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Hybrid algorithms for a family of pseudocontractive mappings

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Abstract

In this paper, we present an iterative algorithm with hybrid technique for a family of pseudocontractive mappings. It is shown that the suggested algorithm strongly converges to a common fixed point of a family of pseudocontractive mappings.

Keywords: Pseudocontractive mappings, hybrid algorithms, fixed point, strong convergence
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1. Introduction

Let $\mathbb{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let $\mathbb{C}$ be a nonempty closed convex subset of $\mathbb{H}$. A mapping $\mathbb{T} : \mathbb{C} \to \mathbb{C}$ is called pseudocontractive (or a pseudocontraction) if

$$\langle \mathbb{T}x^\dagger - \mathbb{T}x, x^\dagger - x \rangle \leq \|x^\dagger - x\|^2 \tag{1.1}$$

for all $x^\dagger, x \in \mathbb{C}$. It is easily seen that $\mathbb{T}$ is pseudocontractive if and only if $\mathbb{T}$ satisfies the condition:

$$\|\mathbb{T}x^\dagger - \mathbb{T}x\|^2 \leq \|x^\dagger - x\|^2 + \|(\mathbb{I} - \mathbb{T})x^\dagger - (\mathbb{I} - \mathbb{T})x\|^2 \tag{1.2}$$

for all $x^\dagger, x \in \mathbb{C}$.

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Interest in pseudocontractive mappings stems mainly from their firm connection with the class of nonlinear monotone or accretive operators. It is a classical result, see Deimling [9], that if $\mathbb{T}$ is an accretive operator, then the solutions of the equations $\mathbb{T}x = 0$ correspond to the equilibrium points of some evolution systems. It is now well-known that Mann's algorithm [11] fails to converge for Lipschitzian pseudocontractions. This explains the importance, from this point of view, of the improvement brought by the Ishikawa iteration which was introduced by Ishikawa [10] in 1974. The original result of Ishikawa is stated in the following.

Theorem 1.1. *Let $\mathbb{C}$ be a convex compact subset of a Hilbert space $\mathbb{H}$ and let $\mathbb{T} : \mathbb{C} \to \mathbb{C}$ be a Lipschitzian pseudocontractive mapping and $x_1 \in \mathbb{C}$. Then the Ishikawa iteration $\{u_n\}$ defined by*

$$\begin{cases} v_n = (1-\eta_n)u_n + \eta_n \mathbb{T}u_n, \\ u_{n+1} = (1-\xi_n)u_n + \xi_n \mathbb{T}v_n \end{cases} \tag{1.3}$$

for all $n \in \mathbb{N}$, where $\{\xi_n\}$, $\{\eta_n\}$ are sequences of positive numbers satisfying

(i) $0 \le \xi_n \le \eta_n \le 1$;
(ii) $\lim_{n\to\infty} \eta_n = 0$;
(iii) $\sum_{n=1}^{\infty} \xi_n \eta_n = \infty$,

converges strongly to a fixed point of $\mathbb{T}$.

However, strong convergence of (1.3) has not been achieved without compactness assumption on $\mathbb{T}$ or $\mathbb{C}$. Consequently, considerable research efforts, especially within the past 40 years or so, have been devoted to iterative methods for approximating fixed points of $\mathbb{T}$ when $\mathbb{T}$ is pseudocontractive (see for example [2], [5]-[7], [13], [15], [16], [18]-[24] and the references therein). On the other hand, some convergence results are obtained by using the hybrid method in mathematical programming, see, for example, [1], [3], [4], [12] ,[14], [17] and [20]. Especially, Cho, Qin and Kang [8] presented a hybrid projection algorithm and proved the following strong convergence theorem.

Theorem 1.2. *Let $\mathbb{C}$ be a nonempty closed and convex subset of a real Hilbert space $\mathbb{H}$. Let Δ be an index set and $\mathbb{T}(t) : \mathbb{C} \to \mathbb{C}$, where $t \in \Delta$, a demicontinuous pseudocontraction. Assume that $\mathfrak{F} := \bigcap_{t\ge 0} Fix(\mathbb{T}(t)) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following iterative process:*

$$\begin{cases} x_0 \in \mathbb{H}, \text{ chosen arbitrarily,} \\ \mathbb{C}_1(t) = \mathbb{C}, \mathbb{C}_1 = \bigcap_{t\in\Delta} \mathbb{C}_1(t), x_1 = proj_{\mathbb{C}_1}(x_0), \\ y_n(t) = \alpha_n(t)x_n + (1-\alpha_n(t))\mathbb{T}(t)y_n(t), \\ \mathbb{C}_{n+1}(t) = \{z \in \mathbb{C}_n(t) : \|y_n(t) - z\|^2 \le \|x_n - z\|^2 - (1-\alpha_n(t))^2\|x_n - \mathbb{T}(t)y_n(t)\|^2\}, \\ \mathbb{C}_{n+1} = \bigcap_{t\in\Delta} \mathbb{C}_{n+1}(t), \\ x_{n+1} = proj_{\mathbb{C}_{n+1}}(x_0), \quad \forall n \ge 1. \end{cases} \tag{1.4}$$

Assume that the sequence $\{\alpha_n(t)\} \subset (0,1)$ satisfies the condition $limsup_{n\to\infty}\alpha_n(t) < 1$ for every $t \in \Delta$. Then the sequence $\{x_n\}$ generated by (1.4) converges strongly to $proj_{\mathfrak{F}}(x_0)$.

Inspired by the above results, the purpose of this article is to construct a new algorithm which couples Ishikawa algorithms with hybrid techniques for finding the fixed points of a family of Lipschitzian pseudocontractive mappings. Strong convergence of the presented algorithm is given without any compactness assumption imposed on the operators.

2. Preliminaries

Recall that a mapping $\mathbb{T} : \mathbb{C} \to \mathbb{C}$ is called ζ−Lipschitzian if there exists $\zeta > 0$ such that

$$\|\mathbb{T}x^\dagger - \mathbb{T}x\| \le \zeta\|x^\dagger - x\|$$

for all $x^\dagger, x \in \mathbb{C}$.

We will use $Fix(\mathbb{T})$ to denote the set of fixed points of $\mathbb{T}$, that is, $Fix(\mathbb{T}) = \{v \in \mathbb{C} : v = \mathbb{T}v\}$. Recall that the (nearest point or metric) projection from $\mathbb{H}$ onto $\mathbb{C}$, denoted $proj_{\mathbb{C}}$, assigns, to each $u \in \mathbb{H}$, the unique point $proj_{\mathbb{C}}(u) \in \mathbb{C}$ with the property

$$\|u - proj_{\mathbb{C}}(u)\| = \inf\{\|u - x\| : x \in \mathbb{C}\}.$$

It is well known that the metric projection $proj_{\mathbb{C}}$ of $\mathbb{H}$ onto $\mathbb{C}$ is characterized by

$$\langle u - proj_{\mathbb{C}}(u), v - proj_{\mathbb{C}}(u)\rangle \leq 0 \tag{2.1}$$

for all $u \in \mathbb{H}$, $v \in \mathbb{C}$. It is well-known that in a real Hilbert space $\mathbb{H}$, the following equality holds:

$$\|\alpha u + (1-\alpha)v\|^2 = \alpha\|u\|^2 + (1-\alpha)\|v\|^2 - \alpha(1-\alpha)\|u - v\|^2 \tag{2.2}$$

for all $u, v \in \mathbb{H}$ and $\alpha \in [0,1]$.

Lemma 2.1. ([24]) *Let $\mathbb{H}$ be a real Hilbert space, $\mathbb{C}$ a closed convex subset of $\mathbb{H}$. Let $\mathbb{T} : \mathbb{C} \to \mathbb{C}$ be a continuous pseudocontractive mapping. Then*

(i) $Fix(\mathbb{T})$ is a closed convex subset of $\mathbb{C}$.
(ii) $(\mathbb{I} - \mathbb{T})$ is demiclosed at zero.

In the sequel we shall use the following notations:

- $\omega_w(u_n) = \{u : \exists u_{n_j} \to u \text{ weakly}\}$ denote the weak ω-limit set of $\{u_n\}$;
- $u_n \rightharpoonup u$ stands for the weak convergence of $\{u_n\}$ to u;
- $u_n \to u$ stands for the strong convergence of $\{u_n\}$ to u.

Lemma 2.2. ([12]) *Let $\mathbb{C}$ be a closed convex subset of $\mathbb{H}$. Let $\{u_n\}$ be a sequence in $\mathbb{H}$ and $u \in \mathbb{H}$. Let $q = proj_{\mathbb{C}}u$. If $\{u_n\}$ is such that $\omega_w(u_n) \subset \mathbb{C}$ and satisfies the condition*

$$\|u_n - u\| \leq \|u - q\| \quad \textit{for all } n \in \mathbb{N}.$$

Then $u_n \to q$.

3. Main results

In this section, we state our main results. Let $\mathbb{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. Let Δ be an index set and $\mathbb{T}(t)_{t\in\Delta} : \mathbb{C} \to \mathbb{C}$ be an η-Lipschitzian pseudocontractive mapping. Assume that $\digamma = \bigcap_{t\in\Delta} Fix(\mathbb{T}(t)) \neq \emptyset$. Firstly, we present our new algorithm which couples Ishikawa'a algorithm (1.3) with the hybrid projection algorithm.

Algorithm 3.1. *Let $x_0 \in \mathbb{H}$. For $\mathbb{C}_1(t) = \mathbb{C}$, $\mathbb{C}_1 = \bigcap_{t\in\Delta} \mathbb{C}_1(t)$ and $x_1 = proj_{\mathbb{C}_1}(x_0)$, define a sequence $\{x_n\}$ of $\mathbb{C}$ as follows:*

$$\begin{cases} y_n(t) = (1 - \varsigma_n(t))x_n + \varsigma_n(t)\mathbb{T}(t)x_n, \\ z_n(t) = \varrho_n(t)x_n + (1 - \varrho_n(t))\mathbb{T}(t)y_n(t), \\ \mathbb{C}_{n+1}(t) = \{x^* \in \mathbb{C}_n(t), \|z_n(t) - x^*\| \leq \|x_n - x^*\|\}, \\ \mathbb{C}_{n+1} = \bigcap_{t\in\Delta} \mathbb{C}_{n+1}(t), \\ x_{n+1} = proj_{\mathbb{C}_{n+1}}(x_0), \end{cases} \tag{3.1}$$

for all $n \geq 1$, where $\{\varsigma_n(t)\}$ and $\{\varrho_n(t)\}$ are two sequences in $[0,1]$.

In the sequel, we assume the sequences $\{\varsigma_n(t)\}$ and $\{\varrho_n(t)\}$ satisfy the following conditions

$$0 < k \leq 1 - \varrho_n(t) \leq \varsigma_n(t) < \frac{1}{\sqrt{1+\eta^2}+1}$$

for all $n \in \mathbb{N}$.

Remark 3.2. Without loss of generality, we can assume that the Lipschitz constant $\eta > 1$. If not, then $\mathbb{T}(t)$ is nonexpansive for all $t \in \Delta$. In this case, Algorithm 3.1 is trivial. So, in this article, we assume $\eta > 1$. It is obvious that $\frac{1}{\sqrt{1+\eta^2}+1} < \frac{1}{\eta}$ for all $n \geq 1$.

We prove the following several lemmas which will support our main theorem below.

Lemma 3.3. $\bigcap_{t\in\Delta} Fix(\mathbb{T}(t)) \subset \mathbb{C}_n$ *for* $n \geq 1$ *and* $\{x_n\}$ *is well defined.*

Proof. We use mathematical induction to prove $\bigcap_{t\in\Delta} Fix(\mathbb{T}(t)) \subset \mathbb{C}_n(t)$ for all $n \in \mathbb{N}$.

(i) $\bigcap_{t\in\Delta} Fix(\mathbb{T}(t)) \subset \mathbb{C}_1(t) = \mathbb{C}$ is obvious.

(ii) Suppose that $\bigcap_{t\in\Delta} Fix(\mathbb{T}(t)) \subset \mathbb{C}_k(t)$ for some $k \in \mathbb{N}$. Take $u \in \bigcap_{t\in\Delta} Fix(\mathbb{T}(t)) \subset \mathbb{C}_k(t)$. From (3.1), we have by using (2.2) that,

$$\begin{aligned}\|z_n(t) - u\|^2 &= \|\varrho_n(t)(x_n - u) + (1-\varrho_n(t))(\mathbb{T}(t)((1-\varsigma_n(t))x_n + \varsigma_n(t)\mathbb{T}(t)x_n) - u)\|^2 \\ &= \varrho_n(t)\|x_n - u\|^2 + (1-\varrho_n(t))\|\mathbb{T}(t)((1-\varsigma_n(t))x_n + \varsigma_n(t)\mathbb{T}(t)x_n) - u\|^2 \\ &\quad - \varrho_n(t)(1-\varrho_n(t))\|x_n - \mathbb{T}(t)((1-\varsigma_n(t))x_n + \varsigma_n(t)\mathbb{T}(t)x_n)\|^2.\end{aligned} \tag{3.2}$$

Since $u \in \bigcap_{t\in\Delta} Fix(\mathbb{T}(t))$, we have from (1.2) that

$$\|\mathbb{T}(t)x - u\|^2 \leq \|x - u\|^2 + \|x - \mathbb{T}(t)x\|^2 \tag{3.3}$$

for all $x \in \mathbb{C}_k(t)$.

From (2.2) and (3.3), we obtain

$$\begin{aligned}&\|\mathbb{T}(t)((1-\varsigma_n(t))x_n + \varsigma_n(t)\mathbb{T}(t)x_n) - u\|^2 \\ &\leq \|(1-\varsigma_n(t))x_n + \varsigma_n(t)\mathbb{T}(t)x_n - \mathbb{T}(t)((1-\varsigma_n(t))x_n + \varsigma_n(t)\mathbb{T}(t)x_n)\|^2 \\ &\quad + \|(1-\varsigma_n(t))x_n + \varsigma_n(t)\mathbb{T}(t)x_n - u\|^2 \\ &= \|(1-\varsigma_n(t))(x_n - \mathbb{T}(t)((1-\varsigma_n(t))x_n + \varsigma_n(t)\mathbb{T}(t)x_n)) \\ &\quad + \varsigma_n(t)(\mathbb{T}(t)x_n - \mathbb{T}(t)((1-\varsigma_n(t))x_n + \varsigma_n(t)\mathbb{T}(t)x_n))\|^2 \\ &\quad + \|(1-\varsigma_n(t))(x_n - u) + \varsigma_n(t)(\mathbb{T}(t)x_n - u)\|^2 \\ &= (1-\varsigma_n(t))\|x_n - \mathbb{T}(t)((1-\varsigma_n(t))x_n + \varsigma_n(t)\mathbb{T}(t)x_n)\|^2 \\ &\quad + \varsigma_n(t)\|\mathbb{T}(t)x_n - \mathbb{T}(t)((1-\varsigma_n(t))x_n + \varsigma_n(t)\mathbb{T}(t)x_n)\|^2 \\ &\quad - \varsigma_n(t)(1-\varsigma_n(t))\|x_n - \mathbb{T}(t)x_n\|^2 + (1-\varsigma_n(t))\|x_n - u\|^2 + \varsigma_n(t)\|\mathbb{T}(t)x_n - u\|^2 \\ &\quad - \varsigma_n(t)(1-\varsigma_n(t))\|x_n - \mathbb{T}(t)x_n\|^2 \\ &\leq (1-\varsigma_n(t))\|x_n - u\|^2 + \varsigma_n(t)(\|x_n - u\|^2 + \|x_n - \mathbb{T}(t)x_n\|^2) \\ &\quad - \varsigma_n(t)(1-\varsigma_n(t))\|x_n - \mathbb{T}(t)x_n\|^2 \\ &\quad + (1-\varsigma_n(t))\|x_n - \mathbb{T}(t)((1-\varsigma_n(t))x_n + \varsigma_n(t)\mathbb{T}(t)x_n)\|^2 \\ &\quad + \varsigma_n(t)\|\mathbb{T}(t)x_n - \mathbb{T}(t)((1-\varsigma_n(t))x_n + \varsigma_n(t)\mathbb{T}(t)x_n)\|^2 \\ &\quad - \varsigma_n(t)(1-\varsigma_n(t))\|x_n - \mathbb{T}(t)x_n\|^2.\end{aligned}$$

Note that $\mathbb{T}(t)$ is η-Lipschitzian for all $t \in \Delta$. It follows that

$$\begin{aligned}
&\|\mathbb{T}(t)((1-\varsigma_n(t))x_n + \varsigma_n(t)\mathbb{T}(t)x_n) - u\|^2 \\
&\le (1-\varsigma_n(t))\|x_n - u\|^2 + \varsigma_n(t)(\|x_n - u\|^2 + \|x_n - \mathbb{T}(t)x_n\|^2) \\
&\quad - \varsigma_n(t)(1-\varsigma_n(t))\|x_n - \mathbb{T}(t)x_n\|^2 \\
&\quad + (1-\varsigma_n(t))\|x_n - \mathbb{T}(t)((1-\varsigma_n(t))x_n + \varsigma_n(t)\mathbb{T}(t)x_n)\|^2 \\
&\quad + \varsigma_n^3(t)\eta^2\|x_n - \mathbb{T}(t)x_n\|^2 \\
&\quad - \varsigma_n(t)(1-\varsigma_n(t))\|x_n - \mathbb{T}(t)x_n\|^2 \\
&= \|x_n - u\|^2 + (1-\varsigma_n(t))\|x_n - \mathbb{T}(t)((1-\varsigma_n(t))x_n + \varsigma_n(t)\mathbb{T}(t)x_n)\|^2 \\
&\quad - \varsigma_n(t)(1 - 2\varsigma_n(t) - \varsigma_n^2(t)\eta^2)\|x_n - \mathbb{T}(t)x_n\|^2.
\end{aligned} \tag{3.4}$$

By condition $\varsigma_n(t) < \frac{1}{\sqrt{1+\eta^2}+1}$, we have $1 - 2\varsigma_n(t) - \varsigma_n^2(t)\eta^2 > 0$. Substituting (3.4) to (3.2), we have

$$\begin{aligned}
\|z_n(t) - u\|^2 &= \varrho_n(t)\|x_n - u\|^2 + (1-\varrho_n(t))\|\mathbb{T}(t)((1-\varsigma_n(t))x_n + \varsigma_n(t)\mathbb{T}(t)x_n) - u\|^2 \\
&\quad - \varrho_n(t)(1-\varrho_n(t))\|x_n - \mathbb{T}(t)((1-\varsigma_n(t))x_n + \varsigma_n(t)\mathbb{T}(t)x_n)\|^2 \\
&\le \varrho_n(t)\|x_n - u\|^2 + (1-\varrho_n(t))[\|x_n - u\|^2 \\
&\quad + (1-\varsigma_n(t))\|x_n - \mathbb{T}(t)((1-\varsigma_n(t))x_n + \varsigma_n(t)\mathbb{T}(t)x_n)\|^2] \\
&\quad - \varrho_n(t)(1-\varrho_n(t))\|x_n - \mathbb{T}(t)((1-\varsigma_n(t))x_n + \varsigma_n(t)\mathbb{T}(t)x_n)\|^2 \\
&= \|x_n - u\|^2 + (1-\varrho_n(t))(1-\varsigma_n(t) - \varrho_n(t))\|x_n - \mathbb{T}(t)((1-\varsigma_n(t))x_n \\
&\quad + \varsigma_n(t)\mathbb{T}(t)x_n)\|^2.
\end{aligned}$$

Since $\varsigma_n(t) + \varrho_n(t) \ge 1$, we deduce

$$\|z_n(t) - u\| \le \|x_n - u\|. \tag{3.5}$$

Hence $u \in \mathbb{C}_{k+1}(t)$. This implies that

$$\bigcap_{t\in\Delta} Fix(\mathbb{T}(t)) \subset \mathbb{C}_n(t)$$

for all $n \in \mathbb{N}$. Therefore,

$$\bigcap_{t\in\Delta} Fix(\mathbb{T}(t)) \subset \bigcap_{t\in\Delta} \mathbb{C}_n(t) = \mathbb{C}_n.$$

Next, we show that $\mathbb{C}_n$ is closed and convex for all $n \in \mathbb{N}$. It suffices to show that, for each fixed but arbitrary $t \in \Delta$, $\mathbb{C}_n(t)$ is closed and convex for each $n \ge 1$. It is obvious that $\mathbb{C}_1(t) = \mathbb{C}$ is closed and convex. Suppose that $\mathbb{C}_k(t)$ is closed and convex for some $k \in \mathbb{N}$. For $u \in \mathbb{C}_k(t)$, it is obvious that $\|z_k(t) - u\| \le \|x_k - u\|$ is equivalent to $\|z_k(t) - x_k\|^2 + 2\langle z_k(t) - x_k, x_k - u\rangle \le 0$. So, $\mathbb{C}_{k+1}(t)$ is closed and convex. Then, for any $n \in \mathbb{N}$, $\mathbb{C}_n(t)$ is closed and convex. This implies that $\{x_n\}$ is well-defined. □

Lemma 3.4. *$\{x_n\}$ is bounded.*

Proof. Using the characterized inequality (2.1) of metric projection, from $x_n = proj_{\mathbb{C}_n}(x_0)$, we have

$$\langle x_0 - x_n, x_n - y\rangle \ge 0 \ \text{ for all } \ y \in \mathbb{C}_n.$$

Since $\bigcap_{t\in\Delta} Fix(\mathbb{T}(t)) \subset \mathbb{C}_n$, we also have

$$\langle x_0 - x_n, x_n - u\rangle \ge 0 \ \text{ for all } \ u \in \bigcap_{t\in\Delta} Fix(\mathbb{T}(t)).$$

So, for $u \in \bigcap_{t\in\Delta} Fix(\mathbb{T}(t))$, we have

$$0 \le \langle x_0 - x_n, x_n - u\rangle$$

$$\begin{aligned}&= \langle x_0 - x_n, x_n - x_0 + x_0 - u\rangle \\ &= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - u\rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - u\|.\end{aligned}$$

Hence,

$$\|x_0 - x_n\| \leq \|x_0 - u\| \text{ for all } u \in \bigcap_{t\in\Delta} Fix(\mathbb{T}(t)). \tag{3.6}$$

This implies that $\{x_n\}$ is bounded. □

Lemma 3.5. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Proof. From $x_n = proj_{\mathbb{C}_n}(x_0)$ and $x_{n+1} = proj_{\mathbb{C}_{n+1}}(x_0) \in \mathbb{C}_{n+1} \subset \mathbb{C}_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0.$$

Hence,

$$\begin{aligned}0 &\leq \langle x_0 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|,\end{aligned}$$

and therefore

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|,$$

which implies that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Thus,

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \\ &\to 0.\end{aligned}$$

□

Theorem 3.6. *The sequence* $\{x_n\}$ *defined by* (3.1) *converges strongly to* $proj_{\bigcap_{t\in\Delta} Fix(\mathbb{T}(t))}(x_0)$.

Remark 3.7. Note that $\bigcap_{t\in\Delta} Fix(\mathbb{T}(t))$ is closed and convex. Thus the projection $proj_{\bigcap_{t\in\Delta} Fix(\mathbb{T}(t))}$ is well defined.

Proof. Since $x_{n+1} \in \mathbb{C}_{n+1} \subset \mathbb{C}_n$, we have

$$\|z_n(t) - x_{n+1}\| \leq \|x_n - x_{n+1}\| \to 0.$$

Further, we have

$$\|z_n(t) - x_n\| \leq \|z_n(t) - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0.$$

From (3.1), we have

$$\begin{aligned}\|x_n - \mathbb{T}(t)x_n\| &\leq \|x_n - z_n(t)\| + \|z_n(t) - \mathbb{T}(t)x_n\| \\ &\leq \|x_n - z_n(t)\| + \varrho_n(t)\|x_n - \mathbb{T}(t)x_n\| + (1 - \varrho_n(t))\|\mathbb{T}(t)y_n(t) - \mathbb{T}(t)x_n\| \\ &\leq \|x_n - z_n(t)\| + \varrho_n(t)\|x_n - \mathbb{T}(t)x_n\| + (1 - \varrho_n(t))\eta\varsigma_n(t)\|x_n - \mathbb{T}(t)x_n\| \\ &= \|x_n - z_n(t)\| + [\varrho_n(t) + (1 - \varrho_n(t))\eta\varsigma_n(t)]\|x_n - \mathbb{T}(t)x_n\|.\end{aligned}$$

Since $0 < k \leq 1 - \varrho_n(t) \leq \varsigma_n(t) < \frac{1}{\sqrt{1+\eta^2}+1}$, $1 - [\varrho_n(t) + (1 - \varrho_n(t))\eta\varsigma_n(t)] > k(1 - \frac{L}{\sqrt{1+\eta^2}+1}) > 0$. It follows that

$$\begin{aligned} \|x_n - \mathbb{T}(t)x_n\| &\leq \frac{1}{1 - [\varrho_n(t) + (1 - \varrho_n(t))\eta\varsigma_n(t)]}\|x_n - z_n(t)\| \\ &\leq \frac{1}{k(1 - \frac{\eta}{\sqrt{1+\eta^2}+1})}\|x_n - z_n(t)\| \to 0. \end{aligned} \tag{3.7}$$

Now (3.7) and Lemma 2.1 guarantee that every weak limit point of $\{x_n\}$ is a fixed point of $\mathbb{T}(t)$. That is, $\omega_w(x_n) \subset \bigcap_{t\in\Delta} Fix(\mathbb{T}(t))$. This fact, the inequality (3.6) and Lemma 2.2 ensure the strong convergence of $\{x_n\}$ to $proj_{\bigcap_{t\in\Delta} Fix(\mathbb{T}(t))}(x_0)$. This completes the proof. □

Corollary 3.8. *Let $\mathbb{C}$ be a nonempty closed and convex subset of a real Hilbert space $\mathbb{H}$. Let $\mathbb{T} : \mathbb{C} \to \mathbb{C}$ be an η-Lipschitzian pseudocontraction. Assume that $Fix(\mathbb{T}) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following iterative process:*

$$\begin{cases} x_0 \in \mathbb{H}, \text{ chosen arbitrarily,} \\ \mathbb{C}_1 = \mathbb{C}, x_1 = proj_{\mathbb{C}_1}(x_0), \\ y_n = (1 - \varsigma_n)x_n + \varsigma_n\mathbb{T}x_n, \\ z_n = \varrho_n x_n + (1 - \varrho_n)\mathbb{T}y_n, \\ \mathbb{C}_{n+1} = \{x^* \in \mathbb{C}_n, \|z_n - x^*\| \leq \|x_n - x^*\|\}, \\ x_{n+1} = proj_{\mathbb{C}_{n+1}}(x_0), \end{cases} \tag{3.8}$$

for all $n \geq 1$, where $\{\varsigma_n\}$ and $\{\varrho_n\}$ are two sequences in $[0, 1]$. Then $\{x_n\}$ generated by (3.8) converges strongly to $proj_{Fix(\mathbb{T})}(x_0)$ provided ς_n and ϱ_n satisfy the conditions

$$0 < k \leq 1 - \varrho_n \leq \varsigma_n < \frac{1}{\sqrt{1 + \eta^2} + 1}$$

for all $n \in \mathbb{N}$.

Remark 3.9. It is easily seen that all of the above results hold for a family of nonexpansive mappings.

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