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Various Suzuki type theorems in b-metric spaces

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Abstract

In this paper, we prove some fixed point results for α-admissible mappings which satisfy Suzuki type contractive condition in the setup of b-metric spaces. Finally, examples are presented to verify the effectiveness and applicability of our main results.

Keywords: Ordered metric space, b-metric space, fixed point.
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1. Introduction

Banach contractive principle or Banach fixed point theorem is the most celebrated result in fixed point theory which illustrates that in a complete metric space, each contractive mapping has a unique fixed point. There is a great number of generalizations of Banach contraction principle by using different forms of contractive conditions in various spaces. Some of such generalizations are obtained by contraction conditions described by rational expressions, (see, [14, 18, 20, 25]).

Ran and Reurings initiated the studying of fixed point results on partially ordered sets in [21], where they gave many useful results in matrix equations. Recently, many researchers have focused on different contractive conditions in complete metric spaces endowed with a partial order or a graph and obtained many fixed point results in such spaces. For more details on fixed point results, their applications, comparison of different contractive conditions and related results in ordered metric spaces and spaces endowed with a graph we refer the reader to [5] and [19].

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Czerwik in [8] introduced the concept of a b-metric space. Since then, several papers dealt with fixed point theory for single-valued and multi-valued operators in b-metric spaces (see, [7, 9, 16, 22]).

Definition 1.1. Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric if, for all $x, y, z \in X$, the following conditions are satisfied:

(b_1) $d(x, y) = 0$ iff $x = y$,

(b_2) $d(x, y) = d(y, x)$,

(b_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space.

Definition 1.2 ([6])**.** Let (X, d) be a b-metric space.

(a) A sequence $\{x_n\}$ in X is called b-convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$, as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(b) $\{x_n\}$ in X is said to be b-Cauchy if and only if $d(x_n, x_m) \to 0$, as $n, m \to \infty$.

(c) The b-metric space (X, d) is b-complete if every b-Cauchy sequence in X is b-converges.

Note that a b-metric need not to be a continuous function. The following example (corrected from [13]) illustrates this fact.

Example 1.3. Let $X = \mathbb{N} \cup \{\infty\}$ and let $d : X \times X \to \mathbb{R}$ be defined by

$$d(m, n) = \begin{cases} 0, & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}|, & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\ 5, & \text{if one of } m, n \text{ is odd and the other is odd (and } m \neq n) \text{ or } \infty, \\ 2, & \text{otherwise.} \end{cases}$$

It can be checked that for all $m, n, p \in X$, we have

$$d(m, p) \leq \frac{5}{2}(d(m, n) + d(n, p)).$$

Thus, (X, d) is a b-metric space (with $s = 5/2$). Let $x_n = 2n$ for each $n \in \mathbb{N}$. Then

$$d(2n, \infty) = \frac{1}{2n} \to 0 \quad \text{as } n \to \infty,$$

that is, $x_n \to \infty$, but $d(x_n, 1) = 2 \not\to 5 = d(\infty, 1)$ as $n \to \infty$.

Lemma 1.4 ([1])**.** *Let (X, d) be a b-metric space with $s \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y, respectively. Then we have*

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x, y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have,

$$\frac{1}{s} d(x, z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s d(x, z).$$

Let $\mathfrak{S}$ denotes the class of all real functions $\beta : [0, +\infty) \to [0, 1)$ satisfying the condition

$$\beta(t_n) \to 1 \text{ implies that } t_n \to 0, \ \textit{as } n \to \infty.$$

In order to generalize the Banach contraction principle, in 1973, Geraghty proved the following.

Theorem 1.5 ([12]). *Let (X, d) be a complete metric space, and let $f : X \to X$ be a self-map. Suppose that there exists $\beta \in \mathfrak{S}$ such that*
$$d(fx, fy) \leq \beta(d(x, y))d(x, y)$$
holds for all $x, y \in X$. Then f has a unique fixed point $z \in X$ and for each $x \in X$ the Picard sequence $\{f^n x\}$ converges to z.

In [10], some fixed point theorems for Geraghty-type contractive mappings in various generalized metric spaces are proved. As in [10], we will consider the class of functions $\mathcal{F}$, where $\beta \in \mathcal{F}$ if $\beta : [0, \infty) \to [0, 1/s)$ has the property
$$\beta(t_n) \to \frac{1}{s} \text{ implies that } \ t_n \to 0, \ \textit{as } n \to \infty.$$

Theorem 1.6 ([10]). *Let $s > 1$, and let (X, D, s) be a complete metric type space. Suppose that a mapping $f : X \to X$ satisfies the condition*
$$D(fx, fy) \leq \beta(D(x, y))D(x, y)$$
for all $x, y \in X$ and some $\beta \in \mathcal{F}$. Then f has a unique fixed point $z \in X$, and for each $x \in X$ the Picard sequence $\{f^n x\}$ converges to z in (X, D, s).

Unification of the recent results of Zabihi and Razani [28] yield the following result.

Theorem 1.7. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b-metric d on X such that (X, d) is a b-complete b-metric space (with parameter $s > 1$). Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that there exists $\beta \in \mathcal{F}$ such that,*
$$sd(fx, fy) \leq \beta(d(x, y))M(x, y) + LN(x, y) \tag{1.1}$$
for all comparable elements $x, y \in X$, where $L \geq 0$,
$$M(x, y) = \max\left\{d(x, y), \frac{d(x, fx)d(y, fy)}{1 + d(fx, fy)}\right\}$$
and
$$N(x, y) = \min\{d(x, fx), d(x, fy), d(y, fx), d(y, fy)\}.$$
If f is continuous, or, whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$, then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

One of the interesting results which generalizes the Banach contraction principle was given by Samet *et al.* [23] by defining α-ψ-contractive mappings.

Definition 1.8 ([23]). Let T be a self-mapping on X and let $\alpha : X \times X \to [0, \infty)$ be a function. We say that T is an α-admissible mapping if
$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Denote with Ψ' the family of all nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the n-th iterate of ψ.

Theorem 1.9 ([23]). *Let (X, d) be a complete metric space and let T be an α-admissible mapping. Assume that*
$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)) \tag{1.2}$$
where $\psi \in \Psi'$. Also, suppose that the following assertions hold:

(i) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(ii) either T is continuous, or, for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ such that $x_n \to x$ as $n \to \infty$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then T has a fixed point.

Definition 1.10 ([17])**.** Let $f : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. We say that f is a triangular α-admissible mapping if

(T1) $\alpha(x, y) \geq 1 \quad$ implies $\quad \alpha(fx, fy) \geq 1, \quad x, y \in X,$

(T2) $\begin{cases} \alpha(x, z) \geq 1 \\ \alpha(z, y) \geq 1 \end{cases}$ implies $\quad \alpha(x, y) \geq 1, \quad x, y, z \in X.$

Example 1.11 ([17])**.** Let $X = \mathbb{R}$, $fx = \sqrt[3]{x}$ and $\alpha(x, y) = e^{x-y}$, then f is a triangular α-admissible mapping. Indeed, if $\alpha(x, y) = e^{x-y} \geq 1$, then $x \geq y$ which implies that $fx \geq fy$, that is, $\alpha(fx, fy) = e^{fx-fy} \geq 1$. Also, if $\begin{cases} \alpha(x, z) \geq 1 \\ \alpha(z, y) \geq 1 \end{cases}$, then $\begin{cases} x - z \geq 0, \\ z - y \geq 0, \end{cases}$ that is, $x - y \geq 0$ and so, $\alpha(x, y) = e^{x-y} \geq 1$.

Lemma 1.12 ([17])**.** *Let f be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. Define sequence $\{x_n\}$ by $x_n = f^n x_0$. Then*

$$\alpha(x_m, x_n) \geq 1 \text{ for all } m, n \in \mathbb{N} \text{ with } m < n.$$

We now recall the concept of (c)-*comparison function* which was introduced by Berinde [4].

Definition 1.13 ([4])**.** A function $\varphi : [0, \infty) \to [0, \infty)$ is said to be a (c)-comparison function if

(c_1) φ is increasing,

(c_2) there exists $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $\varphi^{k+1}(t) \leq a\varphi^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0, \infty)$.

Later, Berinde [3] introduced the notion of a (b)-comparison function as a generalization of the concept of (c)-comparison function.

Definition 1.14 ([3])**.** Let $s \geq 1$ be a real number. A mapping $\varphi : [0, \infty) \to [0, \infty)$ is called a (b)-comparison function if the following conditions are fulfilled

(1) φ is monotone increasing;

(2) there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\varphi^{k+1}(t) \leq as^k\varphi^k(t) + v_k$ for any $k \geq k_0$ and any $t \in [0, \infty)$.

Let Ψ_b be the class of all (b)-comparison functions $\varphi : [0, \infty) \to [0, \infty)$. It is clear that the notion of (b)-comparison function coincide with (c)-comparison function for $s = 1$.

We now recall the following lemma which will simplify the proofs.

Lemma 1.15 ([2])**.** *If $\varphi : [0, \infty) \to [0, \infty)$ is a (b)-comparison function, then we have the following.*

(1) the series $\sum_{k=0}^{\infty} s^k\varphi^k(t)$ converges for any $t \in \mathbb{R}_+$;

(2) the function $b_s : [0, \infty) \to [0, \infty)$ defined by $b_s(t) = \sum_{k=0}^{\infty} s^k\varphi^k(t)$, $t \in [0, \infty)$, is increasing and continuous at 0.

2. Main Results

In 1962, Edelstein [11] proved an interesting version of Banach contraction principle. In 2009, Suzuki [27] proved certain remarkable results to improve the results of Banach and Edelstein (see also [24, 26]).

Now, we are ready to prove the following Suzuki type theorems for nonlinear contractions.

Theorem 2.1. *Let (X, d) be a b-complete b-metric space (with parameter $s > 1$) and let f be a triangular α-admissible mapping. Suppose that there exists $\beta \in \mathcal{F}$ such that,*

$$\frac{1}{2s}d(x,fx) \leq d(x,y) \Longrightarrow s\alpha(x,y)d(fx,fy) \leq \beta(M(x,y))M(x,y) \tag{2.1}$$

for all $x, y \in X$, where

$$\begin{aligned} M(x,y) = \max\Big\{&d(x,y), \frac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{1+s[d(x,y)+d(fx,fy)]}, \\ &\frac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{1+d(x,fy)+d(y,fx)}\Big\}. \end{aligned}$$

Also, suppose that the following assertions hold:

(i) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;

(ii) for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ such that $x_n \to x$ as $n \to \infty$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then, T has a fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, fx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = f^n x_0$ for all $n \in \mathbb{N}$. Since f is an α-admissible mapping and $\alpha(x_0, x_1) = \alpha(x_0, fx_0) \geq 1$, we deduce that $\alpha(x_1, x_2) = \alpha(fx_0, fx_1) \geq 1$. Continuing this process, we get that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. We will do the proof in the following steps.

Step I: We will show that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Since $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$, and $\frac{1}{2s}d(x_{n-1}, fx_{n-1}) \leq d(x_{n-1}, x_n)$ then by (2.1) we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\leq s\alpha(x_{n-1}, x_n)d(fx_{n-1}, fx_n) \\ &\leq \beta(M(x_{n-1}, x_n))M(x_{n-1}, x_n) \\ &\leq \beta(d(x_{n-1}, x_n))d(x_{n-1}, x_n) \\ &< \tfrac{1}{s}d(x_{n-1}, x_n) \\ &\leq d(x_{n-1}, x_n), \end{aligned} \tag{2.2}$$

because

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), \\ &\quad \frac{d(x_{n-1}, fx_{n-1})d(x_{n-1}, fx_n) + d(x_n, fx_n)d(x_n, fx_{n-1})}{1 + s[d(x_{n-1}, x_n) + d(fx_{n-1}, fx_n)]}, \\ &\quad \frac{d(x_{n-1}, fx_{n-1})d(x_{n-1}, fx_n) + d(x_n, fx_n)d(x_n, fx_{n-1})}{1 + d(x_{n-1}, fx_n) + d(x_n, fx_{n-1})}\} \\ &= \max\{d(x_{n-1}, x_n), \\ &\quad \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})d(x_n, x_n)}{1 + s[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]}, \\ &\quad \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})d(x_n, x_n)}{1 + d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}\} \\ &= d(x_{n-1}, x_n). \end{aligned}$$

Therefore, $\{d(x_n, x_{n+1})\}$ is decreasing. Then there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. We will prove that $r = 0$. Suppose on contrary that $r > 0$. Then, letting $n \to \infty$, from (2.2) we have

$$\frac{1}{s}r \leq \lim_{n\to\infty} \beta(d(x_{n-1}, x_n))r$$

which implies that $d(x_{n-1}, x_n) \to 0$. Hence, $r = 0$, a contradiction. So,

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0 \tag{2.3}$$

holds true.

Step II: Now, we prove that the sequence $\{x_n\}$ is a b-Cauchy sequence. Suppose the contrary, *i.e.*, that $\{x_n\}$ is not a b-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \text{ and } d(x_{m_i}, x_{n_i}) \geq \varepsilon. \tag{2.4}$$

This means that

$$d(x_{m_i}, x_{n_i-1}) < \varepsilon. \tag{2.5}$$

From ((2.4)) and using the triangular inequality, we get

$$\varepsilon \leq d(x_{m_i}, x_{n_i}) \leq sd(x_{m_i}, x_{m_i+1}) + sd(x_{m_i+1}, x_{n_i}).$$

Taking the upper limit as $i \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i}). \tag{2.6}$$

Remember that, from (2.2) we get,

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \tag{2.7}$$

for all $n \in \mathbb{N}$. Suppose that there exists $i_0 \in \mathbb{N}$ such that,

$$\frac{1}{2s}d(x_{m_{i_0}}, fx_{m_{i_0}}) > d(x_{m_{i_0}}, x_{n_{i_0}-1})$$

and

$$\frac{1}{2s}d(x_{m_{i_0}+1}, fx_{m_{i_0}+1}) > d(x_{m_{i_0}+1}, x_{n_{i_0}-1}).$$

Then from (2.7) we have,

$$\begin{aligned}
d(x_{m_{i_0}}, x_{m_{i_0}+1}) &\leq s[d(x_{m_{i_0}}, x_{n_{i_0}-1}) + d(x_{m_{i_0}+1}, x_{n_{i_0}-1})] \\
&< s[\frac{1}{2s}d(x_{m_{i_0}}, fx_{m_{i_0}}) + \frac{1}{2s}d(x_{m_{i_0}+1}, fx_{m_{i_0}+1})] \\
&= \frac{1}{2}[d(x_{m_{i_0}}, x_{m_{i_0}+1}) + d(x_{m_{i_0}+1}, x_{m_{i_0}+2})] \\
&\leq \frac{1}{2}[d(x_{m_{i_0}}, x_{m_{i_0}+1}) + d(x_{m_{i_0}}, x_{m_{i_0}+1})] = d(x_{m_{i_0}}, x_{m_{i_0}+1})
\end{aligned}$$

which is a contradiction. Hence, either,

$$\frac{1}{2s}d(x_{m_i}, fx_{m_i}) \leq d(x_{m_i}, x_{n_i-1})$$

or

$$\frac{1}{2s}d(x_{m_i+1}, fx_{m_i+1}) \leq d(x_{m_i+1}, x_{n_i-1})$$

holds for all $i \in \mathbb{N}$.

First suppose that

$$\frac{1}{2s} d(x_{m_i}, fx_{m_i}) \leq d(x_{m_i}, x_{n_i-1}).$$

As from Lemma 1.12, $\alpha(x_{m_i}, x_{n_i-1}) \geq 1$, we obtain that

$$\begin{aligned} s \cdot \frac{\varepsilon}{s} \leq s \cdot \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i}) &\leq \limsup_{i\to\infty} \beta(M(x_{m_i}, x_{n_i-1})) \limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}) \\ &\leq \varepsilon \limsup_{i\to\infty} \beta(M(x_{m_i}, x_{n_i-1})) \end{aligned}$$

because, from the definition of $M(x, y)$ and the above limits,

$$\begin{aligned} \limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}) &= \limsup_{i\to\infty} \max\{d(x_{m_i}, x_{n_i-1}), \\ &\frac{d(x_{m_i}, fx_{m_i})d(x_{m_i}, fx_{n_i-1}) + d(x_{n_i-1}, fx_{n_i-1})d(x_{n_i-1}, fx_{m_i})}{1 + s[d(x_{m_i}, x_{n_i-1}) + d(fx_{m_i}, fx_{n_i-1})]}, \\ &\frac{d(x_{m_i}, fx_{m_i})d(x_{m_i}, fx_{n_i-1}) + d(x_{n_i-1}, fx_{n_i-1})d(x_{n_i-1}, fx_{m_i})}{1 + d(x_{m_i}, fx_{n_i-1}) + d(x_{n_i-1}, fx_{m_i})}\} \\ &= \limsup_{i\to\infty} \max\{d(x_{m_i}, x_{n_i-1}), \\ &\frac{d(x_{m_i}, x_{m_i+1})d(x_{m_i}, x_{n_i}) + d(x_{n_i-1}, x_{n_i})d(x_{n_i-1}, x_{m_i+1})}{1 + s[d(x_{m_i}, x_{n_i-1}) + d(x_{m_i+1}, x_{n_i})]}, \\ &\frac{d(x_{m_i}, x_{m_i+1})d(x_{m_i}, x_{n_i}) + d(x_{n_i-1}, x_{n_i})d(x_{n_i-1}, x_{m_i+1})}{1 + d(x_{m_i}, x_{n_i}) + d(x_{n_i-1}, x_{m_i+1})}\} \\ &\leq \varepsilon, \end{aligned}$$

which implies that $\frac{1}{s} \leq \limsup_{i\to\infty} \beta(M(x_{m_i}, x_{n_i-1}))$. Now, as $\beta \in \mathcal{F}$ we conclude that $M(x_{m_i}, x_{n_i-1}) \to 0$, hence, we get that $d(x_{m_i}, x_{n_i-1}) \to 0$ which implies that $d(x_{m_i}, x_{n_i}) \to 0$, a contradiction to 2.4. So, $\{x_n\}$ is a b-Cauchy sequence. b-Completeness of X yields that $\{x_n\}$ b-converges to a point $x^* \in X$.

On the other hand, from ((2.4)) and using the triangular inequality, we get

$$\frac{\varepsilon}{s} \leq d(x_{m_i}, x_{n_i}) \leq sd(x_{m_i}, x_{m_i+2}) + sd(x_{m_i+2}, x_{n_i}).$$

Taking the upper limit as $i \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{i\to\infty} d(x_{m_i+2}, x_{n_i}). \tag{2.8}$$

Also, from ((2.6)) and using the triangular inequality, we get

$$d(x_{m_i+1}, x_{n_i-1}) \leq sd(x_{m_i+1}, x_{n_i}) + sd(x_{n_i}, x_{n_i-1}).$$

Taking the upper limit as $i \to \infty$, we get

$$\limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i-1}) \leq s\varepsilon. \tag{2.9}$$

Now, let

$$\frac{1}{2s} d(x_{m_i+1}, fx_{m_i+1}) \leq d(x_{m_i+1}, x_{n_i-1}).$$

From Lemma 1.12, $\alpha(x_{m_i+1}, x_{n_i-1}) \geq 1$, so, we have

$$\begin{aligned} s \cdot \frac{\varepsilon}{s} \leq s \cdot \limsup_{i\to\infty} d(x_{m_i+2}, x_{n_i}) &\leq \limsup_{i\to\infty} \beta(M(x_{m_i+1}, x_{n_i-1})) \limsup_{i\to\infty} M(x_{m_i+1}, x_{n_i-1}) \\ &\leq s\varepsilon \limsup_{i\to\infty} \beta(M(x_{m_i+1}, x_{n_i-1})) \end{aligned}$$

because,

$$\begin{aligned}\limsup_{i\to\infty} M(x_{m_i+1}, x_{n_i-1}) &= \limsup_{i\to\infty} \max\{d(x_{m_i+1}, x_{n_i-1}),\\ &\frac{d(x_{m_i+1}, fx_{m_i+1})d(x_{m_i+1}, fx_{n_i-1}) + d(x_{n_i-1}, fx_{n_i-1})d(x_{n_i-1}, fx_{m_i+1})}{1 + s[d(x_{m_i+1}, x_{n_i-1}) + d(fx_{m_i+1}, fx_{n_i-1})]},\\ &\frac{d(x_{m_i+1}, fx_{m_i+1})d(x_{m_i+1}, fx_{n_i-1}) + d(x_{n_i-1}, fx_{n_i-1})d(x_{n_i-1}, fx_{m_i+1})}{1 + d(x_{m_i+1}, fx_{n_i-1}) + d(x_{n_i-1}, fx_{m_i+1})}\}\\ &= \limsup_{i\to\infty} \max\{d(x_{m_i+1}, x_{n_i-1}),\\ &\frac{d(x_{m_i+1}, x_{m_i+2})d(x_{m_i+1}, x_{n_i}) + d(x_{n_i-1}, x_{n_i})d(x_{n_i-1}, x_{m_i+2})}{1 + s[d(x_{m_i+1}, x_{n_i-1}) + d(x_{m_i+2}, x_{n_i})]},\\ &\frac{d(x_{m_i+1}, x_{m_i+2})d(x_{m_i+1}, x_{n_i}) + d(x_{n_i-1}, x_{n_i})d(x_{n_i-1}, x_{m_i+2})}{1 + d(x_{m_i+1}, x_{n_i}) + d(x_{n_i-1}, x_{m_i+2})}\}\\ &\leq s\varepsilon,\end{aligned}$$

which implies that $\frac{1}{s} \leq \limsup_{i\to\infty} \beta(M(x_{m_i+1}, x_{n_i-1}))$. Now, as $\beta \in \mathcal{F}$ we conclude that $M(x_{m_i+1}, x_{n_i-1}) \to 0$, hence, we get that $d(x_{m_i+1}, x_{n_i-1}) \to 0$ which implies that $d(x_{m_i}, x_{n_i}) \to 0$, a contradiction. So, $\{x_n\}$ is a b-Cauchy sequence. b-Completeness of X yields that $\{x_n\}$ b-converges to a point $x^* \in X$.

Remember that, from (2.2) we get,

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \tag{2.10}$$

for all $n \in \mathbb{N}$. Suppose that there exists $n_0 \in \mathbb{N}$ such that,

$$\frac{1}{2s} d(x_{n_0}, fx_{n_0}) > d(x_{n_0}, x^*)$$

and

$$\frac{1}{2s} d(x_{n_0+1}, fx_{n_0+1}) > d(x_{n_0+1}, x^*).$$

Then, from (2.10) we have,

$$\begin{aligned}d(x_{n_0}, x_{n_0+1}) &\leq s[d(x_{n_0}, x^*) + d(x_{n_0+1}, x^*)]\\ &< s[\frac{1}{2s} d(x_{n_0}, fx_{n_0}) + \frac{1}{2s} d(x_{n_0+1}, fx_{n_0+1})]\\ &= \frac{1}{2}[d(x_{n_0}, x_{n_0+1}) + d(x_{n_0+1}, x_{n_0+2})]\\ &\leq \frac{1}{2}[d(x_{n_0}, x_{n_0+1}) + d(x_{n_0}, x_{n_0+1})] = d(x_{n_0}, x_{n_0+1})\end{aligned}$$

which is a contradiction. Hence, either,

$$\frac{1}{2s} d(x_n, fx_n) \leq d(x_n, x^*)$$

or

$$\frac{1}{2s} d(x_{n+1}, fx_{n+1}) \leq d(x_{n+1}, x^*)$$

holds for all $n \in \mathbb{N}$. First, suppose that,

$$\frac{1}{2s} d(x_n, fx_n) \leq d(x_n, x^*)$$

holds for all $n \in \mathbb{N}$. Then from (2.1) we have,

$$\begin{aligned} d(fx^*, x^*) &\leq s[d(fx^*, fx_n) + d(fx_n, x^*)] \\ &= s[d(fx^*, fx_n) + d(x_{n+1}, x^*)] \\ &\leq s[\beta(M(x^*, x_n))M(x^*, x_n) + d(x_{n+1}, x^*)] \end{aligned}$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ in the above inequality we get, $d(fx^*, x^*) = 0$. *i.e.,*

$$fx^* = x^*.$$

By a similar method we can deduce $fx^* = x^*$ when

$$\frac{1}{2}d(x_{n+1}, fx_{n+1}) \leq d(x_{n+1}, x^*).$$

Hence, we proved that x^* is a fixed point of f. □

Theorem 2.2. *Let (X, d) be a b-complete b-metric space and let f be an α-admissible mapping. Suppose that there exists $\psi \in \Psi_b$ such that,*

$$\frac{1}{2s}d(x, fx) \leq d(x, y) \Longrightarrow \alpha(x, y)d(fx, fy) \leq \psi(M(x, y)) \tag{2.11}$$

where,

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, fx)d(y, fy)}{1 + s[d(x, y) + d(x, fy)]}, \frac{d(x, fy)d(x, y)}{1 + s[d(x, fx) + d(y, fy)]}\right\},$$

for all $x, y \in X$.

Also, suppose that the following assertions hold:

(i) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;

(ii) for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ such that $x_n \to x$ as $n \to \infty$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then, f has a fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, fx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = f^n x_0$ for all $n \in \mathbb{N}$. Since f is an α-admissible mapping and $\alpha(x_0, x_1) = \alpha(x_0, fx_0) \geq 1$, we deduce that $\alpha(x_1, x_2) = \alpha(fx_0, fx_1) \geq 1$. Continuing this process, we get that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$ then, $x_{n_0} = fx_{n_0}$ and so we have no thing for prove. Hence, for all $n \in \mathbb{N}$ we assume that $d(x_n, x_{n+1}) > 0$.

On the other hand, we have,

$$\frac{1}{2s}d(x_{n-1}, fx_{n-1}) = \frac{1}{2}d(x_{n-1}, x_n) \leq d(x_{n-1}, x_n)$$

and $\alpha(x_n, x_{n+1}) \geq 1$, so by (2.11) we get,

$$d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) \leq \alpha(x_{n-1}, x_n)d(fx_{n-1}, fx_n) \leq \psi(M(x_{n-1}, x_n)) \tag{2.12}$$

where,

$$\begin{aligned} M(x_{n-1}, x_n) =& \max\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, fx_{n-1})d(x_n, fx_n)}{1 + s[d(x_{n-1}, x_n) + d(x_{n-1}, fx_n)]}, \\ & \frac{d(x_{n-1}, fx_n)d(x_{n-1}, x_n)}{1 + s[d(x_{n-1}, fx_{n-1}) + d(x_n, fx_n)]}\} \\ &= \max\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{1 + s[d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1})]}, \\ & \frac{d(x_{n-1}, x_{n+1})d(x_{n-1}, x_n)}{1 + s[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]}\} \\ &= d(x_{n-1}, x_n). \end{aligned}$$

Hence,

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) < d(x_{n-1}, x_n). \tag{2.13}$$

By induction, we get that

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \leq \psi^2(d(x_{n-2}, x_{n-1})) \leq \cdots \leq \psi^n(d(x_0, x_1)). \tag{2.14}$$

Then, by the triangular inequality and (2.14), we get

$$\begin{aligned} d(x_n, x_m) &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^{m-n-1} d(x_{m-1}, x_m) \\ &\leq \sum_{k=n}^{m-2} s^{k-n+1} \psi^k(d(x_0, x_1)) \\ &\leq \sum_{k=n}^{\infty} s^k \psi^k(d(x_0, x_1)) \longrightarrow 0, \end{aligned}$$

as $n \longrightarrow \infty$.

Hence, $\{x_n\}$ is a b-Cauchy sequence. b-completeness of X yields that $\{x_n\}$ converges to a point $x^* \in X$, that is, $x_n \to x^*$ as $n \to \infty$.

On the other hand, from (2.13) we get,

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \tag{2.15}$$

for all $n \in \mathbb{N}$. Suppose that there exists $n_0 \in \mathbb{N}$ such that,

$$\frac{1}{2s} d(x_{n_0}, fx_{n_0}) > d(x_{n_0}, x^*)$$

and

$$\frac{1}{2s} d(x_{n_0+1}, fx_{n_0+1}) > d(x_{n_0+1}, x^*).$$

Then from (2.15) we have,

$$\begin{aligned} d(x_{n_0}, x_{n_0+1}) &\leq s[d(x_{n_0}, x^*) + d(x_{n_0+1}, x^*)] \\ &< s[\frac{1}{2s} d(x_{n_0}, fx_{n_0}) + \frac{1}{2s} d(x_{n_0+1}, fx_{n_0+1})] \\ &= \frac{1}{2}[d(x_{n_0}, x_{n_0+1}) + d(x_{n_0+1}, x_{n_0+2})] \\ &\leq \frac{1}{2}[d(x_{n_0}, x_{n_0+1}) + d(x_{n_0}, x_{n_0+1})] = d(x_{n_0}, x_{n_0+1}) \end{aligned}$$

which is a contradiction. Hence, either,

$$\frac{1}{2s} d(x_n, fx_n) \leq d(x_n, x^*)$$

or

$$\frac{1}{2s} d(x_{n+1}, fx_{n+1}) \leq d(x_{n+1}, x^*)$$

holds for all $n \in \mathbb{N}$. First, suppose that,

$$\frac{1}{2s} d(x_n, fx_n) \leq d(x_n, x^*)$$

holds for all $n \in \mathbb{N}$. Then from (2.11) and hypothesis (ii), we have,

$$\begin{aligned} d(Tx^*, x^*) &\leq s[d(fx^*, fx_n) + d(fx_n, x^*)] \\ &\leq s[\alpha(x^*, x_n) d(fx^*, fx_n) + d(x_{n+1}, x^*)] \\ &\leq s[\psi(M(x^*, x_n)) + d(x_{n+1}, x^*)] \end{aligned}$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ in the above inequality we get, $d(fx^*, x^*) = 0$. *i.e.*,

$$fx^* = x^*.$$

By a similar method we can deduce $fx^* = x^*$ when

$$\frac{1}{2}d(x_{n+1}, fx_{n+1}) \leq d(x_{n+1}, x^*).$$

Hence, we proved that x^* is a fixed point of f. $\square$

Example 2.3. Let $X = \mathbb{R}^2$. We define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & x, y \in U = \{(0,0), (4,0), (0,4), (4,5), (5,4)\} \\ 0, & \text{otherwise.} \end{cases}$$

Define metric d on X by $d((x_1, x_2), (y_1, y_2)) = (x_1 - y_1)^2 + (x_2 - y_2)^2$. Clearly, $(X, d, 2)$ is a complete b-metric space. Also, define $f : X \to X$ and $\psi : [0, \infty) \to [0, \infty)$ by

$$f(x_1, x_2) = \begin{cases} (x_1, 0), & \text{If} \quad x_1 \leq x_2 \text{ and } x_1, x_2 \in U \\ (0, x_2) & \text{If} \quad x_1 > x_2 \text{ and } x_1, x_2 \in U \\ (2x_1^2, 3x_2^3) & \text{If} \quad x_1, x_2 \in \mathbb{R}^2 \backslash U \end{cases} \quad \text{and } \psi(t) = 0.99t.$$

First we assume that $\frac{1}{4}d(x, fx) \leq d(x, y)$ and $\alpha(x, y) \geq 1$. Then,

$$\begin{aligned} (x,y) \in \quad & \Big\{\Big((0,0),(4,0)\Big), \Big((0,0),(0,4)\Big), \Big((0,0),(4,5)\Big), \Big((0,0),(5,4)\Big), \\ & \Big((4,0),(0,0)\Big), \Big((4,0),(0,4)\Big), \Big((4,0),(5,4)\Big), \Big((4,0),(4,5)\Big), \\ & \Big((0,4),(0,0)\Big), \Big((0,4),(4,0)\Big), \Big((0,4),(5,4)\Big), \Big((0,4),(4,5)\Big), \\ & \Big((4,5),(0,0)\Big), \Big((4,5),(4,0)\Big), \Big((4,5),(0,4)\Big), \Big((4,5),(5,4)\Big) \\ & \Big((5,4),(0,0)\Big), \Big((5,4),(4,0)\Big), \Big((5,4),(0,4)\Big), \Big((5,4),(4,5)\Big)\Big\}. \end{aligned}$$

Since, $d(fx, fy) = d(fy, fx)$ and $d(x, y) = d(y, x)$, hence without any loss of generality we can reduce the above set to the following:

$$\begin{aligned} (x,y) \in \quad & \Big\{\Big((0,0),(4,0)\Big), \Big((0,0),(0,4)\Big), \Big((0,0),(4,5)\Big), \Big((0,0),(5,4)\Big), \\ & \Big((4,0),(0,4)\Big), \Big((4,0),(5,4)\Big), \Big((4,0),(4,5)\Big), \Big((0,4),(5,4)\Big), \Big((0,4),(4,5)\Big)\Big\}. \end{aligned}$$

Now, we consider the following cases:

- Let $(x, y) = \Big((0,0), (4,0)\Big)$, then,

$$d(fx, fy) = d(f(0,0), f(4,0)) = 0 \leq \psi(d(x, y)).$$

- Let $(x, y) = \Big((0,0), (0,4)\Big)$, then,

$$d(fx, fy) = d(f(0,0), f(0,4)) = 0 \leq \psi(d(x, y)).$$

- Let $(x,y) = \Big((0,0),(4,5)\Big)$, then,

$$d(fx,fy) = d(f(0,0),f(4,5)) = 16 \leq 40.59 = 0.99 \times 41 = \psi(d((0,0),(4,5))).$$

- Let $(x,y) = \Big((0,0),(5,4)\Big)$, then,

$$d(fx,fy) = d(f(0,0),f(5,4)) = 16 \leq 40.59 = 0.99 \times 41 = \psi(d((0,0),(4,5))).$$

- Let $(x,y) = \Big((4,0),(0,4)\Big)$, then,

$$d(fx,fy) = d(f(4,0),f(0,4)) = 0 \leq \psi(d(x,y)).$$

- Let $(x,y) = \Big((4,0),(5,4)\Big)$, then,

$$d(fx,fy) = d(f(4,0),f(5,4)) = 16 \leq 16.83 = 0.99 \times 17 = \psi(d((4,0),(5,4))).$$

- Let $(x,y) = \Big((4,0),(4,5)\Big)$, then,

$$d(fx,fy) = d(f(4,0),f(4,5)) = 16 \leq 16.83 = 0.99 \times 17 = \psi(d((4,0),(4,5))).$$

- Let $(x,y) = \Big((0,4),(5,4)\Big)$, then,

$$d(fx,fy) = d(f(0,4),f(5,4)) = 16 \leq 24.75 = 0.99 \times 25 = \psi(d((0,4),(5,4))).$$

- Let $(x,y) = \Big((0,4),(4,5)\Big)$, then,

$$d(fx,fy) = d(f(0,4),f(4,5)) = 16 \leq 16.83 = 0.99 \times 17 = \psi(d((0,4),(4,5))).$$

That is, $\frac{1}{4}d(x,fx) \leq d(x,y)$ and $\alpha(x,y) \geq 1$ implies that $d(fx,fy) \leq \psi(d(x,y))$. Let $\alpha(x,y) \geq 1$, then $x,y \in U$. On the other hand, $fw \in U$ for all $w \in U$. Then, $\alpha(fx,fy) \geq 1$. That is, f is an α-admissible mapping. If $\{x_n\}$ be a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ with $x_n \to x$ as $n \to \infty$, then, $x_n \in U$ for all $n \in \mathbb{N}$. Also, U is a closed set. Then, $x \in [0,+\infty)$. That is, $\alpha(x_n,x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Clearly, $\alpha((0,0),f(0,0)) \geq 1$.

Therefore all conditions of Theorem 2.2 holds and f has a fixed point. Here, $x = (0,0)$ is a fixed point of f.

3. Contractive mappings on b-metric spaces endowed with a graph

Recently, some results have appeared for a mapping to be a Picard Operator where (X,d) is endowed with a graph. The first result in this direction was given by Jachymski [15].

Definition 3.1 ([15])**.** Let (X,d) be a metric space endowed with a graph G. We say that a self-mapping $f : X \to X$ is a Banach G-contraction or simply a G-contraction if f preserves the edges of G, that is,

$$(x,y) \in E(G) \Longrightarrow (fx,fy) \in E(G) \qquad \text{for all } x,\, y \in X$$

and f decreases the weights of the edges of G in the following way:

$$\exists\, \alpha \in (0,1) \text{ such that for all } x,y \in X, \quad (x,y) \in E(G) \Longrightarrow d(fx,fy) \leq \alpha d(x,y).$$

Definition 3.2. Let (X, d) be a b-metric space endowed with a graph G. We say that a self-mapping $f : X \to X$ is a G-ψ-Suzuki type rational contraction if T preserves the edges of G, that is,

$$(x, y) \in E(G) \Longrightarrow (fx, fy) \in E(G), \qquad \text{for all } x, y \in X$$

and f decreases the weights of the edges of G in the following way:

$$\frac{1}{2s} d(x, fx) \leq d(x, y) \Longrightarrow d(fx, fy) \leq \psi(M(x, y)) \tag{3.1}$$

where,

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, fx)d(y, fy)}{1 + s[d(x, y) + d(x, fy)]}, \frac{d(x, fy)d(x, y)}{1 + s[d(x, fx) + d(y, fy)]} \right\}$$

and $\psi \in \Psi_b$ for all $(x, y) \in E(G)$.

Theorem 3.3. *Let (X, d) be a b-complete b-metric space endowed with a graph G. Assume that $T : X \to X$ is a G-ψ-Suzuki type rational contraction such that the following conditions hold:*

(i) there exists an element $x_0 \in X$ such that $(x_0, fx_0) \in E(G)$;

(ii) for any sequence $\{x_n\}$ in X with $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$ such that $x_n \to x$ as $n \to \infty$, we have $(x_n, x) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$.

Then T has a fixed point.

Proof. Define $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in E(G) \\ \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that f is an α-admissible mapping and also, f is an α-ψ-Suzuki type rational contraction. From (i), there exists an $x_0 \in X$ such that $(x_0, fx_0) \in E(G)$, that is, $\alpha(x_0, fx_0) \geq 1$. Hence, all the conditions of Theorem 2.2 are satisfied and hence, f has a fixed point. □

Definition 3.4. Let (X, d) be a b-metric space endowed with a graph G. We say that a self-mapping $f : X \to X$ is a G-Suzuki type rational Geraghty contractive mapping if f preserves the edges of G, that is,

$$(x, y) \in E(G) \Longrightarrow (fx, fy) \in E(G), \qquad \text{for all } x, y \in X$$

and T decreases the weights of the edges of G in the following way:

$$\frac{1}{2s} d(x, fx) \leq d(x, y) \Longrightarrow sd(fx, fy) \leq \beta(M(x, y))M(x, y) \tag{3.2}$$

where,

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, fx)d(y, fy)}{1 + s[d(x, y) + d(x, fy)]}, \frac{d(x, fy)d(x, y)}{1 + s[d(x, fx) + d(y, fy)]} \right\}$$

for all $(x, y) \in E(G)$.

Similarly, using Theorem 3.3, we can prove the following theorem.

Theorem 3.5. *Let (X, d) be a b-complete b-metric space (with parameter $s > 1$). Let f be a triangular α-admissible mapping which is a G-Suzuki type rational Geraghty contractive mapping. Also, suppose that the following assertions hold:*

(i) there exists $x_0 \in X$ such that $(x_0, fx_0) \in E(G)$;

(ii) for any sequence $\{x_n\}$ in X with $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$ such that $x_n \to x$ as $n \to \infty$, we have $(x_n, x) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$.

Then, T has a fixed point.

4. Contractive mappings on ordered b-metric spaces

Definition 4.1. Let $(X, d, \preceq)$ be a partially ordered b-metric space. We say that a self-mapping $f : X \to X$ is an ordered ψ-Suzuki type rational contractive mapping if

$$\frac{1}{2s} d(x, fx) \leq d(x, y) \Longrightarrow d(fx, fy) \leq \psi(M(x, y)) \tag{4.1}$$

where,

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, fx)d(y, fy)}{1 + s[d(x, y) + d(x, fy)]}, \frac{d(x, fy)d(x, y)}{1 + s[d(x, fx) + d(y, fy)]} \right\}.$$

for all $x, y \in X$ whit $x \preceq y$.

Theorem 4.2. *Let $(X, d, \preceq)$ be an ordered b-complete b-metric space. Assume that $f : X \to X$ is an ordered ψ-Suzuki type rational contractive mapping such that the following conditions hold:*

(i) there exists an element $x_0 \in X$ such that $x_0 \preceq fx_0$;

(ii) f is an increasing mapping;

(ii) for any sequence $\{x_n\}$ in X with $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ such that $x_n \to x$ as $n \to \infty$, we have $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$.

Then T has a fixed point.

Proof. Define $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x \preceq y \\ 0, & \text{otherwise.} \end{cases}$$

□

Similarly, using Theorem 2.2, we can prove the following theorem.

Theorem 4.3. *Let $(X, d, \preceq)$ be an ordered b-complete b-metric space (with parameter $s > 1$). Let T be a non-decreasing ordered Suzuki type rational Geraghty contractive mapping, that is, there exists $\beta \in \mathcal{F}$ such that,*

$$\frac{1}{2s} d(x, fx) \leq d(x, y) \Longrightarrow sd(fx, fy) \leq \beta(M(x, y))M(x, y) \tag{4.2}$$

for all comparable elements $x, y \in X$, where

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, fx)d(x, fy) + d(y, fy)d(y, fx)}{1 + s[d(x, y) + d(fx, fy)]}, \frac{d(x, fx)d(x, fy) + d(y, fy)d(y, fx)}{1 + d(x, fy) + d(y, fx)} \right\}.$$

Also, suppose that the following assertions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;

(ii) for any sequence $\{x_n\}$ in X with $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ such that $x_n \to x$ as $n \to \infty$, we have $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$.

Then, T has a fixed point.

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References

[1] A. Aghajani, M. Abbas, J. R. Roshan, *Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces*, Math. Slovaca, **64** (2014), 941–960. 1.4

[2] V. Berinde, *Generalized contractions in quasi-metric spaces*, Seminar on Fixed Point Theory, Preprint no. **3** (1993), 3–9. 1.15

[3] V. Berinde, *Sequences of operators and fixed points in quasi-metric spaces*, Studia Universitatis Babes-Bolyai, Mathematica, **16** (1996), 23–27. 1, 1.14

[4] V. Berinde, *Contracţii generalizate şi aplicaţii*, Editura Club Press, 22, Baia Mare, (1997). 1, 1.13

[5] F. Bojor, *Fixed point theorems for Reich type contraction on metric spaces with a graph*, Nonlinear Anal., **75** (2012) 3895–3901. 1

[6] M. Boriceanu, *Fixed point theory for multivalued generalized contractions on a set with two b-metrics*, Studia Univ. "Babes–Bolyai", Mathematica, Vol. LIV, No. 3, (2009). 1.2

[7] M. Boriceanu, *Strict fixed point theorems for multivalued operators in b-metric spaces*, Int. J. Modern Math., **4** (2009), 285–301. 1

[8] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inf. Univ. Ostrav., **1** (1993), 5–11. 1

[9] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Sem. Mat. Fis. Univ. Modena, **46** (1998), 263–276. 1

[10] D. Dukić, Z. Kadelburg, S. Radenović, *Fixed points of Geraghty-type mappings in various generalized metric spaces*, Abstr. Appl. Anal., **2011** (2011), 13 pages. 1, 1.6

[11] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc., **37** (1962), 74–79. 2

[12] M. Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc., **40** (1973), 604–608. 1.5

[13] N. Hussain, D. Ðorić, Z. Kadelburg, S. Radenović, *Suzuki-type fixed point results in metric type spaces*, Fixed Point Theory Appl., **2012** (2012), 12 pages. 1

[14] D. S. Jaggi, *Some unique fixed point theorems*, Indian J. Pure Appl. Math., **8** (1977), 223–230. 1

[15] J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc., **136** (2008), 1359–1373. 3, 3.1

[16] M. A. Khamsi, N. Hussain, *KKM mappings in metric type spaces*, Nonlinear Anal., **73** (2010), 3123–3129. 1

[17] E. Karapınar, P. Kumam, P. Salimi, *On $\alpha-\psi$-Meir-Keeler contractive mappings*, Fixed Point Theory Appl., **2013** (2013), 12 pages. 1.10, 1.11, 1.12

[18] H. K. Nashine, M. Imdad, M. Hasanc, *Common fixed point theorems under rational contractions in complex valued metric spaces*, J. Nonlinear Sci. Appl., **7** (2014), 42–50. 1

[19] J. J. Nieto, R. R. Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, **22** (2005), 223–239. 1

[20] M. Ozturk, M. Basarir, *On some common fixed point theorems with rational expressions on cone metric spaces over a Banach algebra*, Hacet. J. Math. Stat., **41** (2012), 211–222. 1

[21] A. C. M. Ran, M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some application to matrix equations*, Proc. Amer. Math. Soc., **132** (2004), 1435–1443. 1

[22] J. R. Roshan, V. Parvaneh, S. Sedghi, N. Shobkolaei, W. Shatanawi, *Common fixed points of almost generalized $(\psi,\varphi)_s$-contractive mappings in ordered b-metric spaces*, Fixed Point Theory Appl., **2013** (2013), 23 pages. 1

[23] B. Samet, C. Vetro, P. Vetro, *Fixed point theorem for $\alpha-\psi$-contractive type mappings*, Nonlinear Anal., **75** (2012), 2154–2165. 1, 1.8, 1.9

[24] P. Salimi, E. Karapınar, *Suzuki-Edelstein type contractions via auxiliary functions*, Math. Probl. Eng., **2013** (2013), 8 pages. 2

[25] R. J. Shahkoohi, A. Razani, *Some fixed point theorems for rational Geraghty contractive mappings in ordered b-metric spaces*, J. Inequal. Appl., **2014** (2014), 23 pages. 1

[26] T. Suzuki, *A generalized Banach contraction principle that characterizes metric completeness*, Proc. Amer. Math. Soc., **136** (2008), 1861–1869. 2

[27] T. Suzuki, *A new type of fixed point theorem in metric spaces*, Nonlinear Anal.-TMA, **71** (2009), 5313–5317. 2

[28] F. Zabihi, A. Razani, *Fixed point theorems for Hybrid rational Geraghty contractive mappings in ordered b-metric spaces*, J. Appl. Math., **2014** (2014), 9 pages. 1