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Fixed point theorems for two new types of cyclic weakly contractive mappings in partially ordered Menger PM-spaces

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Abstract

In this paper, we introduce the concepts of cyclic weakly (ψ, ϕ)-contractive mappings and cyclic weakly (C, ψ, φ)-contractive mappings, and prove some fixed point theorems for such two types of mappings in complete partially ordered Menger PM-spaces. Some new results are obtained, which extend and generalize some fixed point results in metric and probabilistic metric spaces. Some examples are given to support our results.

Keywords: Menger PM-space, partially ordered, cyclic weakly contractive, fixed point.
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1. Introduction and Preliminaries

The study of fixed points of mappings satisfying cyclic contractive condition and weakly contractive condition has been at center of vigorous research activity in last years. In 2003, Kirk and Srinvasan [13] proved fixed point theorems for mappings satisfying cyclical contractive conditions in metric spaces. In 2009, Harjani and Sadarangani [8] obtained some fixed point theorems for weakly contractive mappings in complete metric spaces endowed with a partial order. In 2011, Karapinar [11] presented a fixed point theorem for cyclic weak ϕ-contraction in metric spaces. Harjani *et al.* [7] proved some fixed point theorems for nonlinear weakly C-contractive mappings in partially ordered metric spaces. In 2012, Karapinar [12] obtained some fixed point theorems for cyclic generalized weak ϕ-contraction on partial metric spaces. In

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2014, Alsulami [1] proved fixed point theorem for g-weakly C-contractive mappings in partial metric spaces. Meantime, other authors also obtained some corresponding results in this area [5]-[10].

The notion of a probabilistic metric space was introduced and studied by Menger [14]. The idea of Menger was to use distribution functions instead of nonnegative real numbers to describe the distance between two points. It has become an active field since then and many fixed point results for mappings satisfying different conditions have been studied [10]-[20].

The purpose of this paper is to present some fixed point theorems for cyclic weakly (ψ, ϕ)-contractive mappings and cyclic weakly (C, ψ, φ)-contractive mappings in complete partially ordered Menger PM-spaces.

We first recall some definitions from probabilistic metric spaces (see [4, 18]).

Let $\mathbb{R}$ denote the set of reals and $\mathbb{R}^+$ the nonnegative reals. A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is nondecreasing and left continuous with $\inf\limits_{t\in R} F(t) = 0$ and $\sup\limits_{t\in R} F(t) = 1$. We will denote by D the set of all distribution functions and let $D^+ = \{F \in D : F(t) = 0, \forall t \leq 0\}$.

Let H denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

Definition 1.1 ([18]). The mapping $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (for short, a t-norm) if the following conditions are satisfied:
$(\Delta - 1)$ $\Delta(a, 1) = a$, for all $a \in [0, 1]$;
$(\Delta - 2)$ $\Delta(a, b) = \Delta(b, a)$;
$(\Delta - 3)$ $\Delta(a, b) \leq \Delta(c, d)$, for $c \geq a, d \geq b$;
$(\Delta - 4)$ $\Delta(a, \Delta(b, c)) = \Delta(\Delta(a, b), c)$.

Three typical examples of continuous t-norm are $\Delta_1(a, b) = \max\{a + b - 1, 0\}$, $\Delta_2(a, b) = ab$ and $\Delta_M(a, b) = \min\{a, b\}$, for all $a, b \in [0, 1]$.

Definition 1.2 ([18]). A triplet $(X, \mathcal{F}, \Delta)$ is called a Menger probabilistic metric space (for short, a Menger PM-space), if X is a nonempty set, Δ is a t-norm and $\mathcal{F}$ is a mapping from $X \times X \to D$ satisfying the following conditions (for $x, y \in X$, we denote $\mathcal{F}(x, y)$ by $F_{x,y}$):
(MS-1) $F_{x,y}(t) = H(t)$, for all $t \in \mathbb{R}$, if and only if $x = y$;
(MS-2) $F_{x,y}(t) = F_{y,x}(t)$, for all $x, y \in X$ and $t \in \mathbb{R}$;
(MS-3) $F_{x,y}(s + t) \geq \Delta(F_{x,z}(s), F_{z,y}(t))$ for all $x, y, z \in X$ and $s, t \geq 0$.

Remark 1.3. Schweizer and Sklar [18] point out that if $(X, \mathcal{F}, \Delta)$ is a Menger probabilistic metric space and Δ is continuous, then $(X, \mathcal{F}, \Delta)$ is a Hausdorff topological space in the (ε, λ)-topology T, i.e., the family of sets $\{U_x(\varepsilon, \lambda) : \varepsilon > 0, \lambda \in (0, 1]\}$ $(x \in X)$ is a basis of neighborhoods of point x for T, where $U_x(\varepsilon, \lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\}$.

Definition 1.4 ([4]). $(X, \mathcal{F}, \Delta)$ is called a non-Archimedean Menger PM-space(shortly, a N.A Menger PM-space), if $(X, \mathcal{F}, \Delta)$ is a Menger PM-space and Δ satisfies the following condition: for all $x, y, z \in X$ and $t_1, t_2 \geq 0$,

$$F_{x,z}(\max\{t_1, t_2\}) \geq \Delta(F_{x,y}(t_1), F_{y,z}(t_2)). \tag{1.1}$$

Definition 1.5 ([4]). A non-Archimedean Menger PM-space (X, F, Δ) is said to be of type $(D)_g$ if there exists a $g \in \Omega$ such that

$$g(\Delta(s, t)) \leq g(s) + g(t),$$

for all $s, t \in [0, 1]$, where $\Omega = \{g : g : [0, 1] \to [0, \infty)$ is continuous, strictly decreasing, $g(1) = 0\}$.

Example 1.6. Let $(X, \mathcal{F}, \Delta)$ be a N.A Menger PM-space and $\Delta \geq \Delta_2$. Let $g \in \Omega$ satisfy one of the three conditions:

(1) $g(t) = 1 - t^p$ for all $t \in [0, 1]$, where $0 < p < 1$;

(2) $g(t) = |\ln t|$, for all $t \in (0, 1]$, and $g(0) = +\infty$;
(3) $g(t) = a^t - a$, for all $t \in [0, 1]$, where $\frac{1}{e} \leq a < 1$.
Then $(X, \mathcal{F}, \Delta)$ is of type $(D)_g$.

Definition 1.7 ([13]). Let X be a nonempty set, m be a positive integer, $A_1, A_2, \ldots, A_m$ be nonempty subsets of X, $Y = \cup_{i=1}^m A_i$ and a mapping $f : Y \to Y$. Then Y is said to be a cyclic representation of Y with respect to f, if
(i) A_i, $i = 1, 2, \ldots, m$, are nonempty closed sets, and
(ii) $f(A_1) \subseteq A_2$, …, $f(A_{m-1}) \subseteq A_m$, $f(A_m) \subseteq A_1$.

Example 1.8. Let $X = \mathbb{R}^+$. Let $A_1 = [0, \frac{\pi}{2}]$, $A_2 = [1, \frac{\pi}{2}]$, $A_3 = [2 - \sin 1, \frac{\pi}{2}]$ and $Y = \bigcup_{i=1}^3 A_i$. Define $f : Y \to Y$ by $fx = x - \sin x + 1$, for all $x \in Y$.

Clearly $Y = \bigcup_{i=1}^3 A_i$ is a cyclic representation of Y with respect to f.

Definition 1.9 ([10]). A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function, if it is continuous and nondecreasing in $[0, \infty)$, and $\psi(t) = 0$ if and only if $t = 0$.

Lemma 1.10 ([4]). *Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} F_{x_n,x_{n+1}}(t) = 1$ for all $t > 0$, If the sequence $\{x_n\}$ is not a Cauchy sequence in X, then there exist $\varepsilon_0 > 0$, $t_0 > 0$ and two sequences $\{k(i)\}$, $\{m(i)\}$ of positive integers such that*
(1) $m(i) > k(i)$, and $m(i) \to \infty$ as $i \to \infty$;
(2) $F_{x_{m(i)},x_{k(i)}}(t_0) < 1 - \varepsilon_0$ and $F_{x_{m(i)-1},x_{k(i)}}(t_0) \geq 1 - \varepsilon_0$, for $i = 1, 2, \cdots$.

2. Main results

In this section, we first define the concepts of cyclic weakly (ψ, ϕ)-contractive mappings and cyclic weakly (C, ψ, φ)-contractive mappings in partially ordered Menger PM-spaces.

Definition 2.1. Let $(X, \leq)$ be a partially ordered set and $(X, \mathcal{F}, \Delta)$ be a N.A Menger PM-space of type $(D)_g$. Let m be a positive integer, $A_1, A_2, \ldots, A_m$ be nonempty subsets of X and $Y = \cup_{i=1}^m A_i$. A mapping $T : X \to X$ is said to be a cyclic weakly (ψ, ϕ)-contractive, if Y is a cyclic representation of Y with respect to T, and for $k \in \{1, 2, \ldots, m\}$, $A_{m+1} = A_1$, $x \in A_k$ and $y \in A_{k+1}$ are comparable with

$$\psi(g(F_{Tx,Ty}(t))) \leq \psi(g(F_{x,y}(t))) - \phi(g(F_{x,y}(t))), \tag{2.1}$$

for all $t > 0$, where ψ is a altering distance function, $\phi : [0, \infty) \to [0, \infty)$ is a continuous function, such that $\phi(s) = 0$ if and only if $s = 0$.

Definition 2.2. Let $(X, \leq)$ be a partially ordered set and $(X, \mathcal{F}, \Delta)$ be a N.A Menger PM-space of type $(D)_g$. Let m be a positive integer, $A_1, A_2, \ldots, A_m$ be nonempty subsets of X and $Y = \cup_{i=1}^m A_i$. A mapping $T : Y \to Y$ is said to be a cyclic weakly (C, ψ, φ)-contractive, if Y is a cyclic representation of Y with respect to T, and for $k \in \{1, 2, \ldots, m\}$, $A_{m+1} = A_1$, $x \in A_k$ and $y \in A_{k+1}$ are comparable with

$$\psi(g(F_{Tx,Ty}(t))) \leq \psi(\frac{1}{2}(g(F_{x,Ty}(t)) + g(F_{Tx,y}(t)))) - \varphi(g(F_{x,Ty}(t)), g(F_{Tx,y}(t))), \tag{2.2}$$

for all $t > 0$, where ψ is a altering distance function, $\varphi : [0, \infty) \times [0, \infty) \to [0, \infty)$ is a continuous function, such that $\varphi(s, t) = 0$ if and only if $s = t = 0$.

Now, we are ready to state our main results.

Theorem 2.3. *Let $(X, \leq)$ be a partially ordered set and $(X, \mathcal{F}, \Delta)$ be a complete N.A Menger PM-space of type $(D)_g$, m be a positive integer, $A_1, A_2, \ldots, A_m$ be subsets of X and $Y = \cup_{i=1}^m A_i$. Let $T : Y \to Y$ be a cyclic weakly (ψ, ϕ)-contractive mapping satisfying* (2.1)*. Suppose that the following hold:*

(i) T is nondecreasing;

(ii) if a nondecreasing sequence $\{x_n\} \subseteq Y$ such that $x_n \to x$, then $x_n \leq x$, for all $n \in \mathbb{N}$.

If there exists $x_0 \in A_1$ such that $x_0 \leq Tx_0$, then T has a fixed point in Y. Furthermore, if for each $x, y \in Y$, there exists $z \in Y$ which is comparable to x and y, then T has a unique fixed point.

Proof. Let $x_0 \in A_1$ such that $x_0 \leq Tx_0$. Since $T(A_1) \subseteq A_2$, there exists $x_1 \in A_2$, such that $x_1 = Tx_0$. Since $T(A_2) \subseteq A_3$, there exists $x_2 \in A_3$, such that $x_2 = Tx_1$. Continuing this process, we can construct a sequence $\{x_n\}$ in Y, such that $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}$, and there exists $i_n \in \{1, 2, \ldots, m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$.

Since T is nondecreasing and $x_0 \leq Tx_0 = x_1$, we have $x_1 = Tx_0 \leq Tx_1 = x_2$. By induction, we obtain

$$x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots, \text{ for all } n \in \mathbb{N}. \tag{2.3}$$

Since $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$ are comparable, for $i_n \in \{1, 2, \ldots, m\}$, by (2.1) and (2.3), we have

$$\psi[g(F_{x_{n+1},x_n}(t))] \leq \psi[g(F_{x_n,x_{n-1}}(t))] - \phi(g(F_{x_n,x_{n-1}}(t))) \leq \psi[g(F_{x_n,x_{n-1}}(t))], \text{ for all } t > 0. \tag{2.4}$$

Since ψ is nondecreasing, it follows from (2.4) that $\{g(F_{x_{n+1},x_n}(t))\}$ is a decreasing sequence and bounded below, for every $t > 0$. Hence, there exists $r_t \geq 0$, such that $\lim_{n\to\infty} g(F_{x_{n+1},x_n}(t)) = r_t$.

By using the continuities of ψ and ϕ, letting $n \to \infty$ in (2.4), we get $\psi(r_t) \leq \psi(r_t) - \phi(r_t)$, which implies that $\phi(r_t) = 0$. Using the property of ϕ, we obtain $r_t = 0$. Thus, $\lim_{n\to\infty} g(F_{x_{n+1},x_n}(t)) = 0$ and $\lim_{n\to\infty} F_{x_{n+1},x_n}(t) = 1$, for all $t > 0$.

In the sequel, we will prove that $\{x_n\}$ is a Cauchy sequence. In order to prove this fact, we first prove the following claim.

Claim: for every $t > 0$ and $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that $p, q \geq n_0$ with $p - q \equiv 1(m)$ then $F_{x_p,x_q}(t) > 1 - \varepsilon$ and $g(F_{x_p,x_q}(t)) < g(1 - \varepsilon)$.

In fact, in oppose case, there exist $t_0 > 0$ and $\varepsilon_0 > 0$, such that for any $n \in \mathbb{N}$, we can find $p(n) > q(n) \geq n$ with $p(n) - q(n) \equiv 1(m)$ satisfying $F_{x_{p(n)},x_{q(n)}}(t_0) \leq 1 - \varepsilon_0$. Thus, $g(F_{x_{p(n)},x_{q(n)}}(t_0)) \geq g(1 - \varepsilon_0)$.

Now, take $n > 2m$. Then corresponding to $q(n) \geq n$, we can choose $p(n)$ in such a way that it is the smallest integer with $p(n) > q(n)$ satisfying $p(n) - q(n) \equiv 1(m)$ and $g(F_{x_{p(n)},x_{q(n)}}(t_0)) \geq g(1-\varepsilon_0)$. Therefore, $g(F_{x_{p(n)-m},x_{q(n)}}(t_0)) < g(1 - \varepsilon_0)$. Using the triangular inequality, we have

$$\begin{aligned} g(1-\varepsilon_0) &\leq g(F_{x_{q(n)},x_{p(n)}}(t_0)) \leq g(\Delta(F_{x_{q(n)},x_{q(n)+1}}(t_0), F_{x_{q(n)+1},x_{p(n)}}(t_0))) \\ &\leq g(F_{x_{q(n)},x_{q(n)+1}}(t_0)) + g(F_{x_{q(n)+1},x_{p(n)}}(t_0)) \\ &\leq g(F_{x_{q(n)},x_{q(n)+1}}(t_0)) + g(F_{x_{q(n)+1},x_{p(n)+1}}(t_0)) + g(F_{x_{p(n)+1},x_{p(n)}}(t_0)) \\ &\leq 2g(F_{x_{q(n)},x_{q(n)+1}}(t_0)) + g(F_{x_{q(n)},x_{p(n)}}(t_0)) + 2g(F_{x_{p(n)+1},x_{p(n)}}(t_0)) \\ &\leq 2g(F_{x_{q(n)},x_{q(n)+1}}(t_0)) + g(F_{x_{q(n)},x_{p(n)-m}}(t_0)) + g(F_{x_{p(n)-m},x_{p(n)}}(t_0)) + 2g(F_{x_{p(n)+1},x_{p(n)}}(t_0)) \\ &\leq 2g(F_{x_{q(n)},x_{q(n)+1}}(t_0)) + g(1-\varepsilon_0) + \sum_{i=1}^{m} g(F_{x_{p(n)-i},x_{p(n)-i+1}}(t_0)) + 2g(F_{x_{p(n)+1},x_{p(n)}}(t_0)). \end{aligned} \tag{2.5}$$

Since $\lim_{n\to\infty} g(F_{x_n,x_{n+1}}(t)) = 0$, for all $t > 0$, letting $n \to \infty$ in (2.5), we obtain

$$\lim_{n\to\infty} g(F_{x_{q(n)},x_{p(n)}}(t_0)) = \lim_{n\to\infty} g(F_{x_{q(n)+1},x_{p(n)+1}}(t_0)) = g(1 - \varepsilon_0). \tag{2.6}$$

By $p(n) - q(n) \equiv 1(m)$, we know that $x_{p(n)}$ and $x_{q(n)}$ lie in different adjacently labeled sets A_i and A_{i+1}, for $1 \leq i \leq m$. By (2.1) and (2.3), we have

$$\psi[g(F_{x_{q(n)+1},x_{p(n)+1}}(t_0))] \leq \psi[g(F_{x_{q(n)},x_{p(n)}}(t_0))] - \phi(g(F_{x_{q(n)},x_{p(n)}}(t_0))). \tag{2.7}$$

From the continuities of ψ and ϕ, and (2.6), letting $n \to \infty$ in (2.7), we get

$$\psi[g(1 - \varepsilon_0)] \leq \psi[g(1 - \varepsilon_0)] - \phi(g(1 - \varepsilon_0)),$$

which implies that $\phi(g(1 - \varepsilon_0)) = 0$. Hence, $g(1 - \varepsilon_0) = 0$, it follows that $\varepsilon_0 = 0$, which is in contradiction to $\varepsilon_0 > 0$. Therefore, our claim is proved.

Now, we prove that that $\{x_n\}$ is a Cauchy sequence. By the continuity of g and $g(1) = 0$, we have $\lim_{a\to 0^+} g(1 - a\epsilon) = 0$, for any given $\varepsilon > 0$. Since g is strictly decreasing, then there exists $a > 0$, such that $g(1 - a\varepsilon) \leq \frac{g(1-\varepsilon)}{2}$.

For any given $t > 0$ and $\varepsilon > 0$, there exists $a > 0$ such that $g(1 - a\varepsilon) \leq \frac{g(1-\varepsilon)}{2}$. By the claim, we find $n_0 \in N$, such that if $p, q \geq n_0$ with $p - q \equiv 1(m)$, then

$$F_{x_p,x_q}(t) > 1 - a\varepsilon \quad \text{and} \quad g(F_{x_p,x_q}(t)) < g(1 - a\varepsilon) \leq \frac{g(1-\varepsilon)}{2}. \tag{2.8}$$

Since $\lim_{n\to\infty} g(F_{x_{n+1},x_n}(t)) = 0$, we also find $n_1 \in \mathbb{N}$ such that

$$g(F_{x_{n+1},x_n}(t)) \leq \frac{g(1-\varepsilon)}{2m}, \tag{2.9}$$

for all $n > n_1$.

Suppose that $r, s \geq \max\{n_0, n_1\}$ and $s > r$. Then there exists $k \in \{1, 2, \ldots, m\}$ such that $s - r \equiv k(m)$. Therefore, $s - r + j \equiv 1(m)$, for $j = m - k + 1$ and so

$$g(F_{x_r,x_s}(t)) \leq g(F_{x_r,x_{s+j}}(t)) + g(F_{x_{s+j},x_{s+j-1}}(t)) + \cdots + g(F_{x_{s+1},x_s}(t)).$$

By (2.8), (2.9) and the last inequality, we get

$$g(F_{x_r,x_s}(t)) < \frac{g(1-\varepsilon)}{2} + j \cdot \frac{g(1-\varepsilon)}{2m} \leq \frac{g(1-\varepsilon)}{2} + \frac{g(1-\varepsilon)}{2} = g(1-\varepsilon). \tag{2.10}$$

Since g is strictly decreasing, by (2.10), we have $F_{x_r,x_s}(t) > 1 - \varepsilon$. This proves that $\{x_n\}$ is Cauchy sequence. Since $(X, \mathcal{F}, \Delta)$ is a complete PM-space, there exists $x^* \in X$, such that $x_n \to x^*$. Since $\{x_n\} \subseteq P$ and $Y = \cup_{i=0}^m A_i$ is closed, we get that $x^* \in Y$. As $Y = \cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T, then the sequence $\{x_n\}$ has infinite terms in each A_i for $i \in \{1, 2, \ldots, m\}$.

First, suppose that $x^* \in A_i$, then $Tx^* \in A_{i+1}$, and we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$(the existence of this subsequence is guaranteed by the comment above).

Since $x_n \to x^*$ and $\{x_n\}$ is a nondecreasing sequence, by the condition (ii), we have $x_{n_k} \in A_{i-1}$ and $x^* \in A_i$ are comparable, for all $k \in \mathbb{N}$. By (2.1), we have

$$\psi[g(F_{x_{n_k+1},Tx^*}(t))] \leq \psi[g(F_{x_{n_k},x^*}(t))] - \phi(g(F_{x_{n_k},x^*}(t))) \leq \psi[g(F_{x_{n_k},x^*}(t))].$$

Since ψ is nondecreasing and g is strictly decreasing, by the above inequality, we deduce

$$F_{x_{n_k+1},Tx^*}(t) \geq F_{x_{n_k},x^*}(t). \tag{2.11}$$

Let G_0 be the set of all discontinuous points of $F_{x^*,Tx^*}(\cdot)$. Moreover, we know that G_0 is a countable set. Let $G = \mathbb{R}^+ \backslash G_0$. When $t \in G$ (t is a continuous point of $F_{x^*,Tx^*}(\cdot)$), it follows from (2.11) that $F_{x^*,Tx^*}(t) \geq F_{x^*,x^*}(t) = H(t)$. Thus,

$$F_{x^*,Tx^*}(t) = H(t), \quad \text{for all } t \in G. \tag{2.12}$$

When $t \in G_0$ with $t > 0$, by the density of real numbers, there exist $t_1, t_2 \in G$ such that $0 < t_! < t < t_2$. Since the distribution is nondecreasing, we have

$$1 = H(t_1) = F_{x^*,Tx^*}(t_1) \leq F_{x^*,Tx^*}(t) \leq F_{x^*,Tx^*}(t_2) = 1.$$

This shows that, for all $t \in G_0$ with $t > 0$,

$$F_{x^*,Tx^*}(t) = H(t). \tag{2.13}$$

Hence, from (2.12) with (2.13), we obtain $F_{x^*,Tx^*}(t) = H(t)$, for all $t > 0$. Thus, $Tx^* = x^*$ and x^* is a fixed point of T.

Finally, suppose that for each $x, y \in Y$, there exists $z \in Y$ which is comparable to x and y. We prove that fixed point of T is unique. In fact, suppose that there exist $x^*, y^* \in Y$, such that $Tx^* = x^*$ and $Ty^* = y^*$, then we have $x^*, y^* \in \cap_{i=1}^m A_i$. Now, we consider the following cases:

Case 1. If $x^* \in A_i$ and $y^* \in A_{i+1}$ are comparable. By $(2,1)$, we have

$$\psi[g(F_{x^*,y^*}(t))] \leq \psi[g(F_{x^*,y^*}(t))] - \phi(g(F_{x^*,y^*}(t))), \quad \text{for all } t > 0.$$

Hence, $\phi(g(F_{x^*,y^*}(t))) = 0$, that is, $g(F_{x^*,y^*}(t)) = 0$. Thus, $F_{x^*,y^*}(t) = 1$, for all $t > 0$. Then $x^* = y^*$.

Case 2. If x^* and y^* are not comparable, then there exists $z_0 \in Y$ comparable to x^* and y^*.

First, suppose that $z_0 \in A_i$. Define a sequence $\{z_n\}$ in Y, such that $z_{n+1} = Tz_n$, for all $n \in \mathbb{N}$, and there exists $i_n \in \{1, 2, \ldots, m\}$ such that $z_n \in A_{i_n}$ and $z_{n+1} \in A_{i_n+1}$.

Since T is nondecreasing and $z_0 \in A_i$ and $x^* \in A_{i+1}$ are comparable, we have $z_1 = Tz_0$ and $Tx^* = x^*$ are comparable. By induction, we obtain $z_{n+1} = Tz_n$ and $Tx^* = x^*$ are comparable, for all $n \in \mathbb{N}$. Since $x^* \in \cap_{i=1}^m A_i$, we have z_n and x^* lie in different adjacently labeled sets, for all $n \in \mathbb{N}$. By (2.1), we have

$$\psi(g(F_{z_n,x^*}(t))) \leq \psi(g(F_{z_{n-1},x^*}(t))) - \phi(g(F_{z_{n-1},x^*}(t))) \leq \psi(g(F_{z_{n-1},x^*}(t))). \tag{2.14}$$

Since ψ is nondecreasing, we get $g(F_{z_n,x^*}(t)) \leq g(F_{z_{n-1},x^*}(t))$. Hence, $\{g(F_{z_n,x^*}(t))\}$ is a nonnegative decreasing sequence and hence possesses limit r_t, for every $t > 0$.

By the continuities of ψ and ϕ, letting $n \to \infty$ in (2.14), we have $\psi(r_t) \leq \psi(r_t) - \phi(r_t)$. Hence, $\phi(r_t) = 0$, that is, $r_t = 0$. Then $\lim\limits_{n\to\infty} g(F_{z_n,x^*}(t)) = 0$ and $\lim\limits_{n\to\infty} F_{z_n,x^*}(t) = 1$, for all $t > 0$. Thus, $\lim\limits_{n\to\infty} z_n = x^*$. Analogously, $\lim\limits_{n\to\infty} z_n = y^*$. Therefore, $x^* = y^*$.

Therefore, T has a unique fixed point. $\square$

Theorem 2.4. *Let $(X, \leq)$ be a partially ordered set and $(X, \mathcal{F}, \Delta)$ be a complete N.A Menger PM-space of type $(D)_g$, m be a positive integer, $A_1, A_2, \ldots, A_m$ be subsets of X and $Y = \cup_{i=1}^m A_i$. Let $T : Y \to Y$ be a cyclic weakly (C, ψ, φ)-contractive mapping satisfying* (2.2). *Suppose that the following hold:*

(i) T is nondecreasing;

(ii) if a nondecreasing sequence $\{x_n\} \subseteq Y$ such that $x_n \to x$, then $x_n \leq x$, for all $n \in \mathbb{N}$.

If there exists $x_0 \in A_1$ such that $x_0 \leq Tx_0$, then T has a fixed point in Y. Furthermore, if for each $x, y \in Y$, there exists $z \in Y$ which is comparable to x and y, then T has a unique fixed point.

Proof. According to the proof of Theorem 2.3, we can construct a sequence $\{x_n\}$ in Y, such that $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}$, and there exists $i_n \in \{1, 2, \ldots, m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$. Since T is nondecreasing and $x_0 \leq Tx_0 = x_1$, we also have that (2.3) holds.

Since $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$ are comparable, for $i_n \in \{1, 2, \ldots, m\}$, by (2.2) and (2.3), we get

$$\begin{aligned}
\psi[g(F_{x_{n+1},x_n}(t))] &= \psi[g(F_{Tx_n,Tx_{n-1}}(t))] \\
&\leq \psi[\frac{1}{2}(g(F_{x_n,Tx_{n-1}}(t)) + g(F_{Tx_n,x_{n-1}}(t)))] - \varphi(g(F_{x_n,Tx_{n-1}}(t)), g(F_{Tx_n,x_{n-1}}(t))) \\
&= \psi[\frac{1}{2}(g(F_{x_n,x_n}(t)) + g(F_{x_{n+1},x_{n-1}}(t)))] - \varphi(g(F_{x_n,x_n}(t)), g(F_{x_{n+1},x_{n-1}}(t))) \\
&\leq \psi[\frac{1}{2}(0 + g(F_{x_{n+1},x_{n-1}}(t)))].
\end{aligned} \tag{2.15}$$

Since h is nondecreasing, by (2.15), we have

$$\begin{aligned}
g(F_{x_{n+1},x_n}(t)) &\leq \frac{1}{2} g(F_{x_{n+1},x_{n-1}}(t)) \\
&\leq \frac{1}{2} g(\Delta(F_{x_{n+1},x_n}(t), F_{x_n,x_{n-1}}(t))) \\
&\leq \frac{1}{2}(g(F_{x_{n+1},x_n}(t)) + g(F_{x_n,x_{n-1}}(t))),
\end{aligned} \tag{2.16}$$

that is,

$$g(F_{x_{n+1},x_n}(t)) \le g(F_{x_n,x_{n-1}}(t)). \tag{2.17}$$

From (2.17), it implies that$\{g(F_{x_{n+1},x_n}(t)\}$ is a decreasing sequence and bounded below, for every given $t > 0$. Hence, there exists $r_t \ge 0$ such that

$$\lim_{n\to\infty} g(F_{x_{n+1},x_n}(t)) = r_t. \tag{2.18}$$

By (2.18), letting $n \to \infty$ in (2.16), we get

$$r_t \le \lim_{n\to\infty} \frac{1}{2} g(F_{x_{n+1},x_{n-1}}(t)) \le \frac{1}{2}(r_t + r_t),$$

that is,

$$\lim_{n\to\infty} g(F_{x_{n+1},x_{n-1}}(t)) = 2r_t.$$

By using the continuities of ψ and φ, letting $n \to \infty$ in (2.15), we get

$$\psi(r_t) \le \psi(\frac{1}{2}(2r_t)) - \varphi(0, 2r_t) = \psi(r_t) - \varphi(0, 2r_t),$$

which implies that $\varphi(0, 2r_t) = 0$. Thus, $r_t = 0$. Then $\lim\limits_{n\to\infty} g(F_{x_{n+1},x_n}(t)) = 0$ and $\lim\limits_{n\to\infty} F_{x_{n+1},x_n}(t) = 1$, for all $t > 0$.

In the sequel, we will prove that $\{x_n\}$ is a Cauchy sequence. In order to prove this fact, we first prove the following claim.

Claim: for every $t > 0$ and $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that $p, q \ge n_0$ with $p - q \equiv 1(m)$, then $F_{x_p,x_q}(t) > 1 - \varepsilon$ and $g(F_{x_p,x_q}(t)) < g(1 - \varepsilon)$.

In fact, in the oppose case, there exist $t_0 > 0$ and $\varepsilon_0 > 0$, such that for any $n \in \mathbb{N}$, we can find $p(n) > q(n) \ge n$ with $p(n) - q(n) \equiv 1(m)$ satisfying $F_{x_{p(n)},x_{q(n)}}(t_0) \le 1 - \varepsilon_0$. Thus, $g(F_{x_{p(n)},x_{q(n)}}(t_0)) \ge g(1 - \varepsilon_0)$.

Now, take $n > 2m$. Then corresponding to $q(n) \ge n$, we can choose $p(n)$ in such away that it is the smallest integer with $p(n) > q(n)$ satisfying $p(n) - q(n) \equiv 1(m)$ and $g(F_{x_{p(n)},x_{q(n)}}(t_0)) \ge g(1-\varepsilon_0)$. Therefore, $g(F_{x_{p(n)-m},x_{q(n)}}(t_0)) < g(1 - \varepsilon_0)$. Using the triangular inequality, we have

$$\begin{aligned} g(1 - \varepsilon_0) &\le g(F_{x_{q(n)},x_{p(n)}}(t_0)) \le g(\Delta(F_{x_{q(n)},x_{q(n)+1}}(t_0), F_{x_{q(n)+1},x_{p(n)}}(t_0))) \\ &\le g(F_{x_{q(n)},x_{q(n)+1}}(t_0)) + g(F_{x_{q(n)+1},x_{p(n)}}(t_0)) \\ &\le g(F_{x_{q(n)},x_{q(n)+1}}(t_0)) + g(F_{x_{q(n)+1},x_{p(n)-1}}(t_0)) + g(F_{x_{p(n)-1},x_{p(n)}}(t_0)) \\ &\le g(F_{x_{q(n)},x_{q(n)+1}}(t_0)) + g(F_{x_{q(n)+1},x_{p(n)}}(t_0)) + 2g(F_{x_{p(n)-1},x_{p(n)}}(t_0)) \\ &\le 2g(F_{x_{q(n)},x_{q(n)+1}}(t_0)) + g(F_{x_{q(n)},x_{p(n)-m}}(t_0)) + g(F_{x_{p(n)-m},x_{p(n)}}(t_0)) + 2g(F_{x_{p(n)-1},x_{p(n)}}(t_0)) \\ &\le 2g(F_{x_{q(n)},x_{q(n)+1}}(t_0)) + g(1 - \varepsilon_0) + \sum_{i=1}^{m} g(F_{x_{p(n)-i},x_{p(n)-i+1}}(t_0)) + 2g(F_{x_{p(n)-1},x_{p(n)}}(t_0)). \end{aligned} \tag{2.19}$$

Since $\lim\limits_{n\to\infty} g(F_{x_{n+1},x_n}(t)) = 0$, for all $t > 0$, letting $n \to \infty$ in (2.19), we obtain

$$\lim_{n\to\infty} g(F_{x_{q(n)},x_{p(n)}}(t_0)) = \lim_{n\to\infty} g(F_{x_{q(n)+1},x_{p(n)-1}}(t_0)) = \lim_{n\to\infty} g(F_{x_{q(n)+1},x_{p(n)}}(t_0)) = g(1 - \varepsilon_0). \tag{2.20}$$

By $p(n) - q(n) \equiv 1(m)$, we know that $x_{p(n)}$ and $x_{q(n)}$ lie in different adjacently labeled sets A_i and A_{i+1}, for $1 \le i \le m$. By (2.2) and (2.3), we have

$$\begin{aligned} \psi[g(F_{x_{p(n)},x_{q(n)+1}}(t_0))] \le& \psi[\frac{1}{2}(g(F_{x_{p(n)-1},x_{q(n)+1}}(t_0)) + g(F_{x_{p(n)},x_{q(n)}}(t_0)))] \\ &- \varphi(g(F_{x_{p(n)-1},x_{q(n)+1}}(t_0)), g(F_{x_{p(n)},x_{q(n)}}(t_0))). \end{aligned}$$

By the continuities of ψ and φ, and (2.20), letting $n \to \infty$ in last inequality, we have

$$\psi[g(1-\varepsilon_0)] \leq \psi[g(1-\varepsilon_0)] - \varphi(g(1-\varepsilon_0), g(1-\varepsilon_0)),$$

which implies that $\varphi(g(1-\varepsilon_0)), g(1-\varepsilon_0)) = 0$. Thus, $g(1-\varepsilon_0) = 0$. Then $\varepsilon_0 = 0$, which is in contradiction to $\varepsilon_0 > 0$. Therefore, our claim is proved.

By the claim and using the same arguments in the proof of Theorem 2.3, we know that $\{x_n\}$ is a Cauchy sequence. Since $(X, \mathcal{F}, \Delta)$ is a complete PM-space, there exists $x^* \in X$, such that $x_n \to x^*$. Since $\{x_n\} \subseteq P$ and $Y = \cup_{i=0}^m A_i$ is closed, we get that $x^* \in Y$. As $Y = \cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T, the sequence $\{x_n\}$ has infinite terms in each A_i for $i \in \{1, 2, \ldots, m\}$.

First, suppose that $x^* \in A_i$, then $Tx^* \in A_{i+1}$, and we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$.

Since $\{x_n\}$ is a nondecreasing sequence and $x_n \to x^*$, by the condition (ii), we have $x_{n_k} \in A_{i-1}$ and $x^* \in A_i$ are comparable, for all $k \in N$. By (2.2), we have

$$\begin{aligned}\psi[g(F_{x_{n_k+1},Tx^*}(t))] &\leq \psi[\frac{1}{2}(g(F_{x_{n_k},Tx^*}(t)) + g(F_{x_{n_k+1},x^*}(t)))] - \varphi(g(F_{x_{n_k},Tx^*}(t)), g(F_{x_{n_k+1},x^*}(t))) \\ &\leq \psi[\frac{1}{2}(g(F_{x_{n_k},Tx^*}(t)) + g(F_{x_{n_k+1},x^*}(t)))].\end{aligned}$$

Since ψ is nondecreasing, by the above inequality, we get

$$g(F_{x_{n_k+1},Tx^*}(t)) \leq \frac{1}{2}(g(F_{x_{n_k},Tx^*}(t)) + g(F_{x_{n_k+1},x^*}(t))). \tag{2.21}$$

Let G_0 be the set of all discontinuous points of $F_{x^*,Tx^*}(\cdot)$. Since g is continuous and strictly decreasing, we get that G_0 also is the set of all discontinuous points of $g(F_{x^*,Tx^*}(\cdot))$. Moreover, we know that G_0 is a countable set. Let $G = \mathbb{R}^+ \backslash G_0$. When $t \in G$ (t is a continuous point of $F_{x^*,Tx^*}(\cdot)$), it follows from (2.21) that $g(F_{x^*,Tx^*}(t)) \leq \frac{1}{2}[g(F_{x^*,x^*}(t)) + g(F_{x^*,Tx^*}(t))] = \frac{1}{2}g(F_{x^*,Tx^*}(t))$. Thus, $g(F_{x^*,Tx^*}(t)) = 0$. Then

$$F_{x^*,Tx^*}(t) = H(t), \quad \text{for all } t \in G. \tag{2.22}$$

When $t \in G_0$ with $t > 0$, by the density of real numbers, there exist $t_1, t_2 \in G$ such that $0 < t_! < t < t_2$. Since the distribution is nondecreasing, we have $1 = H(t_1) = F_{x^*,Tx^*}(t_1) \leq F_{x^*,Tx^*}(t) \leq F_{x^*,Tx^*}(t_2) = 1$. This shows that, for all $t \in G_0$ with $t > 0$,

$$F_{x^*,Tx^*}(t) = H(t). \tag{2.23}$$

Hence, from (2.22) with (2.23), we have $F_{x^*,Tx^*}(t) = H(t)$, for all $t > 0$. Thus, $Tx^* = x^*$ and x^* is a fixed point of T.

Finally, suppose that for each $x, y \in Y$, there exists $z \in Y$ which is comparable to x and y. We prove that fixed point of T is unique. In fact, suppose that there exist $x^*, y^* \in Y$, such that $Tx^* = x^*$ and $Ty^* = y^*$, then we have $x^*, y^* \in \cap_{i=1}^m A_i$. Now, we consider the following cases:

Case 1. If $x^* \in A_i$ and $y^* \in A_{i+1}$ are comparable. By (2.2), we have

$$\psi[g(F_{x^*,y^*}(t))] \leq \psi[\frac{1}{2}(g(F_{x^*,y^*}(t)) + g(F_{x^*,y^*}(t)))] - \varphi(g(F_{x^*,y^*}(t)), g(F_{x^*,y^*}(t))), \quad \text{for all } t > 0,$$

which implies that $\varphi(g(F_{x^*,y^*}(t), g(F_{x^*,y^*}(t)) = 0$. Thus, $g(F_{x^*,y^*}(t))) = 0$ and $F_{x^*,y^*}(t) = H(t)$ for all $t > 0$. Then $x^* = y^*$.

Case 2. If x^* and y^* are not comparable, then there exists $z_0 \in Y$ comparable to x^* and y^*.

First, suppose that $z_0 \in A_i$. By the proof of Theorem 2.3, we can construct a sequence $\{z_n\}$ in Y, such that $z_{n+1} = Tz_n$, for all $n \in \mathbb{N}$, there exists $i_n \in \{1, 2, \ldots, m\}$ such that $z_n \in A_{i_n}$ and $z_{n+1} \in A_{i_n+1}$, and we have $z_{n+1} = Tz_n$ and $Tx^* = x^*$ are comparable, for all $n \in N$. Since $x^* \in \cap_{i=1}^m A_i$, we have z_n and x^* lie in different adjacently labeled sets, for all $n \in \mathbb{N}$. By (2.2), we have

$$\begin{aligned}\psi[g(F_{z_n,x^*}(t))] &= \psi[g(F_{Tz_{n-1},Tx^*}(t))]\\ &\le \psi[\frac{1}{2}(g(F_{z_{n-1},x^*}(t)) + g(F_{z_n,x^*}(t)))] - \varphi(g(F_{z_{n-1},x^*}(t)), g(F_{z_n,x^*}(t)))\\ &\le \psi[\frac{1}{2}(g(F_{z_{n-1},x^*}(t)) + g(F_{z_n,x^*}(t)))].\end{aligned} \tag{2.24}$$

Since ψ is nondecreasing, by (2.24), we get

$$g(F_{z_n,x^*}(t)) \le \frac{1}{2}(g(F_{z_{n-1},x^*}(t)) + g(F_{z_n,x^*}(t))).$$

Hence, $g(F_{z_n,x^*}(t)) \le g(F_{z_{n-1},x^*}(t))$, which implies that $\{g(F_{z_n,x^*}(t))\}$ is a nonnegative decreasing sequence and hence possesses limit r_t, for every $t > 0$.

By the continuities of ψ and φ, letting $n \to \infty$ in (2.24), then $\psi(r_t) \le \psi(r_t) - \varphi(r_t, r_t)$. Hence, $\varphi(r_t, r_t) = 0$, that is, $r_t = 0$. Thus, $\lim_{n\to\infty} g(F_{z_n,x^*}(t)) = 0$ and $\lim_{n\to\infty} F_{z_n,x^*}(t) = 1$, for all $t > 0$. Then $\lim_{n\to\infty} z_n = x^*$. Analogously, $\lim_{n\to\infty} z_n = y^*$. Therefore, $x^* = y^*$.

Then T has a unique fixed point. □

Remark 2.5. Theorem 2.3 and Theorem 2.4 extend the fixed point theorems of weakly contractive mappings in metric spaces to probabilistic metric spaces, and they extend and generalize many existing fixed point theorems in the literature [1-6] and [12-14].

Now, in order to support the usability of our results, we present the following examples.

Example 2.6. Let $X = \mathbb{R}^+$, $\Delta = \Delta_2$, $g(t) = |\ln t|$, for all $t \in (0, 1]$ and $g(0) = +\infty$. Define $\mathcal{F} : X\times X \to D^+$ by

$$\mathcal{F}(x, y)(t) = F_{x,y}(t) = \begin{cases} e^{-\frac{|x-y|}{t}}, & t > 0;\\ 0, & t \le 0.\end{cases}$$

for all $x, y \in X$. Then $(X, \mathcal{F}, \Delta)$ is a complete N.A Menger PM-space. In fact, for all $t_1, t_2 > 0$,

$$\begin{aligned}F_{x,y}(\max\{t_1, t_2\}) &= e^{-\frac{|x-y|}{\max\{t_1,t_2\}}} \ge e^{-\frac{|x-z|+|z-y|}{\max\{t_1,t_2\}}}\\ &\ge e^{-\frac{|x-z|}{t_1}} \cdot e^{-\frac{|z-y|}{t_2}} = \Delta(F_{x,z}(t_1), F_{z,y}(t_2)).\end{aligned}$$

By Example 1.6, we obtain $(X, \mathcal{F}, \Delta)$ is of type $(D)_g$.

Suppose that $A_1 = [1, 27]$, $A_2 = [1, 9]$, $A_3 = [1, 3]$, and $Y = \cup_{i=1}^3 A_i = [1, 27]$. Define $T : Y \to Y$ by $Tx = x^{\frac{1}{3}}$, for all $x \in Y$. Then T is nondecreasing and Y is a cyclic representation of Y with respect to T

Let $\psi(t) = \frac{t}{2}$, $\phi(t) = \frac{t}{3}$, for all $t \in [0, \infty)$. Now, we verify inequality (2.1) in Theorem 2.3. By the definitions of F, g, ψ and ϕ, we only need to prove that

$$\frac{1}{2} \cdot \frac{|Tx - Ty|}{t} \le \frac{1}{2} \cdot \frac{|x - y|}{t} - \frac{1}{3} \cdot \frac{|x - y|}{t},$$

for all $t > 0$, that is,

$$|Tx - Ty| \le \frac{1}{3}|x - y|, \text{ for all } x, y \in Y. \tag{2.25}$$

For all $x, y \in Y = [1, 27]$, we have $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} \ge 3$. According to the definition of T, we get

$$|Tx - Ty| = |x^{\frac{1}{3}} - y^{\frac{1}{3}}| \le \frac{x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}}{3} \cdot |x^{\frac{1}{3}} - y^{\frac{1}{3}}| = \frac{1}{3}|x - y|,$$

which implies that (2.25) holds.

Also, conditions (i) and (ii) of Theorem 2.3 are satisfied. Therefore, from Theorem 2.3, we obtain that T has a fixed point in Y, indeed, $x = 1$ is a fixed point of T.

Example 2.7. Let $X = \mathbb{R}^+$, $\Delta = \Delta_2$, $g : [0, 1] \to [0, +\infty)$ and $\mathcal{F} : X \times X \to D^+$ be the same as the ones in Example 2.6. Then $(X, \mathcal{F}, \Delta)$ is a complete N.A Menger PM-space of type $(D)_g$.

Suppose that $A_1 = [1, 4]$, $A_2 = [1, 3]$, $A_3 = [1, 2]$, and $Y = \cup_{i=1}^3 A_i = [1, 4]$. Define $T : Y \to Y$ by $Tx = x^{\frac{1}{2}}$, for all $x \in Y$. Then T is nondecreasing and Y is a cyclic representation of Y with respect to T

Let $\psi(t) = t$, $\varphi(s, t) = \frac{s+t}{6}$, for all $s, t \in [0, \infty)$. Now, we verify inequality (2.2) in Theorem 2.4. By the definitions of F, g, ψ and φ, we only need to prove that

$$\frac{|Tx - Ty|}{t} \leq \frac{1}{2}[\frac{|x - Ty|}{t} + \frac{|Tx - y|}{t}] - \frac{1}{6}[\frac{|x - Ty|}{t} + \frac{|Tx - y|}{t}],$$

for all $t > 0$, that is,

$$|Tx - Ty| \leq \frac{1}{3}[|x - Ty| + |Tx - y|], \text{ for all } \ x, y \in Y. \tag{2.26}$$

We consider the following cases:

First, suppose that $x \leq y$, $x, y \in Y$. For all $x, y \in Y = [1, 27]$, we have $x^{\frac{1}{2}} + y^{\frac{1}{2}} \geq 2$.

Case 1. If $x \leq y^{\frac{1}{2}} \leq y$, then $x^{\frac{1}{2}} \leq x \leq y^{\frac{1}{2}} \leq y$. Hence,

$$\begin{aligned}
&\frac{1}{3}[|x - Ty| + |Tx - y|] - |Tx - Ty| = \frac{1}{3}[(y - x^{\frac{1}{2}}) + (y^{\frac{1}{2}} - x)] - (y^{\frac{1}{2}} - x^{\frac{1}{2}}) \\
=&\frac{1}{3}(y - x) - \frac{2}{3}(y^{\frac{1}{2}} - x^{\frac{1}{2}}) = \frac{1}{3}(y^{\frac{1}{2}} - x^{\frac{1}{2}})(y^{\frac{1}{2}} + x^{\frac{1}{2}} - 2) \geq 0,
\end{aligned}$$

which implies that (2.26) holds.

Case 2. If $y^{\frac{1}{2}} \leq x \leq y$, then $x^{\frac{1}{2}} \leq y^{\frac{1}{2}} \leq x \leq y$. Let

$$\begin{aligned}
h(y) =&\frac{1}{3}[|x - Ty| + |Tx - y|] - |Tx - Ty| = \frac{1}{3}[(y - x^{\frac{1}{2}}) + (x - y^{\frac{1}{2}})] - (y^{\frac{1}{2}} - x^{\frac{1}{2}}) \\
=&\frac{1}{3}y + \frac{1}{3}x - \frac{4}{3}y^{\frac{1}{2}} + \frac{2}{3}x^{\frac{1}{2}},
\end{aligned}$$

for all $y \in [x, \min\{x^2, 4\}]$. Hence, $h'(y) = \frac{1}{3} - \frac{2}{3}y^{-\frac{1}{2}} \leq \frac{1}{3} - \frac{2}{3} \cdot 4^{-\frac{1}{2}} = 0$, for all $y \in [x, \min\{x^2, 4\}]$. Thus, h is decreasing. If $\min\{x^2, 4\} = 4$, then $x \geq 2$. Hence,

$$h_{\min} = h(4) = \frac{4}{3} + \frac{x}{3} + \frac{2}{3}x^{\frac{1}{2}} - \frac{4}{3} \cdot 2 \geq \frac{2}{3} + \frac{2\sqrt{2}}{3} - \frac{4}{3} \geq 0.$$

If $\min\{x^2, 4\} = x^2$, then $1 \leq x \leq 2$ and $y \leq x^2 \leq 4$. Hence,

$$h_{\min} = h(x^2) = \frac{x^2}{3} + \frac{x}{3} + \frac{2}{3}x^{\frac{1}{2}} - \frac{4x}{3} = \frac{x^2}{3} + \frac{2}{3}x^{\frac{1}{2}} - x \geq 0.$$

Thus, $h(y) \geq 0$, for all $y \in [x, \min\{x^2, 4\}]$, which implies that (2.26) holds.

Similarly, If $x > y$, for $x, y \in Y$. we also have (2.26) holds.

Also, conditions (i) and (ii) of Theorem 2.4 are satisfied. Therefore, from Theorem 2.4, we obtain that T has a fixed point in Y, indeed, $x = 1$ is a fixed point of T.

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