, Q) \le \frac{1}{1-\alpha} d_2(f, \mathcal{J}_2 f) \le \frac{\alpha}{16(1-\alpha)}.$$

The rest of the proof is similar to the proof of Theorem 4.10. $\square$

Corollary 4.13. *Let r, θ be positive real numbers with $r > 4$. Suppose that $f : X \to Y$ is an even mapping satisfying (3.17) and $f(0) = 0$ for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quartic mapping $Q : X \to Y$ satisfying (3.48) for all $x = [x_{ij}] \in M_n(X)$.*

Proof. The proof follows immediately by taking $\varphi(a,b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$ and choosing $\alpha = 2^{4-r}$ in Theorem 4.12. $\square$

5. Stability of the functional equation (1.1) in matrix fuzzy normed spaces

Throughout this section, let $(X, \{N_n\})$ be a matrix fuzzy normed space, $(Y, \{Y_n\})$ be a matrix fuzzy Banach space and let n be a fixed positive integer. Using the fixed point method, we prove the Hyers-Ulam stability of the ACQ-functional equation (1.1) in matrix fuzzy normed spaces. We need the following Lemma:

Lemma 5.1 ([22]). *Let $(X, \{N_n\})$ be a matrix fuzzy normed space. Then*
(1) *$N_n(E_{kl} \otimes x, t) = N(x, t)$ for all $t > 0$ and $x \in X$;*
(2) *For all $[x_{ij}] \in M_n(X)$ and $t = \sum_{i,j=1}^{n} t_{ij}$,*

$$N(x_{kl}, t) \ge N_n([x_{ij}], t) \ge \min\{N(x_{ij}, t_{ij}) : i, j = 1, 2, \ldots, n\},$$
$$N(x_{kl}, t) \ge N_n([x_{ij}], t) \ge \min\{N(x_{ij}, \frac{t}{n^2}) : i, j = 1, 2, \ldots, n\};$$

(3) *$\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} x_{ijn} = x_{ij}$ for $x_n = [x_{ijn}], x = [x_{ij}] \in M_k(X)$.*

Theorem 5.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(a, b) \leq 8\alpha\varphi(\frac{a}{2}, \frac{b}{2}) \tag{5.1}$$

for all $a, b \in X$. Suppose that $f : X \to Y$ is an odd mapping satisfying

$$N_n(Df_n([x_{ij}], [y_{ij}]), t) \geq \frac{t}{t + \sum\limits_{i,j=1}^{n} \varphi(x_{ij}, y_{ij})} \tag{5.2}$$

for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\begin{aligned} &N_n(f_n(2[x_{ij}]) - 2f_n([x_{ij}]) - C_n([x_{ij}]), t) \\ &\qquad \geq \frac{(264 - 264\alpha)t}{(264 - 264\alpha)t + 17n^2 \sum\limits_{i,j=1}^{n} (\varphi(2x_{ij}, x_{ij}) + \varphi(0, x_{ij}))} \end{aligned} \tag{5.3}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof. Let $n = 1$ in (5.2). Then (5.2) is equivalent to

$$N(Df(a, b), t) \geq \frac{t}{t + \varphi(a, b)} \tag{5.4}$$

for all $t > 0$ and $a, b \in X$. By the same reasoning as in the proof of [12, Theorem 3], one can show that there exists a unique cubic mapping $C : X \to Y$ such that

$$N(f(2a) - 2f(a) - C(a), t) \geq \frac{(264 - 264\alpha)t}{(264 - 264\alpha)t + 17(\varphi(2a, a) + \varphi(0, a))} \tag{5.5}$$

for all $t > 0$ and $a \in X$. The mapping $C : X \to Y$ is given by

$$C(a) = N - \lim_{l\to\infty} \frac{f(2^{l+1}a) - 2f(2^l a)}{8^l}$$

for all $a \in X$.

By Lemma 5.1 and (5.5),

$$\begin{aligned} &N_n(f_n(2[x_{ij}]) - 2f_n([x_{ij}]) - C_n([x_{ij}]), t) \\ &\geq \min\{N(f(2x_{ij}) - 2f(x_{ij}) - C(x_{ij}), \frac{t}{n^2}) : i, j = 1, 2, \ldots, n\} \\ &\geq \min\{\frac{(264 - 264\alpha)t}{(264 - 264\alpha)t + 17n^2(\varphi(2x_{ij}, x_{ij}) + \varphi(0, x_{ij}))} : i, j = 1, 2, \ldots, n\} \\ &\geq \frac{(264 - 264\alpha)t}{(264 - 264\alpha)t + 17n^2 \sum\limits_{i,j=1}^{n} (\varphi(2x_{ij}, x_{ij}) + \varphi(0, x_{ij}))} \end{aligned}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$. Thus $C : X \to Y$ a unique cubic mapping satisfying (5.3), as desired. This completes the proof of the theorem. □

Corollary 5.3. *Let r, θ be positive real numbers with $r < 3$. Suppose that $f : X \to Y$ is an odd mapping satisfying*

$$N_n(Df_n([x_{ij}], [y_{ij}]), t) \geq \frac{t}{t + \sum\limits_{i,j=1}^{n} \theta(\|x_{ij}\|^r + \|y_{ij}\|^r)} \tag{5.6}$$

for all $t > 0$ *and* $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. *Then there exists a unique cubic mapping* $C : X \to Y$ *such that*

$$N_n(f_n(2[x_{ij}]) - 2f_n([x_{ij}]) - C_n([x_{ij}]), t) \geq \frac{(264 - 33 \cdot 2^r)t}{(264 - 33 \cdot 2^r)t + 17n^2(2^r + 2) \sum_{i,j=1}^{n} \theta \|x_{ij}\|^r} \tag{5.7}$$

for all $t > 0$ *and* $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Proof. The proof follows immediately by taking $\varphi(a, b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$ and choosing $\alpha = 2^{r-3}$ in Theorem 5.2. □

Theorem 5.4. *Let* $\varphi : X^2 \to [0, \infty)$ *be a function such that there exists an* $\alpha < 1$ *with*

$$\varphi(a, b) \leq \frac{\alpha}{8} \varphi(2a, 2b) \tag{5.8}$$

for all $a, b \in X$. *Suppose that* $f : X \to Y$ *is an odd mapping satisfying (5.2) for all* $t > 0$ *and* $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. *Then there exists a unique cubic mapping* $C : X \to Y$ *such that*

$$N_n(f_n(2[x_{ij}]) - 2f_n([x_{ij}]) - C_n([x_{ij}]), t) \geq \frac{(264 - 264\alpha)t}{(264 - 264\alpha)t + 17n^2\alpha \sum_{i,j=1}^{n} (\varphi(2x_{ij}, x_{ij}) + \varphi(0, x_{ij}))} \tag{5.9}$$

for all $t > 0$ *and* $x = [x_{ij}] \in M_n(X)$.

Proof. The proof is similar to the proof of Theorem 5.2. □

Corollary 5.5. *Let* r, θ *be positive real numbers with* $r > 3$. *Suppose that* $f : X \to Y$ *is an odd mapping satisfying (5.6) for all* $t > 0$ *and* $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. *Then there exists a unique cubic mapping* $C : X \to Y$ *such that*

$$N_n(f_n(2[x_{ij}]) - 2f_n([x_{ij}]) - C_n([x_{ij}]), t) \geq \frac{(33 \cdot 2^r - 264)t}{(33 \cdot 2^r - 264)t + 17n^2(2^r + 2) \sum_{i,j=1}^{n} \theta \|x_{ij}\|^r} \tag{5.10}$$

for all $t > 0$ *and* $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Proof. By choosing $\varphi(a, b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$ and $\alpha = 2^{3-r}$ in Theorem 5.4, we obtain the inequality (5.10). □

Theorem 5.6. *Let* $\varphi : X^2 \to [0, \infty)$ *is a function such that there exists an* $\alpha < 1$ *with*

$$\varphi(a, b) \leq 2\alpha \varphi(\frac{a}{2}, \frac{b}{2}) \tag{5.11}$$

for all $a, b \in X$. *Suppose that* $f : X \to Y$ *is an odd mapping satisfying (5.2) for all* $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. *Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$N_n(f_n(2[x_{ij}]) - 8f_n([x_{ij}]) - A_n([x_{ij}]), t) \geq \frac{(66 - 66\alpha)t}{(66 - 66\alpha)t + 17n^2 \sum_{i,j=1}^{n} (\varphi(2x_{ij}, x_{ij}) + \varphi(0, x_{ij}))} \tag{5.12}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. Let $n = 1$ in (5.2). Then (5.2) is equivalent to (5.4) for all $t > 0$ and $a, b \in X$. By the same reasoning as in the proof of [12, Theorem 5], one can show that there exists a unique additive mapping $A : X \to Y$ such that

$$N(f(2a) - 8f(a) - A(a), t) \geq \frac{(66 - 66\alpha)t}{(66 - 66\alpha)t + 17(\varphi(2a, a) + \varphi(0, a))} \tag{5.13}$$

for all $t > 0$ and $a \in X$. The mapping $C : X \to Y$ is given by

$$A(a) = N - \lim_{l\to\infty} \frac{f(2^{l+1}a) - 8f(2^l a)}{2^l}$$

for all $a \in X$.

By Lemma 5.1 and (5.13),

$$\begin{aligned}
&N_n(f_n(2[x_{ij}]) - 8f_n([x_{ij}]) - A_n([x_{ij}]), t) \\
&\geq \min\{N(f(2x_{ij}) - 8f(x_{ij}) - A(x_{ij}), \frac{t}{n^2}) : i, j = 1, 2, \ldots, n\} \\
&\geq \min\{\frac{(66 - 66\alpha)t}{(66 - 66\alpha)t + 17n^2(\varphi(2x_{ij}, x_{ij}) + \varphi(0, x_{ij}))} : i, j = 1, 2, \ldots, n\} \\
&\geq \frac{(66 - 66\alpha)t}{(66 - 66\alpha)t + 17n^2 \sum_{i,j=1}^{n} (\varphi(2x_{ij}, x_{ij}) + \varphi(0, x_{ij}))}
\end{aligned}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$. Thus $A : X \to Y$ a unique additive mapping satisfying (5.12), as desired. This completes the proof of the theorem. □

Corollary 5.7. *Let r, θ be positive real numbers with $r < 1$. Suppose that $f : X \to Y$ is an odd mapping satisfying (5.6) for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\begin{aligned}
&N_n(f_n(2[x_{ij}]) - 8f_n([x_{ij}]) - A_n([x_{ij}]), t) \\
&\qquad \geq \frac{(66 - 33 \cdot 2^r)t}{(66 - 33 \cdot 2^r)t + 17n^2(2^r + 2) \sum_{i,j=1}^{n} \theta\|x_{ij}\|^r}
\end{aligned} \tag{5.14}$$

for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Proof. The proof follows immediately by taking $\varphi(a, b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$ and choosing $\alpha = 2^{r-1}$ in Theorem 5.6. □

Theorem 5.8. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(a, b) \leq \frac{\alpha}{2}\varphi(2a, 2b) \tag{5.15}$$

for all $a, b \in X$. Suppose that $f : X \to Y$ is an odd mapping satisfying (5.2) for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\begin{aligned}
&N_n(f_n(2[x_{ij}]) - 8f_n([x_{ij}]) - A_n([x_{ij}]), t) \\
&\qquad \geq \frac{(66 - 66\alpha)t}{(66 - 66\alpha)t + 17n^2\alpha \sum_{i,j=1}^{n} (\varphi(2x_{ij}, x_{ij}) + \varphi(0, x_{ij}))}
\end{aligned} \tag{5.16}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. The proof is similar to the proof of Theorem 5.6. □

Corollary 5.9. *Let r, θ be positive real numbers with $r > 1$. Suppose that $f : X \to Y$ is an odd mapping satisfying (5.6) for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$N_n(f_n(2[x_{ij}]) - 8f_n([x_{ij}]) - A_n([x_{ij}]), t) \geq \frac{(33 \cdot 2^r - 66)t}{(33 \cdot 2^r - 66)t + 17n^2(2^r + 2) \sum_{i,j=1}^{n} \theta \|x_{ij}\|^r} \tag{5.17}$$

for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Proof. The proof follows immediately by taking $\varphi(a, b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$ and choosing $\alpha = 2^{1-r}$ in Theorem 5.8. □

Theorem 5.10. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(a, b) \leq 16\alpha\varphi(\frac{a}{2}, \frac{b}{2}) \tag{5.18}$$

for all $a, b \in X$. Suppose that $f : X \to Y$ is an even mapping satisfying (5.2) and $f(0) = 0$ for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quartic mapping $Q : X \to Y$ such that

$$N_n(f_n([x_{ij}]) - Q_n([x_{ij}]), t) \geq \frac{(352 - 352\alpha)t}{(352 - 352\alpha)t + 13n^2 \sum_{i,j=1}^{n} (\varphi(x_{ij}, x_{ij}) + \varphi(0, x_{ij}))} \tag{5.19}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. Let $n = 1$ in (5.2). Then (5.2) is equivalent to (5.4) for all $t > 0$ and $a, b \in X$. By the same reasoning as in the proof of [12, Theorem 7], one can show that there exists a unique quartic mapping $Q : X \to Y$ such that

$$N(f(a) - Q(a), t) \geq \frac{(352 - 352\alpha)t}{(352 - 352\alpha)t + 13(\varphi(a, a) + \varphi(0, a))} \tag{5.20}$$

for all $t > 0$ and $a \in X$. The mapping $C : X \to Y$ is given by

$$Q(a) = N - \lim_{l\to\infty} \frac{f(2^l a)}{16^l}$$

for all $a \in X$.

By Lemma 5.1 and (5.13),

$$\begin{aligned}
&N_n(f_n([x_{ij}]) - Q_n([x_{ij}]), t) \\
&\geq \min\{N(f(x_{ij}) - Q(x_{ij}), \frac{t}{n^2}) : i, j = 1, 2, \ldots, n\} \\
&\geq \min\{\frac{(352 - 352\alpha)t}{(352 - 352\alpha)t + 13n^2(\varphi(x_{ij}, x_{ij}) + \varphi(0, x_{ij}))} : i, j = 1, 2, \ldots, n\} \\
&\geq \frac{(352 - 352\alpha)t}{(352 - 352\alpha)t + 13n^2 \sum_{i,j=1}^{n} (\varphi(x_{ij}, x_{ij}) + \varphi(0, x_{ij}))}
\end{aligned}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$. Thus $Q : X \to Y$ a unique quartic mapping satisfying (5.19), as desired. This completes the proof of the theorem. □

Corollary 5.11. *Let r, θ be positive real numbers with $r < 4$. Suppose that $f : X \to Y$ is an even mapping satisfying (5.6) and $f(0) = 0$ for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quartic mapping $Q : X \to Y$ such that*

$$N_n(f_n([x_{ij}]) - Q_n([x_{ij}]), t) \geq \frac{(352 - 22 \cdot 2^r)t}{(352 - 22 \cdot 2^r)t + 39n^2 \sum_{i,j=1}^{n} \theta \|x_{ij}\|^r} \tag{5.21}$$

for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Proof. By choosing $\varphi(a, b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$ and $\alpha = 2^{r-4}$ in Theorem 5.10, we obtain the inequality (5.21). □

Theorem 5.12. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(a, b) \leq \frac{\alpha}{16} \varphi(2a, 2b) \tag{5.22}$$

for all $a, b \in X$. Suppose that $f : X \to Y$ is an even mapping satisfying (5.2) and $f(0) = 0$ for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quartic mapping $Q : X \to Y$ such that

$$N_n(f_n([x_{ij}]) - Q_n([x_{ij}]), t) \geq \frac{(352 - 352\alpha)t}{(352 - 352\alpha)t + 13n^2\alpha \sum_{i,j=1}^{n} (\varphi(x_{ij}, x_{ij}) + \varphi(0, x_{ij}))} \tag{5.23}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. The proof is similar to the proof of Theorem 5.10. □

Corollary 5.13. *Let r, θ be positive real numbers with $r > 4$. Suppose that $f : X \to Y$ is an even mapping satisfying (5.6) and $f(0) = 0$ for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique quartic mapping $Q : X \to Y$ such that*

$$N_n(f_n([x_{ij}]) - Q_n([x_{ij}]), t) \geq \frac{(22 \cdot 2^r - 352)t}{(22 \cdot 2^r - 352)t + 39n^2 \sum_{i,j=1}^{n} \theta \|x_{ij}\|^r} \tag{5.24}$$

for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Proof. The proof follows from Theorem 5.12 by taking asserted $\varphi(a, b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$. Then we can choose $\alpha = 2^{4-r}$ and we get the desired result. □

Acknowledgements

The research work was done during the first author's study to University of Louisville as a Postdoctoral Fellow in 2013-14.

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