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Volterra composition operators from generally weighted Bloch spaces to Bloch-type spaces on the unit ball

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Dedicated to George A Anastassiou on the occasion of his sixtieth birthday

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Abstract

Let φ be a holomorphic self-map of the open unit ball $\mathbb{B}$, $g \in H(\mathbb{B})$. In this paper, the boundedness and compactness of the Volterra composition operator T_g^φ from generally weighted Bloch spaces to Bloch-type spaces are investigated.

Keywords: Volterra composition operator; generally weighted Bloch space; boundedness; compactness.
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1. Introduction and preliminaries

Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^n$ and $H(\mathbb{B})$ the class of all holomorphic functions on $\mathbb{B}$. Let $z = (z_1, z_2, \cdots, z_n), w = (w_1, w_2, \cdots, w_n)$ be points in $\mathbb{C}^n$ and $\langle z, w\rangle = \sum_{j=1}^n z_j\overline{w_j}$. Let $\mathcal{R}f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)$ be the radial derivative of $f \in H(\mathbb{B})$, see for more details in [14].

For any $0 < \alpha < \infty$, we define the generally weighted Bloch space $\mathcal{B}^\alpha_{\log}$ of holomorphic functions such that

$$\|f\|_{\mathcal{B}^\alpha_{\log}} = |f(0)| + \sup_{z\in\mathbb{B}}(1 - |z|^2)^\alpha|\mathcal{R}f(z)| \log \frac{4}{1 - |z|^2} < \infty.$$

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When $\alpha = 1$, for the case of the unit disk the logarithmic Bloch space has appeared for the first time in characterizing of the multipliers of the Bloch space in [1], for more details in [12] and [13]. In [3], [4] and [5], we studied composition operator on generally weighted Bloch spaces.

A positive continuous function μ on the interval [0,1) is called normal ([8]) if there are $\delta \in$[0,1) and a and b, $0 < a < b$ such that

$$\frac{\mu(r)}{(1-r)^a} \text{ is decreasing on } [0,1) \text{ and } \lim_{r\to 1}\frac{\mu(r)}{(1-r)^a} = 0;$$

$$\frac{\mu(r)}{(1-r)^b} \text{ is increasing on } [0,1) \text{ and } \lim_{r\to 1}\frac{\mu(r)}{(1-r)^b} = 0.$$

If we say that a function $\mu : \mathbb{B} \to [0,\infty)$ is normal, then we will also assume that $\mu(z) = \mu(|z|)$, $z \in \mathbb{B}$.

The Bloch-type space $B_\mu(\mathbb{B})$ consists of analytic functions $f : \mathbb{B} \to C$ such that

$$\|f\|_\mu = \sup_{z\in\mathbb{B}} \mu(z)|\mathcal{R}f(z)| < \infty,$$

where μ is normal.

In [9] and [10], it was shown that $B_\mu(\mathbb{B})$ is a Banach space with the norm $\|f\|_{\mathcal{B}_\mu} = |f(0)| + \|f\|_\mu$.

The little Bloch-type space $B_{\mu,0}(\mathbb{B})$ consists of analytic functions $f : \mathbb{B} \to C$ such that

$$\lim_{|z|\to 1} \mu(z)|\mathcal{R}f(z)| = 0.$$

Let φ be a holomorphic self-map of $\mathbb{B}$. The composition operator C_φ as usual is defined by

$$(C_\varphi f)(z) = (f \circ \varphi)(z),\ f \in H(\mathbb{B}),\ z \in \mathbb{B}.$$

For some results on composition operators, see [2] or [7].

Suppose that $g : \mathbb{B} \to \mathbb{C}$ is a holomorphic map, define

$$T_g f(z) = \int_0^1 f(tz)\frac{dg(tz)}{dt} = \int_0^1 f(tz)\mathcal{R}g(tz)\frac{dt}{t},\ f \in H(\mathbb{B}), z \in \mathbb{B}. \tag{1.1}$$

This operator is called the Riemann-Stieltjes operator or extended Cesàro operator, see for example [11].

The Volterra composition operator is defined by

$$T_g^\varphi f(z) = \int_0^1 f(\varphi(tz))\frac{dg(tz)}{dt} = \int_0^1 f(\varphi(tz))\mathcal{R}g(tz)\frac{dt}{t},\ f \in H(\mathbb{B}), z \in \mathbb{B}. \tag{1.2}$$

When $\varphi(z) = z$, by (1.1) and (1.2), then $T_g^\varphi f(z) = T_g f(z)$. The Volterra composition operator is a natural extension of the Riemann-Stieltjes or extended Cesàro operator. The Volterra composition operator on the unit disk is considered in [6]. The Volterra composition operators on logarithmic Bloch spaces on $\mathbb{B}$ are studied in [15].

In this paper, we give the characterization of the boundedness and compactness of Volterra composition operator T_g^φ from generally weighted Bloch spaces to Bloch-type spaces. Throughout the remainder of this paper C will denote a positive constant, the exact value of which will vary from one appearance to the next.

2. The boundedness and compactness of $T_g^\varphi : \mathcal{B}^\alpha_{\log} \to B_\mu$

In the beginning, we introduce some auxiliary results which will be needed in our proofs of the theorems.

Lemma 2.1. *Let $f \in \mathcal{B}^{\alpha}_{\log}(\mathbb{B})$ and $z \in \mathbb{B}$, then we have*

(a) $|f(z)| \leq (1 + \frac{1}{(1-\alpha)\log 4})\|f\|_{\mathcal{B}^{\alpha}_{\log}}$, *when* $0 < \alpha < 1$;

(b) $|f(z)| \leq C\big(\log\log\frac{4}{1-|z|^2}\big)\|f\|_{\mathcal{B}^{1}_{\log}}$, *when* $\alpha = 1$;

(c) $|f(z)| \leq (1 + A(|z|))\|f\|_{\mathcal{B}^{\alpha}_{\log}}$, *when* $\alpha > 1$,

where $A(|z|) = \int_0^{|z|} \frac{dt}{(1-s^2)^{\alpha}\log\frac{4}{1-s^2}}$.

Proof. Using the integral representation for $\mathcal{R}$ differential operator, we have

$$
\begin{aligned}
|f(z)| &= \left| f(0) + \int_0^1 \frac{\mathcal{R}f(tz)dt}{t} \right| \\
&\leq |f(0)| + \int_0^1 \frac{|z|dt}{(1-|tz|^2)^{\alpha}\log\frac{4}{1-|tz|^2}} \cdot \|f\|_{\mathcal{B}^{\alpha}_{\log}} \\
&\leq |f(0)| + \int_0^{|z|} \frac{ds}{(1-s^2)^{\alpha}\log\frac{4}{1-s^2}} \cdot \|f\|_{\mathcal{B}^{\alpha}_{\log}} \\
&\leq \|f\|_{\mathcal{B}^{\alpha}_{\log}} + \int_0^{|z|} \frac{ds}{(1-s^2)^{\alpha}\log\frac{4}{1-s^2}} \cdot \|f\|_{\mathcal{B}^{\alpha}_{\log}}.
\end{aligned}
$$

For $0 < \alpha < 1$, $\alpha > 1$, then (a) and (c) hold.

For $\alpha = 1$,

$$
\begin{aligned}
|f(z)| &\leq \|f\|_{\mathcal{B}^{1}_{\log}} + 2\int_0^{|z|} \frac{ds}{(1-s)^{\alpha}\log\frac{4}{1-s}} \cdot \|f\|_{\mathcal{B}^{1}_{\log}} \\
&\leq \left(2\log(2\log\frac{4}{1-|z|^2}) + 1 - 2\log\log 4\right)\|f\|_{\mathcal{B}^{1}_{\log}} \\
&\leq \left(2\log\log\frac{4}{1-|z|^2} + \log 4 + 1 - 2\log\log 4\right)\|f\|_{\mathcal{B}^{1}_{\log}} \\
&\leq C(\log\log\frac{4}{1-|z|^2})\|f\|_{\mathcal{B}^{1}_{\log}}.
\end{aligned}
$$

□

Proposition 2.2. *Let φ be a holomorphic self-map of the open unit ball $\mathbb{B}$, $g \in H(\mathbb{B})$, and $\alpha > 0$. Then $T^{\varphi}_g : \mathcal{B}^{\alpha}_{\log}(\mathbb{B}) \to B_{\mu}(\mathbb{B})$ is compact if and only if for any bounded sequence $(f_j)_{j\in N}$ in $\mathcal{B}^{\alpha}_{\log}(\mathbb{B})$ which converges to zero uniformly on compact subsets of $\mathbb{B}$ as $j \to \infty$, $\|T^{\varphi}_g f_j\|_{\mathcal{B}_{\mu}} \to 0$ as $j \to \infty$.*

Proof. Assume that T^{φ}_g is compact and that $(f_j)_{j\in N}$ is a bounded sequence in $B^{\alpha}_{\log}$ with $f_j \to 0$ uniformly on compact subsets of $\mathbb{B}$. By the compactness of T^{φ}_g, we have that the sequence $(T^{\varphi}_g f_j)_{j\in N}$ has a subsequence $(T^{\varphi}_g f_{j_m})_{m\in N}$ which converges to f in $\mathcal{B}_{\mu}$. By Lemma 2.1 and $|f(0)| \leq \|f\|_{B^{\alpha}_{\log}}$, then for any compact $K \subset \mathbb{B}$, there is a $C \geq 0$ such that

$$
|T^{\varphi}_g f_{j_m}(z) - f(z)| \leq C\|T^{\varphi}_g f_{j_m} - f\|_{\mathcal{B}_{\mu}}, \ \forall\, z \in K.
$$

This implies that $T_g^\varphi f_{j_m}(z) - f(z) \to 0$ uniformly on compact subsets of $\mathbb{B}$ as $m \to \infty$. Since $f_{j_m} \to 0$ on compact subsets of $\mathbb{B}$, by the definition of the operator T_g^φ, it is easy to see that for each $z \in \mathbb{B}$, $\lim_{m\to\infty} T_g^\varphi f_{j_m}(z) = 0$. Hence $f = 0$. By the arbitrary of $(f_j)_{j\in N}$, we obtain that $T_g^\varphi f_j \to 0$ in $\mathcal{B}_\mu$ as $j \to \infty$.

Conversely, let $\{h_j\}$ be any sequence in the ball $K_M = B_{B^\alpha_{\log}}(0, M)$ (at the center of zero with the radius M) of the space $B^\alpha_{\log}$. Since $\|h_j\|_{B^\alpha_{\log}} \le M < \infty$, by Lemma 2.1, $\{h_j\}$ is uniformly bounded on compact subsets of $\mathbb{B}$ and hence normal by Montel's theorem. Hence we may extract a subsequence $\{h_{j_m}\}$ which converges uniformly on compact subsets of $\mathbb{B}$ to some $h \in B^\alpha_{\log}$, moreover $h \in B^\alpha_{\log}$ and $\|h\|_{B^\alpha_{\log}} \le M$. It follows that $(h_{j_m} - h)$ is that $\|h_{j_m} - h\|_{B^\alpha_{\log}} \le 2M < \infty$ and converges to zero on compact subsets of $\mathbb{B}$, by the hypothesis, we have that $T_g^\varphi h_{j_m} \to T_g^\varphi h$ in $\mathcal{B}_\mu$. Thus the set $T_g^\varphi(K)$ is relatively compact. Hence $T_g^\varphi : B^\alpha_{\log} \to \mathcal{B}_\mu$ is compact. □

Here we only consider respectively the following two cases: $0 < \alpha < 1$; $\alpha > 1$. Obviously, $\mathcal{R}(T_g^\varphi f)(z) = f(\varphi(z))\mathcal{R}g(z)$.

Theorem 2.3. *Let φ be a holomorphic self-map of the open unit ball $\mathbb{B}$, $g \in H(\mathbb{B}), 0 < \alpha < 1$. Then the following statements are equivalent.*

(i) $T_g^\varphi : \mathcal{B}^\alpha_{\log}(\mathbb{B}) \to \mathcal{B}_\mu(\mathbb{B})$ is bounded;

(ii) $T_g^\varphi : \mathcal{B}^\alpha_{\log}(\mathbb{B}) \to \mathcal{B}_\mu(\mathbb{B})$ is compact;

(iii) $g \in \mathcal{B}_\mu$.

Proof. (ii)⇒(i) By (ii) and the compactness of $T_g^\varphi : \mathcal{B}^\alpha_{\log}(\mathbb{B}) \to \mathcal{B}_\mu(\mathbb{B})$, then (i) holds.

(i)⇒(iii) By (i), then there exists a positive constant C such that $\|T_g^\varphi f\|_{\mathcal{B}_\mu} \le C\|f\|_{\mathcal{B}^\alpha_{\log}}$. By taking the test function $f = 1$ which is in $\mathcal{B}^\alpha_{\log}$, then

$$\sup_{z\in\mathbb{B}} \mu(z)|\mathcal{R}g(z)| \le C.$$

Then (iii) holds.

(iii)⇒(i) For any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $B^\alpha_{\log}$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{B}$,

$$\begin{aligned}
\|T_g^\varphi f_k\| &= \sup_{z\in\mathbb{B}} \mu(z)|\mathcal{R}(T_g^\varphi f_k)(z)| \\
&= \sup_{z\in\mathbb{B}} \mu(z)|f_k(\varphi(z))||\mathcal{R}g(z)| \\
&\le \sup_{z\in\mathbb{B}} \mu(z)(1 + \frac{1}{(1-\alpha)\log 4})|\mathcal{R}g(z)|\|f_k\|_{\mathcal{B}^\alpha_{\log}} \\
&= \sup_{z\in\mathbb{B}} \mu(z)|\mathcal{R}g(z)| \cdot \{(1 + \frac{1}{(1-\alpha)\log 4})\|f_k\|_{\mathcal{B}^\alpha_{\log}}\}.
\end{aligned}$$

By (iii), $g \in \mathcal{B}_\mu$, moreover, $f_k \in B^\alpha_{\log}$, then (i) holds. □

Theorem 2.4. *Let φ be a holomorphic self-map of the open unit ball $\mathbb{B}$, $g \in H(\mathbb{B}), \alpha > 1$. Then the following statements are equivalent.*

(i) $T_g^\varphi : \mathcal{B}^\alpha_{\log}(\mathbb{B}) \to \mathcal{B}_\mu(\mathbb{B})$ is bounded;

(ii) $g \in \mathcal{B}_\mu(\mathbb{B})$ and

$$\sup_{z\in\mathbb{B}} \mu(z)A(|\varphi(z)|)|\mathcal{R}g(z)| < \infty. \tag{2.1}$$

Proof. (ii)⇒(i) Assume that (ii) holds. Then for $f \in \mathcal{B}^\alpha_{\log}$

$$\begin{aligned}\|T_g^\varphi f\| &= \sup_{z\in\mathbb{B}} \mu(z)|\mathcal{R}(T_g^\varphi f)(z)| \\ &= \sup_{z\in\mathbb{B}} \mu(z)|f(\varphi(z))||\mathcal{R}g(z)| \\ &\le \sup_{z\in\mathbb{B}} \mu(z)(1+A(|\varphi(z)|))|\mathcal{R}g(z)|\|f\|_{\mathcal{B}^\alpha_{\log}} < \infty.\end{aligned}$$

By (ii), then we have $T_g^\varphi : \mathcal{B}^\alpha_{\log}(\mathbb{B}) \to \mathcal{B}_\mu(\mathbb{B})$ is bounded.

Conversely, let

$$f_k(z) = \int_0^{\langle z,\varphi(z_k)\rangle} \frac{dt}{(1-t)^\alpha \log\frac{4}{1-t}},\ k\in\mathbb{N},\ \text{then}\ f_k \in \mathcal{B}^\alpha_{\log}.$$

$$\begin{aligned}\infty > \|T_g^\varphi f_k\| &= \sup_{z\in\mathbb{B}} \mu(z)|f_k(\varphi(z))||\mathcal{R}g(z)| \\ &\ge \mu(z_k)|f_k(\varphi(z_k))||\mathcal{R}g(z_k)| \\ &\ge C\mu(z_k)A(|\varphi(z_k)|)|\mathcal{R}g(z_k)|.\end{aligned}$$

Then (2.1) holds. By taking the test function $f = 1$ which is in $\mathcal{B}^\alpha_{\log}$, then $g \in \mathcal{B}_\mu$. □

Theorem 2.5. *Let φ be a holomorphic self-map of the open unit ball $\mathbb{B}$, $g \in H(\mathbb{B}), \alpha > 1$. Then the following statements are equivalent.*

(i) $T_g^\varphi : \mathcal{B}^\alpha_{\log}(\mathbb{B}) \to \mathcal{B}_\mu(\mathbb{B})$ is compact;

(ii) $g \in \mathcal{B}_\mu(\mathbb{B})$ and

$$\lim_{|\varphi(z)|\to 1} \mu(z)A(|\varphi(z)|)|\mathcal{R}g(z)| = 0. \tag{2.2}$$

Proof. Assume that $T_g^\varphi : \mathcal{B}^\alpha_{\log} \to B_\mu$ is compact. Let $(\varphi(z_k))_{k\in\mathbb{N}}$ be a sequence in $\mathbb{B}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Let

$$f_k(z) = \left(\int_0^{\langle z,\varphi(z_k)\rangle} \frac{dt}{(1-t)^\alpha \log\frac{4}{1-t}}\right)^2 \left(\int_0^{|\varphi(z_k)|^2} \frac{dt}{(1-t)^\alpha \log\frac{4}{1-t}}\right)^{-1}, z\in\mathbb{B}.$$

Then $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in $\mathcal{B}^\alpha_{\log}$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{B}$,

$$\|T_g^\varphi f_k\| \ge \mu(z_k)|f_k(\varphi(z_k))||\mathcal{R}g(z_k)| \ge C\mu(z_k)A(|\varphi(z_k)|)|\mathcal{R}g(z_k)|.$$

Then (2.2) holds by letting $k \to \infty$. By taking the test function $f = 1$ which is in $\mathcal{B}^\alpha_{\log}$, then $g \in \mathcal{B}_\mu$.

Conversely, by (2.2), for any given $\varepsilon > 0$ there exists a positive number δ ($\delta < 1$) such that

$$\mu(z)A(|\varphi(z)|)|\mathcal{R}g(z_k)| < \varepsilon \text{ whenever} \delta < |\varphi(z)| < 1.$$

For any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{B}^\alpha_{\log}$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{B}$,

$$\begin{aligned}\|T_g^\varphi f_k\|_{\mathcal{B}_\mu} &= \sup_{z\in\mathbb{B}} \mu(z)|f_k(\varphi(z))||\mathcal{R}g(z)| \\ &\le \sup_{|\varphi(z)|\le\delta} \mu(z)|f_k(\varphi(z))||\mathcal{R}g(z)| \\ &\quad + \sup_{|\varphi(z)|>\delta} \mu(z)|f_k(\varphi(z))||\mathcal{R}g(z)| \\ &\le \sup_{|\varphi(z)|\le\delta} \mu(z)|\mathcal{R}g(z)| \cdot \sup_{|\varphi(z)|\le\delta} |f_k(\varphi(z))| \\ &\quad + C\|f_k\|_{\mathcal{B}^\alpha_{\log}} \sup_{|\varphi(z)|>\delta} \mu(z)|\mathcal{R}g(z)|(1+A(|\varphi(z)|)).\end{aligned}$$

Then $\|T_g^\varphi f_k\|_{\mathcal{B}_\mu} \to 0$ as $k \to \infty$. □

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