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New common fixed point theorem for a family of non-self mappings in cone metric spaces

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Abstract

In this paper, we prove a common fixed point theorem for a family of non-self mappings satisfying generalized contraction condition of Ciric type in cone metric spaces (over the cone which is not necessarily normal). Our result generalizes and extends all the recent results related to non-self mappings in the setting of cone metric space.

Keywords: Cone metric spaces, Common fixed point, Non-self mappings, Contraction condition of Ciric type.
2010 MSC: 47H10, 54H25.

1. Introduction and Preliminaries

The existing literature of fixed point theory contains many results enunciating fixed point theorems for self-mappings in metric and Banach spaces. Recently, Huang and Zhang [9] generalized the concept of a metric space, replacing the set of real numbers by ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions. Subsequently, the study of fixed point theorems in such spaces is followed by some other mathematicians, see [1, 2, 11, 14, 15, 17, 20, 25, 26]. The study of fixed point theorems for non-self mappings in metrically convex metric spaces was initiated by Assad and Kirk [4]. Utilizing the induction method of Assad and Kirk [4], many authors like Assad [3], Ciric [5], Hadzic [7], Hadzic and Gajic [8], Imdad and Kumar [12], Rhoades [21, 22, 23] have obtained common fixed point in metrically convex spaces. Recently, Ciric and Ume [6] defined a wide class of multi-valued non-self mappings which satisfy a generalized contraction condition and proved a fixed point theorem which generalize the results of Itoh [13] and Khan [16].

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Very recently, Radenovic and Rhoades [17] extended the fixed point theorem of Imdad and Kumar [12] for a pair of non-self mappings to non-normal cone metric spaces. Jankovic et al. [15] proved new common fixed point results for a pair of non-self mappings defined on a closed subset of metrically convex cone metric space which is not necessarily normal by adapting Assad-Kirks method. Huang et al.[10] proved a fixed point theorem for a family of non-self mappings in cone metric spaces which generalizes the result of Jankovic et al. [15]. Sumitra et al. [24] generalized the fixed point theorems of Ciric and Ume [6] for a pair of non-self mappings to non-normal cone metric spaces. In the same time, Sumitra et al.'s [24] results extended the results of Jankovic et al. [15] and Radenovic and Rhoades [17]. Motivated by Sumitra et al. [24], we prove a common fixed point theorem for a family of non-self mappings on cone metric spaces in which the cone need not be normal and the condition is weaker. This result generalizes the result of Sumitra et al. [24] and Huang et al.[10].

Consistent with Huang and Zhang [9], the following definitions and results will be needed in the sequel.

Let E be a real Banach space. A subset P of E is called a cone if and only if:

(a) P is closed, nonempty and $P \neq \{\theta\}$;
(b) $a, b \in R, a, b \geq 0, x, y \in P$ implies $ax + by \in P$;
(c) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \text{ implies } \| x \| \leq K \| y \|.$$

The least positive number satisfying the above inequality is called the normal constant of P, while $x \ll y$ stands for $y - x \in intP$ (interior of P).

Definition 1.1 ([6]). Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d1) $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space.

Definition 1.2 ([9]). Let (X, d) be a cone metric space. We say that $\{x_n\}$ is:

(e) a Cauchy sequence if for every $c \in E$ with $\theta \ll c$, there is an N such that for all $n, m > N, d(x_n, x_m) \ll c$;
(f) a Convergent sequence if for every $c \in E$ with $\theta \ll c$, there is an N such that for all $n > N, d(x_n, x) \ll c$ for some fixed $x \in X$.

A cone metric space X is said to be complete if for every Cauchy sequence in X is convergent in X. It is known that $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to \theta$ as $n \to \infty$. It is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta(n, m \to \infty)$.

Remark 1.3 ([27]). Let E be an ordered Banach (normed) space. Then c is an interior point of P, if and only if $[-c, c]$ is a neighborhood of θ.

Corollary 1.4 ([19]). *(1) If $a \leq b$ and $b \ll c$, then $a \ll c$. Indeed, $c - a = (c - b) + (b - a) \geq c - b$ implies $[-(c - a), c - a] \supseteq [-(c - b), c - b]$.*

(2) If $a \ll b$ and $b \ll c$, then $a \ll c$. Indeed, $c - a = (c - b) + (b - a) \geq c - b$ implies $[-(c - a), c - a] \supseteq [-(c - b), c - b]$.

(3) If $\theta \leq u \ll c$ for each $c \in intP$ then $u = \theta$.

Remark 1.5 ([14, 17]). If $c \in intP, \theta \leq a_n$ and $a_n \to \theta$, then there exists an n_0 such that for all $n > n_0$ we have $a_n \ll c$.

Remark 1.6 ([19, 20]). If E is a real Banach space with cone P and if $a \leq ka$ where $a \in P$ and $0 < k < 1$, then $a = \theta$.

Definition 1.7 ([1]). Let f and g be self maps on a set X (i.e., $f, g : X \to X$). If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. Self maps f and g are said to be weakly compatible if they commute at their coincidence point; i.e., if $fx = gx$ for some $x \in X$, then $fgx = gfx$.

2. Main result

The following theorem is Sumitra et,al [24] generalization of Ciric and Ume's [6] result in cone metric spaces.

Theorem 2.1. *Let (X, d) be a complete cone metric space, C a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ (the boundary of C) such that*

$$d(x, z) + d(z, y) = d(x, y).$$

Suppose that $f, g : C \to X$ are two non-self mappings satisfying for all $x, y \in C$ with $x \neq y$,

$$d(gx, gy) \leq \alpha d(fx, fy) + \beta u + \gamma v \tag{2.1}$$

where $u \in \{d(fx, gx), d(fy, gy)\}$, $v \in \{d(fx, gx) + d(fy, gy), d(fx, gy) + d(fy, gx)\}$, and α, β, γ are nonnegative real numbers such that

$$\alpha + 2\beta + 3\gamma + \alpha\gamma < 1. \tag{2.2}$$

Also assume that

(i) $\partial C \subseteq fC, gC \cap C \subseteq fC$,
(ii) $fx \in \partial C$ implies that $gx \in C$,
(iii) fC is closed in X.

Then the pair (f, g) has a coincidence point in C. Moreover, if pair (f, g) is weakly compatible, then f and g have a unique common fixed point in C.

Remark 2.2. From the proof of this theorem, it is easy to see that condition (2.2) can be weakened to $\alpha + \beta + 2\gamma < 1$.

The purpose of this paper is to extend above theorem for a family of non-self mappings in cone metric spaces with weaker condition.

We state and prove our main result as follows.

Theorem 2.3. *Let (X, d) be a complete cone metric space, C a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x, z) + d(z, y) = d(x, y).$$

Suppose that $\{F_n\}_{n=1}^{\infty}, S, T : C \to X$ are a family of non-self mappings satisfying for all $i = 2n - 1, j = 2n (n \in N)$ and $x, y \in C$ with $x \neq y$,

$$d(F_i x, F_j y) \leq \alpha d(Tx, Sy) + \beta u + \gamma v \tag{2.3}$$

where $u \in \{d(Tx, F_i x), d(Sy, F_j y)\}$, $v \in \{d(Tx, F_i x) + d(Sy, F_j y), d(Tx, F_j y) + d(Sy, F_i x)\}$ and α, β, γ are nonnegative real numbers such that

$$\alpha + \beta + 2\gamma < 1. \tag{2.4}$$

Also assume that

(I) $\partial C \subseteq SC \cap TC, F_iC \cap C \subseteq SC, F_jC \cap C \subseteq TC,$

(II) $Tx \in \partial C$ *implies that* $F_ix \in C, Sx \in \partial C$ *implies that* $F_jx \in C,$

(III) SC *and* TC *(or* F_iC *and* F_jC*) are closed in* X.

Then

(IV) (F_i, T) *has a point of coincidence,*

(V) (F_j, S) *has a point of coincidence.*

Moreover, if (F_i, T) *and* (F_j, S) *are weakly compatible pairs for all* $i = 2n-1, j = 2n (n \in N)$, *then* $\{F_n\}_{n=1}^{\infty}, S$ *and* T *have a unique common fixed point.*

Proof. Let $x \in \partial C$ be arbitrary. Then (due to $\partial C \subseteq TC$) there exists a point $x_0 \in C$ such that $x = Tx_0$. From the implication if $Tx_0 \in \partial C$, then $F_1x_0 \in F_1C \cap C \subseteq SC$. Thus, there exist $x_1 \in C$ such that $y_1 = Sx_1 = F_1x_0 \in C$. Since $y_1 = F_1x_0$ there exists a point $y_2 = F_2x_1$ such that

$$d(y_1, y_2) = d(F_1x_0, F_2x_1).$$

Suppose $y_2 \in C$. Then $y_2 \in F_2C \cap C \subseteq TC$ which implies that there exists a point $x_2 \in C$ such that $y_2 = Tx_2$. Otherwise, if $y_2 \notin C$, then there exists a point $p \in \partial C$ such that

$$d(Sx_1, p) + d(p, y_2) = d(Sx_1, y_2).$$

Since $p \in \partial C \subseteq TC$ there exists a point $x_2 \in C$ with $p = Tx_2$ so that

$$d(Sx_1, Tx_2) + d(Tx_2, y_2) = d(Sx_1, y_2).$$

Let $y_3 = F_3x_2$ be such that $d(y_2, y_3) = d(F_2x_1, F_3x_2)$. Thus, repeating the foregoing arguments, one obtains two sequences $\{x_n\}$ and $\{y_n\}$ such that

(a) $y_{2n} = F_{2n}x_{2n-1}, y_{2n+1} = F_{2n+1}x_{2n},$

(b) $y_{2n} \in C$ implies that $y_{2n} = Tx_{2n}$ or $y_{2n} \notin C$ implies that $Tx_{2n} \in \partial C$ and

$$d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, y_{2n}) = d(Sx_{2n-1}, y_{2n}).$$

(c) $y_{2n+1} \in C$ implies that $y_{2n+1} = Sx_{2n+1}$ or $y_{2n+1} \notin C$ implies that $Sx_{2n+1} \in \partial C$ and

$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1}).$$

We denote

$$\begin{aligned} P_0 &= \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} = y_{2i}\}, \\ P_1 &= \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} \neq y_{2i}\}, \\ Q_0 &= \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} = y_{2i+1}\}, \\ Q_1 &= \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} \neq y_{2i+1}\}. \end{aligned}$$

Note that $(Tx_{2n}, Sx_{2n+1}) \notin P_1 \times Q_1$, as if $Tx_{2n} \in P_1$, then $y_{2n} \neq Tx_{2n}$ and one infers that $Tx_{2n} \in \partial C$ which implies that $y_{2n+1} = F_{2n+1}x_{2n} \in C$. Hence $y_{2n+1} = Sx_{2n+1} \in Q_0$. Similarly, one can argue that $(Sx_{2n-1}, Tx_{2n}) \notin Q_1 \times P_1$.

Now, we distinguish the following three cases.

Case 1. If $(Tx_{2n}, Sx_{2n+1}) \in P_0 \times Q_0$, then from (2.3)

$$d(Tx_{2n}, Sx_{2n+1}) = d(F_{2n+1}x_{2n}, F_{2n}x_{2n-1}) \leq \alpha d(Tx_{2n}, Sx_{2n-1}) + \beta u_{2n} + \gamma v_{2n},$$

where

$$u_{2n} \in \{d(Tx_{2n}, F_{2n+1}x_{2n}), d(Sx_{2n-1}, F_{2n}x_{2n-1})\} = \{d(Tx_{2n}, y_{2n+1}), d(Sx_{2n-1}, y_{2n})\},$$

$$v_{2n} \in \{d(Tx_{2n}, F_{2n+1}x_{2n}) + d(Sx_{2n-1}, F_{2n}x_{2n-1}), d(Tx_{2n}, F_{2n}x_{2n-1}) + d(Sx_{2n-1}, F_{2n+1}x_{2n})$$
$$= \{d(Tx_{2n}, y_{2n+1}) + d(Sx_{2n-1}, y_{2n}), d(Sx_{2n-1}, y_{2n+1})\}.$$

Clearly, there are infinitely many n such that at least one of the following four cases holds:
(1) If $u_{2n} = d(Tx_{2n}, y_{2n+1})$ and $v_{2n} = d(Tx_{2n}, y_{2n+1}) + d(Sx_{2n-1}, y_{2n})$, then

$$\begin{aligned} d(Tx_{2n}, Sx_{2n+1}) &\leq \alpha d(Tx_{2n}, Sx_{2n-1}) + \beta d(Tx_{2n}, y_{2n+1}) + \gamma(d(Tx_{2n}, y_{2n+1}) + d(Sx_{2n-1}, y_{2n})) \\ &= \alpha d(Tx_{2n}, Sx_{2n-1}) + \beta d(Tx_{2n}, Sx_{2n+1}) + \gamma d(Tx_{2n}, Sx_{2n+1}) + \gamma d(Sx_{2n-1}, Tx_{2n}). \end{aligned}$$

This implies that $d(Tx_{2n}, Sx_{2n+1}) \leq \frac{\alpha+\gamma}{1-\beta-\gamma} d(Sx_{2n-1}, Tx_{2n})$.
(2) If $u_{2n} = d(Tx_{2n}, y_{2n+1})$ and $v_{2n} = d(Sx_{2n-1}, y_{2n+1})$, then

$$\begin{aligned} d(Tx_{2n}, Sx_{2n+1}) &\leq \alpha d(Tx_{2n}, Sx_{2n-1}) + \beta d(Tx_{2n}, y_{2n+1}) + \gamma d(Sx_{2n-1}, y_{2n+1}) \\ &\leq \alpha d(Tx_{2n}, Sx_{2n-1}) + \beta d(Tx_{2n}, y_{2n+1}) + \gamma(d(Sx_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})) \\ &= \alpha d(Tx_{2n}, Sx_{2n-1}) + \beta d(Tx_{2n}, Sx_{2n+1}) + \gamma d(Sx_{2n-1}, Tx_{2n}) + \gamma d(Tx_{2n}, Sx_{2n+1}). \end{aligned}$$

This implies that $d(Tx_{2n}, Sx_{2n+1}) \leq \frac{\alpha+\gamma}{1-\beta-\gamma} d(Sx_{2n-1}, Tx_{2n})$.
(3) If $u_{2n} = d(Sx_{2n-1}, y_{2n})$ and $v_{2n} = d(Tx_{2n}, y_{2n+1}) + d(Sx_{2n-1}, y_{2n})$, then

$$\begin{aligned} d(Tx_{2n}, Sx_{2n+1}) &\leq \alpha d(Tx_{2n}, Sx_{2n-1}) + \beta d(Sx_{2n-1}, y_{2n}) + \gamma(d(Tx_{2n}, y_{2n+1}) + d(Sx_{2n-1}, y_{2n})) \\ &= \alpha d(Tx_{2n}, Sx_{2n-1}) + \beta d(Sx_{2n-1}, Tx_{2n}) + \gamma d(Tx_{2n}, Sx_{2n+1}) + \gamma d(Sx_{2n-1}, Tx_{2n}). \end{aligned}$$

This implies that $d(Tx_{2n}, Sx_{2n+1}) \leq \frac{\alpha+\beta+\gamma}{1-\gamma} d(Sx_{2n-1}, Tx_{2n})$.
(4) If $u_{2n} = d(Sx_{2n-1}, y_{2n})$ and $v_{2n} = d(Sx_{2n-1}, y_{2n+1})$, then

$$\begin{aligned} d(Tx_{2n}, Sx_{2n+1}) &\leq \alpha d(Tx_{2n}, Sx_{2n-1}) + \beta d(Sx_{2n-1}, y_{2n}) + \gamma d(Sx_{2n-1}, y_{2n+1}) \\ &\leq \alpha d(Tx_{2n}, Sx_{2n-1}) + \beta d(Sx_{2n-1}, y_{2n}) + \gamma(d(Sx_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})) \\ &= \alpha d(Tx_{2n}, Sx_{2n-1}) + \beta d(Sx_{2n-1}, Tx_{2n}) + \gamma d(Sx_{2n-1}, Tx_{2n}) + \gamma d(Tx_{2n}, Sx_{2n+1}). \end{aligned}$$

This implies that $d(Tx_{2n}, Sx_{2n+1}) \leq \frac{\alpha+\beta+\gamma}{1-\gamma} d(Sx_{2n-1}, Tx_{2n})$.
From (1), (2), (3), (4) it follows that

$$d(Tx_{2n}, Sx_{2n+1}) \leq \lambda d(Sx_{2n-1}, Tx_{2n}), \tag{2.5}$$

where $\lambda = max\{\frac{\alpha+\gamma}{1-\beta-\gamma}, \frac{\alpha+\beta+\gamma}{1-\gamma}\} < 1$ by (2.4).
Similarly, if $(Sx_{2n+1}, Tx_{2n+2}) \in Q_0 \times P_0$, we have

$$d(Sx_{2n+1}, Tx_{2n+2}) = d(F_{2n+1}x_{2n}, F_{2n+2}x_{2n+1}) \leq \lambda d(Tx_{2n}, Sx_{2n+1}). \tag{2.6}$$

If $(Sx_{2n-1}, Tx_{2n}) \in Q_0 \times P_0$, we have

$$d(Sx_{2n-1}, Tx_{2n}) = d(F_{2n-1}x_{2n-2}, F_{2n}x_{2n-1}) \leq \lambda d(Tx_{2n-2}, Sx_{2n-1}). \tag{2.7}$$

Case 2. If $(Tx_{2n}, Sx_{2n+1}) \in P_0 \times Q_1$, then $Sx_{2n+1} \in Q_1$ and

$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1}) \tag{2.8}$$

which in turns yields

$$d(Tx_{2n}, Sx_{2n+1}) \leq d(Tx_{2n}, y_{2n+1}) = d(y_{2n}, y_{2n+1}) \tag{2.9}$$

and hence

$$d(Tx_{2n}, Sx_{2n+1}) \leq d(y_{2n}, y_{2n+1}) = d(F_{2n+1}x_{2n}, F_{2n}x_{2n-1}). \tag{2.10}$$

Now, proceeding as in Case 1, we have (2.5) holds.

If $(Sx_{2n+1}, Tx_{2n+2}) \in Q_1 \times P_0$, then $Tx_{2n} \in P_0$. We show that

$$d(Sx_{2n+1}, Tx_{2n+2}) \leq \lambda d(Tx_{2n}, Sx_{2n-1}). \tag{2.11}$$

Using (2.8), we get

$$\begin{aligned} d(Sx_{2n+1}, Tx_{2n+2}) &\leq d(Sx_{2n+1}, y_{2n+1}) + d(y_{2n+1}, Tx_{2n+2}) \\ &= d(Tx_{2n}, y_{2n+1}) - d(Tx_{2n}, Sx_{2n+1}) + d(y_{2n+1}, Tx_{2n+2}). \end{aligned} \tag{2.12}$$

By noting that $Tx_{2n+2}, Tx_{2n} \in P_0$, one can conclude that

$$d(y_{2n+1}, Tx_{2n+2}) = d(y_{2n+1}, y_{2n+2}) = d(F_{2n+1}x_{2n}, F_{2n+2}x_{2n+1}) \leq \lambda d(Tx_{2n}, Sx_{2n+1}), \tag{2.13}$$

and

$$d(Tx_{2n}, y_{2n+1}) = d(y_{2n}, y_{2n+1}) = d(F_{2n+1}x_{2n}, F_{2n}x_{2n-1}) \leq \lambda d(Sx_{2n-1}, Tx_{2n}), \tag{2.14}$$

in view of Case 1.

Thus,

$$d(Sx_{2n+1}, Tx_{2n+2}) \leq \lambda d(Sx_{2n-1}, Tx_{2n}) - (1-\lambda) d(Tx_{2n}, Sx_{2n+1}) \leq \lambda d(Sx_{2n-1}, Tx_{2n}),$$

and we proved (2.11).

Case 3. If $(Tx_{2n}, Sx_{2n+1}) \in P_1 \times Q_0$, then $Sx_{2n-1} \in Q_0$. We show that

$$d(Tx_{2n}, Sx_{2n+1}) \leq \lambda d(Sx_{2n-1}, Tx_{2n-2}). \tag{2.15}$$

Since $Tx_{2n} \in P_1$, then

$$d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, y_{2n}) = d(Sx_{2n-1}, y_{2n}). \tag{2.16}$$

From this, we get

$$\begin{aligned} d(Tx_{2n}, Sx_{2n+1}) &\leq d(Tx_{2n}, y_{2n}) + d(y_{2n}, Sx_{2n+1}) \\ &= d(Sx_{2n-1}, y_{2n}) - d(Sx_{2n-1}, Tx_{2n}) + d(y_{2n}, Sx_{2n+1}). \end{aligned} \tag{2.17}$$

By noting that $Sx_{2n+1}, Sx_{2n-1} \in Q_0$, one can conclude that

$$d(y_{2n}, Sx_{2n+1}) = d(y_{2n}, y_{2n+1}) = d(F_{2n+1}x_{2n}, F_{2n}x_{2n-1}) \leq \lambda d(Sx_{2n-1}, Tx_{2n}), \tag{2.18}$$

and

$$d(Sx_{2n-1}, y_{2n}) = d(y_{2n-1}, y_{2n}) = d(F_{2n-1}x_{2n-2}, F_{2n}x_{2n-1}) \leq \lambda d(Sx_{2n-1}, Tx_{2n-2}), \tag{2.19}$$

in view of Case 1.

Thus,

$$d(Tx_{2n}, Sx_{2n+1}) \leq \lambda d(Sx_{2n-1}, Tx_{2n-2}) - (1-\lambda) d(Sx_{2n-1}, Tx_{2n}) \leq \lambda d(Sx_{2n-1}, Tx_{2n-2}),$$

and we proved (2.15).

Similarly, If $(Sx_{2n+1}, Tx_{2n+2}) \in Q_0 \times P_1$, then $Tx_{2n+2} \in P_1$, and

$$d(Sx_{2n+1}, Tx_{2n+2}) + d(Tx_{2n+2}, y_{2n+2}) = d(Sx_{2n+1}, y_{2n+2}).$$

From this, we have

$$\begin{aligned} d(Sx_{2n+1}, Tx_{2n+2}) &\le d(Sx_{2n+1}, y_{2n+2}) + d(y_{2n+2}, Tx_{2n+2}) \\ &\le d(Sx_{2n+1}, y_{2n+2}) + d(Sx_{2n+1}, y_{2n+2}) - d(Sx_{2n+1}, Tx_{2n+2}) \\ &= 2d(Sx_{2n+1}, y_{2n+2}) - d(Sx_{2n+1}, Tx_{2n+2}) \\ &\Rightarrow d(Sx_{2n+1}, Tx_{2n+2}) \le d(Sx_{2n+1}, y_{2n+2}). \end{aligned}$$

By noting that $Sx_{2n+1} \in Q_0$, one can conclude that

$$d(Sx_{2n+1}, Tx_{2n+2}) \le d(Sx_{2n+1}, y_{2n+2}) = d(F_{2n+1}x_{2n}, F_{2n+2}x_{2n+1}) \le \lambda d(Tx_{2n}, Sx_{2n+1}), \tag{2.20}$$

in view of Case 1.
Thus, in all case 1-3, there exists $w_{2n} \in \{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n-2}, Sx_{2n-1})\}$ such that

$$d(Tx_{2n}, Sx_{2n+1}) \le \lambda w_{2n}$$

and exists $w_{2n+1} \in \{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n}, Sx_{2n+1})\}$ such that

$$d(Sx_{2n+1}, Tx_{2n+2}) \le \lambda w_{2n+1}.$$

Following the procedure of Assad and Kirk [4], it can easily be shown by induction that, for $n \ge 1$, there exists $w_2 \in \{d(Tx_0, Sx_1), d(Sx_1, Tx_2)\}$ such that

$$d(Tx_{2n}, Sx_{2n+1}) \le \lambda^{n-\frac{1}{2}} w_2 \quad \text{and} \quad d(Sx_{2n+1}, Tx_{2n+2}) \le \lambda^n w_2. \tag{2.21}$$

From (2.21) and by the triangle inequality, for $n > m$ we have

$$\begin{aligned} d(Tx_{2n}, Sx_{2m+1}) &\le d(Tx_{2n}, Sx_{2n-1}) + d(Sx_{2n-1}, Tx_{2n-2}) + \cdots + d(Tx_{2m+2}, Sx_{2m+1}) \\ &\le (\lambda^m + \lambda^{m+\frac{1}{2}} + \cdots + \lambda^{n-1}) w_2 \le \frac{\lambda^m}{1-\sqrt{\lambda}} w_2 \to \theta, \quad \text{as} \quad m \to \infty. \end{aligned}$$

From Remark 1.5 and Corollary 1.4 (1) $d(Tx_{2n}, Sx_{2m+1}) \ll c$.

Thus, the sequence $\{Tx_0, Sx_1, Tx_2, Sx_3, \cdots, Sx_{2n-1}, Tx_{2n}, Sx_{2n-1}, \cdots\}$ is a Cauchy sequence. Then, as note in [8], there exists at least one subsequence $\{Tx_{2n_k}\}$ or $\{Sx_{2n_k+1}\}$ which is contained in P_0 or Q_0 respectively and finds its limit $z \in C$. Furthermore, subsequences $\{Tx_{2n_k}\}$ and $\{Sx_{2n_k+1}\}$ both converge to $z \in C$ as C is a closed subset of complete cone metric space (X, d). We assume that there exists a subsequence $\{Tx_{2n_k}\} \subseteq P_0$ for each $k \in N$, and TC as well as SC are closed in X. Since $\{Tx_{2n_k}\}$ is Cauchy sequence in TC, it converges to a point $z \in TC$. Let $w \in T^{-1}z$, then $Tw = z$. Similarly, $\{Sx_{2n_k+1}\}$ being a subsequence of Cauchy sequence $\{Tx_0, Sx_1, Tx_2, Sx_3, \cdots, Sx_{2n-1}, Tx_{2n}, Sx_{2n-1}, \cdots\}$ also converges to z as SC is closed. Using (2.3), one can write

$$\begin{aligned} d(F_i w, z) &\le d(F_i w, F_j x_{2n_k-1}) + d(F_j x_{2n_k-1}, z) \le \alpha d(Tw, Sx_{2n_k-1}) + \beta u_w + \gamma v_w + d(F_j x_{2n_k-1}, z) \\ &= \alpha d(z, Sx_{2n_k-1}) + \beta u_w + \gamma v_w + d(F_j x_{2n_k-1}, z), \end{aligned}$$

where

$$u_w \in \{d(Tw, F_i w), d(Sx_{2n_k-1}, F_j x_{2n_k-1})\} = \{d(z, F_i w), d(Sx_{2n_k-1}, F_j x_{2n_k-1})\},$$

$$v_w \in \{d(Tw, F_i w) + d(Sx_{2n_k-1}, F_j x_{2n_k-1}), d(Tw, F_j x_{2n_k-1}) + d(F_i w, Sx_{2n_k-1})\}$$

$$= \{d(z, F_iw) + d(Sx_{2n_k-1}, F_jx_{2n_k-1}), d(z, F_jx_{2n_k-1}) + d(F_iw, Sx_{2n_k-1})\}.$$

Let $\theta \ll c$. Clearly at least one of the following four cases holds for infinitely many n.

(1) If $u_w = d(z, F_iw)$ and $v_w = d(z, F_iw) + d(Sx_{2n_k-1}, F_jx_{2n_k-1})$, then

$$\begin{aligned}
d(F_iw, z) &\leq \alpha d(z, Sx_{2n_k-1}) + \beta d(z, F_iw) + \gamma(d(z, F_iw) + d(Sx_{2n_k-1}, F_jx_{2n_k-1})) + d(F_jx_{2n_k-1}, z) \\
&\leq \alpha d(z, Sx_{2n_k-1}) + \beta d(z, F_iw) + \gamma d(z, F_iw) + \gamma(d(Sx_{2n_k-1}, z) + d(z, F_jx_{2n_k-1})) + d(F_jx_{2n_k-1}, z) \\
&= (\alpha+\gamma)d(z, Sx_{2n_k-1}) + (\beta+\gamma)d(z, F_iw) + (\gamma+1)d(F_jx_{2n_k-1}, z) \\
&\Rightarrow d(F_iw, z) \leq \frac{\alpha+\gamma}{1-\beta-\gamma}d(z, Sx_{2n_k-1}) + \frac{\gamma+1}{1-\beta-\gamma}d(F_jx_{2n_k-1}, z) \\
&\ll \frac{\alpha+\gamma}{1-\beta-\gamma}\frac{c}{2\frac{\alpha+\gamma}{1-\beta-\gamma}} + \frac{\gamma+1}{1-\beta-\gamma}\frac{c}{2\frac{\gamma+1}{1-\beta-\gamma}} = c;
\end{aligned}$$

(2) If $u_w = d(z, F_iw)$ and $v_w = d(z, F_jx_{2n_k-1}) + d(F_iw, Sx_{2n_k-1})$, then

$$\begin{aligned}
d(F_iw, z) &\leq \alpha d(z, Sx_{2n_k-1}) + \beta d(z, F_iw) + \gamma(d(z, F_jx_{2n_k-1}) + d(F_iw, Sx_{2n_k-1})) + d(F_jx_{2n_k-1}, z) \\
&\leq \alpha d(z, Sx_{2n_k-1}) + \beta d(z, F_iw) + \gamma d(z, F_jx_{2n_k-1}) + \gamma(d(F_iw, z) + d(z, Sx_{2n_k-1})) + d(F_jx_{2n_k-1}, z) \\
&= (\alpha+\gamma)d(z, Sx_{2n_k-1}) + (\beta+\gamma)d(z, F_iw) + (\gamma+1)d(F_jx_{2n_k-1}, z) \\
&\Rightarrow d(Fw, z) \leq \frac{\alpha+\gamma}{1-\beta-\gamma}d(z, Sx_{2n_k-1}) + \frac{\gamma+1}{1-\beta-\gamma}d(F_jx_{2n_k-1}, z) \\
&\ll \frac{\alpha+\gamma}{1-\beta-\gamma}\frac{c}{2\frac{\alpha+\gamma}{1-\beta-\gamma}} + \frac{\gamma+1}{1-\beta-\gamma}\frac{c}{2\frac{\gamma+1}{1-\beta-\gamma}} = c;
\end{aligned}$$

(3) If $u_w = d(Sx_{2n_k-1}, F_jx_{2n_k-1})$ and $v_w = d(z, F_iw) + d(Sx_{2n_k-1}, F_jx_{2n_k-1})$, then

$$\begin{aligned}
d(F_iw, z) &\leq \alpha d(z, Sx_{2n_k-1}) + \beta d(Sx_{2n_k-1}, F_jx_{2n_k-1}) + \gamma(d(z, F_iw) + d(Sx_{2n_k-1}, F_jx_{2n_k-1})) + d(F_jx_{2n_k-1}, z) \\
&\leq \alpha d(z, Sx_{2n_k-1}) + \beta(d(Sx_{2n_k-1}, z) + d(z, F_jx_{2n_k-1})) + \gamma d(z, F_iw) \\
&\quad + \gamma(d(Sx_{2n_k-1}, z) + d(z, F_jx_{2n_k-1})) + d(F_jx_{2n_k-1}, z) \\
&= (\alpha+\beta+\gamma)d(z, Sx_{2n_k-1}) + \gamma d(z, F_iw) + (\beta+\gamma+1)d(F_jx_{2n_k-1}, z) \\
&\Rightarrow d(F_iw, z) \leq \frac{\alpha+\beta+\gamma}{1-\gamma}d(z, Sx_{2n_k-1}) + \frac{\beta+\gamma+1}{1-\gamma}d(F_jx_{2n_k-1}, z) \\
&\ll \frac{\alpha+\beta+\gamma}{1-\gamma}\frac{c}{2^{\frac{\alpha+\beta+\gamma}{1-\gamma}}} + \frac{\beta+\gamma+1}{1-\gamma}\frac{c}{2^{\frac{\beta+\gamma+1}{1-\gamma}}} = c;
\end{aligned}$$

(4) If $u_w = d(Sx_{2n_k-1}, F_jx_{2n_k-1})$ and $v_w = d(z, F_jx_{2n_k-1}) + d(F_iw, Sx_{2n_k-1})$, then

$$\begin{aligned}
d(F_iw, z) &\leq \alpha d(z, Sx_{2n_k-1}) + \beta d(Sx_{2n_k-1}, F_jx_{2n_k-1}) + \gamma(d(z, F_jx_{2n_k-1}) + d(F_iw, Sx_{2n_k-1})) + d(F_jx_{2n_k-1}, z) \\
&\leq \alpha d(z, Sx_{2n_k-1}) + \beta(d(Sx_{2n_k-1}, z) + d(z, F_jx_{2n_k-1})) + \gamma d(z, F_jx_{2n_k-1}) \\
&\quad + \gamma(d(F_iw, z) + d(z, Sx_{2n_k-1})) + d(F_jx_{2n_k-1}, z) \\
&= (\alpha+\beta+\gamma)d(z, Sx_{2n_k-1}) + \gamma d(z, F_iw) + (\beta+\gamma+1)d(F_jx_{2n_k-1}, z) \\
&\Rightarrow d(F_iw, z) \leq \frac{\alpha+\beta+\gamma}{1-\gamma}d(z, Sx_{2n_k-1}) + \frac{\beta+\gamma+1}{1-\gamma}d(F_jx_{2n_k-1}, z) \\
&\ll \frac{\alpha+\beta+\gamma}{1-\gamma}\frac{c}{2^{\frac{\alpha+\beta+\gamma}{1-\gamma}}} + \frac{\beta+\gamma+1}{1-\gamma}\frac{c}{2^{\frac{\beta+\gamma+1}{1-\gamma}}} = c.
\end{aligned}$$

In all the cases we obtain $d(F_iw, z) \ll c$ for each $c \in intP$, Using Corollary Corollary 1.4 (3) it follows that $d(F_iw, z) = \theta$ or $F_iw = z$. Thus, $F_iw = z = Tw$, that is z is a coincidence point of F_i, T.

Further, since Cauchy sequence $\{Tx_0, Sx_1, Tx_2, Sx_3, \cdots, Sx_{2n-1}, Tx_{2n}, Sx_{2n-1}, \cdots\}$ converges to $z \in C$ and $z = F_iw, z \in F_iC \cap C \subseteq SC$, there exists $b \in C$ such that $Sb = z$. Again using (2.3), we get

$$d(Sb, F_jb) = d(z, F_jb) = d(F_iw, F_jb) \leq \alpha d(Tw, Sb) + \beta u_w + \gamma v_w = \beta u_w + \gamma v_w,$$

where

$$u_w \in \{d(Tw, F_iw), d(Sb, F_jb)\} = \{\theta, d(Sb, F_jb)\},$$

$$v_w \in \{d(Tw, F_iw) + d(Sb, F_jb), d(Tw, F_jb) + d(Sb, F_iw)\} = \{d(Sb, F_jb), d(z, F_jb)\} = \{d(Sb, F_jb)\}.$$

Hence, we get the following cases:

$$d(Sb, F_jb) \leq \beta\theta + \gamma d(Sb, F_jb) = \gamma d(Sb, F_jb) \text{ and } d(Sb, F_jb) \leq (\beta + \gamma) d(Sb, F_jb)$$

Since $0 \leq \gamma \leq \beta + \gamma < 1 - \alpha - \gamma \leq 1$, using Remark 1.6 and Corollary 1.4 (3), it follows that $Sb = F_jb$, therefore, $Sb = z = F_jb$, that is z is a coincidence point of (F_j, S).

In case F_iC and F_jC are closed in X, then $z \in F_iC \cap C \subseteq SC$ or $z \in F_jC \cap C \subseteq TC$. The analogous arguments establish (IV) and (V). If we assume that there exists a subsequence $\{Sx_{2n_k+1}\} \subseteq Q_0$ with TC as well SC are closed in X, then noting that $\{Sx_{2n_k+1}\}$ is a Cauchy sequence in SC, foregoing arguments establish (IV) and (V).

Suppose now that (F_i, T) and (F_j, S) are coincidentally commuting pairs, then

$$z = F_iw = Tw \Rightarrow F_iz = F_iTw = TF_iw = Tz \quad \text{and} \quad z = F_jb = Sb \Rightarrow F_jz = F_jSb = SF_jb = Sz$$

Then, from (2.3),

$$d(F_iz, z) = d(F_iz, F_jb) \leq \alpha d(Tz, Sb) + \beta u + \gamma v = \alpha d(F_iz, z) + \beta u + \gamma v,$$

where

$$u \in \{d(Tz, F_iz), d(Sb, F_jb)\} = \{d(F_iz, F_iz), d(z, z)\} = \{\theta\},$$

$$v \in \{d(Tz, F_iz) + d(Sb, F_jb), d(Tz, F_jb) + d(Sb, F_iz)\} = \{\theta, d(F_iz, z) + d(z, F_iz)\} = \{\theta, 2d(F_iz, z)\}.$$

Hence, we get the following cases:

$$d(F_iz, z) \leq \alpha d(F_iz, z) \text{ and } d(F_iz, z) \leq \alpha d(F_iz, z) + 2\gamma d(F_iz, z) = (\alpha + 2\gamma) d(z, F_iz)$$

Since $0 \leq \alpha \leq \alpha + 2\gamma < 1 - \beta \leq 1$, using Remark 1.6 and Corollary 1.4 (3), it follows that $F_iz = z$. Thus, $F_iz = z = Tz$

Similarly, we can prove $F_jz = z = Sz$. Therefore $z = F_iz = F_jz = Sz = Tz$, that is, z is a common fixed point of F_n, S and T.

Uniqueness of the common fixed point follows easily from (2.3). □

Remark 2.4. Setting $F_i = F$ and $F_j = G$ in Theorem 2.3, we obtain the following result:

Corollary 2.5. *Let (X, d) be a complete cone metric space, C a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x, z) + d(z, y) = d(x, y).$$

Suppose that $F, G, S, T : C \to X$ are two pairs of non-self mappings satisfying for all $x, y \in C$ with $x \neq y$,

$$d(Fx, Gy) \leq \alpha d(Tx, Sy) + \beta u + \gamma v \tag{2.22}$$

where $u \in \{d(Tx, Fx), d(Sy, Gy)\}$, $v \in \{d(Tx, Fx) + d(Sy, Gy), d(Tx, Gy) + d(Sy, Fx)\}$, and α, β, γ are nonnegative real numbers such that

$$\alpha + \beta + 2\gamma < 1. \tag{2.23}$$

Also assume that

(I) $\partial C \subseteq SC \cap TC, FC \cap C \subseteq SC, GC \cap C \subseteq TC$,
(II) $Tx \in \partial C$ *implies that* $Fx \in C, Sx \in \partial C$ *implies that* $Gx \in C$,
(III) SC *and* TC *(or* FC *and* GC*) are closed in* X.

Then

(IV) (F, T) *has a point of coincidence,*
(V) (G, S) *has a point of coincidence.*

Moreover, if (F, T) *and* (G, S) *are weakly compatible pairs, then* F, G, S *and* T *have a unique common fixed point.*

Remark 2.6. 1. Setting $F_i = F_j = f$ and $T = S = g$ in Theorem 2.3, one deduces Theorem 2.1 due to Ciric and Ume's [6] with $\alpha + \beta + 2\gamma < 1$.

2. Setting $F_i = F_j = f$ and $T = S = I_X$ in Theorem 2.3, we obtain the following result:

Corollary 2.7. *Let* (X, d) *be a complete cone metric space, and* C *a nonempty closed subset of* X *such that for each* $x \in C$ *and* $y \notin C$ *there exists a point* $z \in \partial C$ *such that*

$$d(x, z) + d(z, y) = d(x, y).$$

Suppose that $g : C \to X$ *satisfying for all* $x, y \in C$ *with* $x \neq y$,

$$d(fx, fy) \leq \alpha d(x, y) + \beta u + \gamma,$$

where

$$u \in \{d(x, gx), d(y, gy)\},\ \ v \in \{d(x, gx) + d(y, gy), d(x, gy) + d(y, gx)\},$$

and α, β, γ *are nonnegative real numbers such that* $\alpha + \beta + 2\gamma < 1$ *and* g *has the additional property that for each* $x \in \partial C, gx \in C$*, then* g *has a unique fixed point in* C.

We now list some corollaries of Theorems 2.3.

Corollary 2.8. *Let* (X, d) *be a complete cone metric space,* C *a nonempty closed subset of* X *such that for each* $x \in C$ *and* $y \notin C$ *there exists a point* $z \in \partial C$ *such that*

$$d(x, z) + d(z, y) = d(x, y).$$

Let $\{F_n\}_{n=1}^{\infty}, S, T : C \to X$ *be such that for all* $i = 2n - 1, j = 2n (n \in N)$ *and* $x, y \in C$ *with* $x \neq y$,

$$d(F_i x, F_j y) \leq \alpha d(Tx, Sy), \tag{2.24}$$

where $\alpha \in (0, 1)$.

Suppose, further, that $\{F_n\}_{n=1}^{\infty}, S, T$ *and* C *satisfy the following conditions:*

(I) $\partial C \subseteq SC \cap TC, F_i C \cap C \subseteq SC, F_j C \cap C \subseteq TC$,
(II) $Tx \in \partial C$ *implies that* $F_i x \in C, Sx \in \partial C$ *implies that* $F_j x \in C$,
(III) SC *and* TC *(or* $F_i C$ *and* $F_j C$*) are closed in* X.

Then

(IV) (F_i, T) *has a point of coincidence,*
(V) (F_j, S) *has a point of coincidence.*

Moreover, if (F_i, T) *and* (F_j, S) *are weakly compatible pairs for all* $i = 2n - 1, j = 2n (n \in N)$*, then* $\{F_n\}_{n=1}^{\infty}, S$ *and* T *have a unique common fixed point.*

Corollary 2.9. *Let (X, d) be a complete cone metric space, C a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x, z) + d(z, y) = d(x, y).$$

Let $\{F_n\}_{n=1}^{\infty}, S, T : C \to X$ be such that for all $i = 2n-1, j = 2n(n \in N)$ and $x, y \in C$ with $x \neq y$,

$$d(F_i x, Fjy) \leq \gamma(d(Tx, F_i x) + d(Sy, F_j y)), \tag{2.25}$$

where $\gamma \in (0, \frac{1}{2})$.

Suppose, further, that $\{F_n\}_{n=1}^{\infty}, S, T$ and C satisfy the following conditions:

(I) $\partial C \subseteq SC \cap TC, F_i C \cap C \subseteq SC, F_j C \cap C \subseteq TC$,
(II) $Tx \in \partial C$ implies that $F_i x \in C, Sx \in \partial C$ implies that $F_j x \in C$,
(III) SC and TC (or $F_i C$ and $F_j C$) are closed in X.

Then

(IV) (F_i, T) has a point of coincidence,
(V) (F_j, S) has a point of coincidence.

Moreover, if (F_i, T) and (F_j, S) are weakly compatible pairs for all $i = 2n-1, j = 2n(n \in N)$, then $\{F_n\}_{n=1}^{\infty}, S$ and T have a unique common fixed point.

Corollary 2.10. *Let (X, d) be a complete cone metric space, C a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x, z) + d(z, y) = d(x, y).$$

Let $\{F_n\}_{n=1}^{\infty}, S, T : C \to X$ be such that for all $i = 2n-1, j = 2n(n \in N)$ and $x, y \in C$ with $x \neq y$,

$$d(F_i x, F_j y) \leq \gamma(d(Tx, F_j y) + d(F_i x, Sy)), \tag{2.26}$$

where $\gamma \in (0, \frac{1}{2})$.

Suppose, further, that $\{F_n\}_{n=1}^{\infty}, S, T$ and C satisfy the following conditions:

(I) $\partial C \subseteq SC \cap TC, F_i C \cap C \subseteq SC, F_j C \cap C \subseteq TC$,
(II) $Tx \in \partial C$ implies that $F_i x \in C, Sx \in \partial C$ implies that $F_j x \in C$,
(III) SC and TC (or $F_i C$ and $F_j C$) are closed in X.

Then

(IV) (F_i, T) has a point of coincidence,
(V) (F_j, S) has a point of coincidence.

Moreover, if (F_i, T) and (F_j, S) are weakly compatible pairs for all $i = 2n-1, j = 2n(n \in N)$, then $\{F_n\}_{n=1}^{\infty}, S$ and T have a unique common fixed point.

Remark 2.11. Setting $F_i = F_j = f$ and $T = S = g$ in Corollary 2.8–2.10, we obtain the following results:

Corollary 2.12. *Let (X, d) be a complete cone metric space, C a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x, z) + d(z, y) = d(x, y).$$

Let $f, g : C \to X$ be such that

$$d(fx, fy) \leq \alpha d(gx, gy), \tag{2.27}$$

for some $\alpha \in (0, 1)$ and for all $x, y \in C$. Suppose, further, that f, g and C satisfy the following conditions:

(I) $\partial C \subseteq gC, fC \cap C \subseteq gC$,
(II) $gx \in \partial C$ *implies that* $fx \in C$,
(III) gC *is closed in* X.

Then the pair (f, g) *has a coincidence point in* C. *Moreover, if pair* (f, g) *is weakly compatible, then* f *and* g *have a unique common fixed point in* C.

Corollary 2.13. *Let* (X, d) *be a complete cone metric space,* C *a nonempty closed subset of* X *such that for each* $x \in C$ *and* $y \notin C$ *there exists a point* $z \in \partial C$ *such that*

$$d(x, z) + d(z, y) = d(x, y).$$

Let $f, g : C \to X$ *be such that*

$$d(fx, fy) \leq \gamma(d(fx, gx) + d(fy, gy)), \tag{2.28}$$

for some $\gamma \in (0, \frac{1}{2})$ *and for all* $x, y \in C$. *Suppose, further, that* f, g *and* C *satisfy the following conditions:*

(I) $\partial C \subseteq gC, fC \cap C \subseteq gC$,
(II) $gx \in \partial C$ *implies that* $fx \in C$,
(III) gC *is closed in* X.

Then the pair (f, g) *has a coincidence point in* C. *Moreover, if pair* (f, g) *is weakly compatible, then* f *and* g *have a unique common fixed point in* C.

Corollary 2.14. *Let* (X, d) *be a complete cone metric space,* C *a nonempty closed subset of* X *such that for each* $x \in C$ *and* $y \notin C$ *there exists a point* $z \in \partial C$ *such that*

$$d(x, z) + d(z, y) = d(x, y).$$

Let $f, g : C \to X$ *be such that*

$$d(fx, fy) \leq \gamma(d(fx, gy) + d(fy, gx)), \tag{2.29}$$

for some $\gamma \in (0, \frac{1}{2})$ *and for all* $x, y \in C$. *Suppose, further, that* f, g *and* C *satisfy the following conditions:*

(I) $\partial C \subseteq gC, fC \cap C \subseteq gC$,
(II) $gx \in \partial C$ *implies that* $fx \in C$,
(III) gC *is closed in* X.

Then the pair (f, g) *has a coincidence point in* C. *Moreover, if pair* (f, g) *is weakly compatible, then* f *and* g *have a unique common fixed point in* C.

Remark 2.15. Corollaries 2.12–2.14 are the corresponding theorems of Abbas and Jungck from [1] in the case that f, g are non-self mappings.

3. Illustrative examples

The following example shows that in general F_n, S and T satisfying the hypotheses of Theorem 2.3 need not have a common coincidence justifying two separate conclusions (IV) and (V).

Example 3.1. Let $E = C^1([0, 1], R), P = \{\varphi \in E : \varphi(t) \geq 0, t \in [0, 1]\}, X = [0, +\infty), C = [0, 2]$ and $d : X \times X \to E$ defined by $d(x, y) = |x - y|\varphi$, where $\varphi \in P$ is a fixed function, e.g., $\varphi(t) = e^t$. Then (X, d) is a complete cone metric space with a non-normal cone having the nonempty interior. Define F_i, F_j, S and $T : C \to X$ as

$$F_i x = x + \frac{4}{5}, i = 2n - 1, \ F_j x = x^2 + \frac{4}{5}, j = 2n, \ Tx = 5x \ \text{ and } \ Sx = 5x^2, x \in C.$$

Since $\partial C = \{0, 2\}$. Clearly, for each $x \in C$ and $y \notin C$ there exists a point $z = 2 \in \partial C$ such that $d(x, z) + d(z, y) = d(x, y)$. Further, $SC \cap TC = [0, 20] \cap [0, 10] = [0, 10] \supset \{0, 2\} = \partial C$, $F_iC \cap C = [\frac{4}{5}, \frac{14}{5}] \cap [0, 2] = [\frac{4}{5}, 2] \subset SC$, $F_jC \cap C = [\frac{4}{5}, \frac{24}{5}] \cap [0, 2] = [\frac{4}{5}, 2] \subset TC$, and, SC, TC, F_iC and F_jC are closed in X.

Also,

$$T0 = 0 \in \partial C \Rightarrow F_i0 = \frac{4}{5} \in C, \quad S0 = 0 \in \partial C \Rightarrow F_j0 = \frac{4}{5} \in C.$$

$$T(\frac{2}{5}) = 2 \in \partial C \Rightarrow F_i(\frac{2}{5}) = \frac{6}{5} \in C, \quad S(\sqrt{\frac{2}{5}}) = 2 \in \partial C \Rightarrow F_j(\sqrt{\frac{2}{5}}) = \frac{6}{5} \in C.$$

Moreover, for each $x, y \in C$,

$$d(F_ix, F_jy) = |x - y^2|\varphi = \frac{1}{5}d(Tx, Sy)$$

that is (2.3) is satisfied with $\alpha = \frac{1}{5}, \beta = \gamma = 0$.

Evidentally, $1 = T(\frac{1}{5}) = F_i(\frac{1}{5}) \neq \frac{1}{5}$ and $1 = S(\frac{1}{\sqrt{5}}) = F_j(\frac{1}{\sqrt{5}}) \neq \frac{1}{\sqrt{5}}$. Notice that two separate coincidence points are not common fixed points as $F_iT(\frac{1}{5}) \neq TF_i(\frac{1}{5})$ and $SF_j(\frac{1}{\sqrt{5}}) \neq F_jS(\frac{1}{\sqrt{5}})$ which shows the necessity of weakly compatible property in Theorem 2.3.

Next, we furnish an illustrate example in support of our result. In doing so, we are essentially inspired by Imdad and Kumar[12].

Example 3.2. Let $E = C^1([0, 1], R), P = \{\varphi \in E : \varphi(t) \geq 0, t \in [0, 1]\}, X = [1, +\infty), C = [1, 3]$ and $d : X \times X \to E$ defined by $d(x, y) = |x - y|\varphi$, where $\varphi \in P$ is a fixed function, e.g., $\varphi(t) = e^t$. Then (X, d) is a complete cone metric space with a non-normal cone having the nonempty interior. Define F_i, F_j, S and $T : C \to X$ as

$$F_ix = \begin{cases} \frac{x^2-1+n}{n} & \text{if } 1 \leq x \leq 2 \\ \frac{n+1}{n} & \text{if } 2 < x \leq 3 \end{cases} \quad i = 2n - 1 (n \geq 1), \quad Tx = \begin{cases} x^4 & \text{if } 1 \leq x \leq 2 \\ 4 & \text{if } 2 < x \leq 3 \end{cases},$$

$$F_jx = \begin{cases} \frac{x^3-1+n}{n} & \text{if } 1 \leq x \leq 2 \\ \frac{n+1}{n} & \text{if } 2 < x \leq 3 \end{cases} \quad j = 2n (n \geq 1), \quad \text{and} \quad Sx = \begin{cases} x^6 & \text{if } 1 \leq x \leq 2 \\ 4 & \text{if } 2 < x \leq 3 \end{cases}.$$

Since $\partial C = \{1, 3\}$. Clearly, for each $x \in C$ and $y \notin C$ there exists a point $z = 3 \in \partial C$ such that $d(x, z) + d(z, y) = d(x, y)$. Further, $SC \cap TC = [1, 64] \cap [1, 16] = [1, 16] \supset \{1, 3\} = \partial C$, $F_iC \cap C = [1, \frac{n+3}{n}] \cap [1, 3] \subset SC$ and $F_jC \cap C = [1, \frac{n+7}{n}] \cap [1, 3] \subset TC$.

Also,

$$T1 = 1 \in \partial C \Rightarrow F_i1 = 1 \in C, \quad S1 = 1 \in \partial C \Rightarrow F_j1 = 1 \in C.$$

$$T(\sqrt[4]{3}) = 3 \in \partial C \Rightarrow F_i(\sqrt[4]{3}) = \frac{\sqrt{3} - 1}{n} + 1 \in C, \quad S(\sqrt[6]{3}) = 3 \in \partial C \Rightarrow F_j(\sqrt[6]{3}) = \frac{\sqrt{3} - 1}{n} + 1 \in C.$$

Moreover, if $x \in [1, 2]$ and $y \in [2, 3]$, then

$$d(F_ix, F_jy) = \frac{1}{n}|x^2 - 2|\varphi = \frac{|x^4 - 4|}{n|x^2 + 2|}\varphi = \frac{|x^4 - 4|}{n|x^2 + 2|}\varphi = \frac{1}{n(x^2 + 2)}d(Tx, Sy).$$

Next, if $x, y \in (2, 3]$, then

$$d(F_ix, F_jy) = 0 = \alpha \cdot d(Tx, Sy).$$

Finally, if $x, y \in [1, 2]$, then

$$d(F_ix, F_jy) = \frac{1}{n}|x^2 - y^3|\varphi = \frac{|x^4 - y^6|}{n|x^2 + y^3|}\varphi = \frac{|x^4 - y^6|}{n|x^2 + y^3|}\varphi = \frac{1}{n(x^2 + y^3)}d(Tx, Sy).$$

Therefore, condition (2.3) is satisfied if we choose $\alpha = \max\{\frac{1}{n(x^2+2)}, \frac{1}{n(x^2+y^3)}\} \in (0, \frac{1}{2}), \beta = \gamma = 0$. Moreover 1 is a point of coincidence as $T1 = F_i1$ as well as $S1 = F_j1$ whereas both the pairs (F_i, T) and (F_j, S) are weakly compatible as $TF_i1 = 1 = F_iT1$ and $SF_j1 = 1 = F_jS1$. Also, SC, TC, F_iC and F_jC are closed in X. Thus, all the conditions of the Theorem 2.3 are satisfied and 1 is the unique common fixed point of F_i, F_j, S and T. One may note that 1 is also a point of coincidence for both the pairs (F_i, T) and (F_j, S).

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