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Fixed point theorems of multi-valued decreasing operators on cones

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Abstract

In this paper, some fixed point theorems for multi-valued decreasing operators are established on cones.

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1. Introduction

Single-valued increasing operators and mixed monotone operators have been widely investigated. Fixed point theorems for these operators are established and have found various applications to nonlinear integral equations and differential equations. For details, we can refer to [1, 4, 5] and the reference therein.

It is natural to extend this study to multi-valued case. In fact, fixed point theorems for multi-valued increasing operators and mixed monotone operators had been established and applied to differential equations and differential inclusions, we can refer to [2, 6, 7] and the reference therein for details.

Noting that multi-valued decreasing operator is a natural extension of single-valued decreasing operator[10], which is important in economic models, [3], and fixed point theorems on cones are important in obtaining the positive solutions of differential inclusions [8], we will establish some fixed point theorems of multi-valued decreasing operators on cones.

2. Preliminaries

At the beginning of this section, let us recall some concepts of the theory of cones in Banach spaces. These concepts play an important role in the remainder of this paper.

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Let X be a Banach space, a closed convex set $P \subset X$ is called a *cone*, if $x \in P$ and $x \neq 0$ implies $\alpha x \in P$ for $\alpha \geq 0$ and $\alpha x \notin P$ for $\alpha < 0$. A cone defines a *partial order* in the Banach space X: we write $x \leq y$ or $y \geq x$ if $y - x \in P$. The relation enjoys the following properties: inequalities may be multiplied by a nonnegative numbers; inequalities of the same kind may be added by terms; one may pass to limit in inequalities; $x \leq y$ and $y \leq z$ implies $x = y$.

It is well known that if X be a partially ordered Banach space endowed with partial order $\leq$, then the subset $P = \{x \in X \mid 0 \leq x\}$ is a cone.

Definition 2.1 ([9])**.** A set $M \subset X$ is *order bounded* with respect to a cone P if there is a $y \in X$ such that $x \leq y$ for all $x \in M$; the element y is called an *upper bound* for M. In the same way we can define a *lower bound*.

If the set of upper bounds of M has a minimal elements z, then z is called the *least upper bound* of M; it is denoted by $\sup M$. In the same way we can define the *greatest lower bound*, $\inf M$.

A cone P is called *minihedral*, if each two-element set $M = \{x, y\}$ has a least upper bound, $\sup\{x, y\}$.

A cone P is called *normal*, if there is a $L > 0$, such that $0 \leq x \leq y$ implies $\| x \| \leq N \| y \|$ and L does not depends on x and y. Any such $L = L(P)$ called a *normal constant* of P.

For the details of cone theory, see in [9] and references therein.

Definition 2.2 ([2])**.** Let X be a topology space, $' \leq'$ be a partial order endowed on X, let A, B be two nonempty subsets of X, the relations between A and B are defined as follow:

(1) If for every $a \in A$, there exists $b \in B$ such that $a \leq b$, then $A \prec_1 B$;
(2) If for every $b \in B$, there exists $a \in A$ such that $a \leq b$, then $A \prec_2 B$;
(3) If $A \prec_1 B$ and $A \prec_2 B$, then $A \prec B$.

Definition 2.3. (1) A multi-valued operator $T{:}X \to 2^X \setminus \{\phi\}$, is called decreasing, if for $\forall x, y \in X,\ x \leq y$ implies $Ty \prec Tx$;
(2) A single-valued operator $A{:}X \to X$,is called decreasing, if for $\forall x, y \in X,\ x \leq y$ implies $Ay \leq Ax$.

Definition 2.4. (1) A multi-valued operator $T{:}X \to 2^X \setminus \{\phi\}$, is called convex, if for $\forall x, y \in P, x \leq y$ implies $T(tx + (1-t)y) \prec tTx + (1-t)Ty,\ \forall t \in [0, 1]$; if $-T$ is convex, then T is called concave;
(2) A single-valued operator $A{:}X \to X$, is called convex, if for $\forall x, y \in X\ , x \leq y$ implies $A(tx + (1-t)y) \leq tAx + (1-t)Ay,\ \forall t \in [0, 1]$; if $-A$ is convex, then A is called concave.

We say that a multi-valued operator $T{:}X \to 2^X \setminus \{\phi\}$ has a closed graph, if $u_n \to u_0, v_n \to v_0, v_n \in Tu_n$ imply $v_0 \in Tu_0$.

Definition 2.5. Let T be a multi-valued operator, $x_0 \in X$ is called a fixed point of T, if $x_0 \in Tx_0$.

3. Main results

In this section, some fixed theorems for multi-valued decreasing operators are proved in partial ordered Banach space.

Theorem 3.1. *Let P be a strongly minihedral normal cone, $T{:}P \to 2^P \setminus \{\phi\}$, be a multi-valued convex and decreasing operator. If*

(1) $\frac{1}{2}T\theta \prec_1 T^2\theta$,where $T^2\theta = \bigcup_{x \in T\theta} Tx$ and θ denotes the zero element of E;
(2) $\sup Tx \in Tx$ for $\forall x \in P$.

Then T has a fixed point in P.

Proof. Define a single-valued operator $A : P \to P$ as $Ax = \sup Tx$. Since T is convex and decreasing, then A also is convex and decreasing. In fact

(1) A is convex, for $\forall x, y \in P, x \leq y$,

$$A(tx + (1-t)y) = \sup T(tx + (1-t)y) \leq t \sup Tx + (1-t)\sup Ty = tAx + (1-t)Ay;$$

(2) A is decreasing, for $\forall x, y \in P, x \leq y$, $Ty \prec Tx$, then $\sup Ty \leq \sup Tx$, i.e. $Ay \leq Ax$.

If follows from $\frac{1}{2}T\theta \prec_1 T^2\theta$, that $\frac{1}{2}\sup T\theta \leq \sup T^2\theta$. Since $\forall u \in T\theta$, then $\sup T\theta \leq u$. Hence $T(T\theta) \prec_1 T(\sup T\theta)$, which implies $\sup T^2\theta \leq \sup T(\sup T\theta)$, i.e.,

$$\frac{1}{2}A\theta = \frac{1}{2}\sup T\theta \leq \sup T^2\theta \leq \sup T(\sup T\theta) = A^2\theta.$$

Now we show A has a fixed point in P.

Let $u_n = A^n\theta, n = 1, 2, \cdots$. Since A is decreasing and $\theta \leq A^2\theta$, then $A^3\theta \leq A\theta$,i.e., $u_3 \leq u_1$, then we have $A^2\theta \leq A^4\theta$, i.e., $u_2 \leq u_4$, in the same way, $A^5\theta \leq A^3\theta$, i.e., $u_5 \leq u_3$. Continue this process, we have

$$\frac{1}{2}u_1 \leq u_2 \leq u_4 \leq \cdots \leq u_{2n} \leq \cdots \leq u_{2n-1} \leq \cdots \leq u_3 \leq u_1 \tag{3.1}$$

Let $\lambda_n = \sup\{\lambda | \lambda u_{2n-1} \leq u_{2n}\}$, according to (3.1), we know λ_n do exist, and

$$\lambda_n \geq \frac{1}{2}, \lambda_n u_{2n-1} \leq u_{2n}, n = 1, 2, \cdots \tag{3.2}$$

By the definition of λ_n and (3.1), we obtain $\{\lambda_n\}$ is increasing and $\{\lambda_n\} \in [\frac{1}{2}, 1]$, then $\{\lambda_n\}$ is convergent. Let $\lambda = \lim_{n\to\infty} \lambda_n$, $\lambda \in [\frac{1}{2}, 1]$.

In what follows, we prove $\lambda = 1$, in fact, since A is convex and decreasing, then

$$\begin{aligned} u_{2n+1} = Au_{2n} &\leq A(\lambda_n u_{2n-1}) \\ &= A(\lambda_n u_{2n-1} + (1-\lambda_n)\theta) \\ &\leq \lambda_n A u_{2n-1} + (1-\lambda_n)A\theta \\ &= \lambda_n u_{2n} + (1-\lambda_n)u_1 \\ &\leq \lambda_n u_{2n} + 2(1-\lambda_n)u_{2n} \\ &= (2-\lambda_n)u_{2n} \end{aligned}$$

so $\frac{1}{2-\lambda_n}u_{2n+1} \leq u_{2n}$, by the definition of λ_n, we have $\lambda_{n+1} \geq \frac{1}{2-\lambda_n}$, hence, we obtain

$\lambda \geq \frac{1}{2-\lambda}$, i.e. $-(1-\lambda)^2 \geq 0$, then $\lambda = 1$.

If follows from (3.1) and (3.2) that for arbitrary q, $u_{2n} \leq u_{2n+q} \leq u_{2n-1}$, so we have

$$u_{2n+q} - u_{2n} \leq u_{2n-1} - u_{2n} \leq (1-\lambda_n)u_{2n-1} \leq (1-\lambda_n)u_1. \tag{3.3}$$

Since P is a normal cone and $\lambda_n \to 1$, we have the sequences $\{u_{2n}\}$ and $\{u_{2n-1}\}$ are convergence, let $u_{2n} \to u^*, u_{2n-1} \to u^{**}$,

It follows from (3.2) that $u_{2n} \leq u_{2n+q} \leq u_{2n-1+q} \leq u_{2n-1}$, let $q \to \infty$, we obtain

$$u_{2n} \leq u^* \leq u^{**} \leq u_{2n-1}, n = 1, 2, \cdots \tag{3.4}$$

and (3.3), that

$$u^{**} - u^* \leq u_{2n-1} - u_{2n} \leq (1-\lambda_n)u_1 \to 0, \lambda_n \to 1$$

so that $u^{**} = u^* = x^*$, then $u_{2n} \leq x^* \leq u_{2n-1}$, then $u_{2n} \leq Ax^* \leq u_{2n-1}$, and since $u_{2n} \to u^*, u_{2n-1} \to u^{**}$, there exists $x^* \in P$, such that $x^* = Ax^*$.

By $\sup Tx \in Tx$, we have $x^* \in Tx^*$. □

Remark 3.2. The following example shows that Theorem 3.1 does not hold without condition of convexity.
Let $E = R, P = R^+, T : P \to 2^P \setminus \{\phi\}$ is defined by

$$T(x) = \begin{cases} \{y | \frac{8}{13} - \frac{2x}{13} \le y \le \frac{9}{13} - \frac{2x}{13}\}, & 0 \le x < \frac{1}{2}, \\ \{y | \frac{7}{13} - \frac{3x}{13} \le y \le \frac{7}{13} - \frac{2x}{13}\}, & \frac{1}{2} \le x \le 1, \\ \{y | \frac{4}{13} \le y \le \frac{5}{13}\}, & x > 1. \end{cases}$$

Then T satisfied all conditions of Theorem 3.1, but the convexity. It is obviously that T does not have any fixed point.

Remark 3.3. In Theorem 3.1, the condition $\sup Tx \in Tx$ was essential, for example, let $E = R, P = R^+, T : P \to 2^P \setminus \{\phi\}$ is defined by

$$T(x) = \begin{cases} (0,1), & x = 0, x \ge 1, \\ (0,x) \cup (x,1), & 0 < x < 1. \end{cases}$$

Then T satisfied all conditions of Theorem 3.1, but $\sup Tx \notin Tx$. It is obvious that T has no fixed point.

Remark 3.4. In Theorem 3.1, if the condition $\frac{1}{2}T\theta \prec_1 T^2\theta$ was be substituted by $\frac{1}{2}T\theta \prec_2 T^2\theta$, and $\sup Tx \in Tx$ be substituted by $\inf Tx \in Tx$, then the existence result still hold.

Theorem 3.5. *Let P be a normal cone, T:$P \to 2^P \setminus \{\phi\}$ be a multi-valued decreasing operator such that*

(1) T has a closed graph;
(2) $\{x_0\} \prec_1 Tx_0$, for some $x_0 \in P$;
(3) There exists a number $q \in [0,1)$ satisfying for $x, y \in P, x \le y$ that

$$Ty \subset Tx - P \cap \overline{B}(\theta, q\|y - x\|)$$

$$Tx \subset Ty + P \cap \overline{B}(\theta, q\|y - x\|),$$

where $\overline{B}(\theta, r) = \{x | \ \|x\| \le r\}$.

Then T has a fixed point in P.

Proof. From the hypothesis (2) we can find an element $x_1 \in Tx_0$ such that $x_0 \le x_1$. From the hypothesis (3), we have

$$Tx_0 \subset Tx_1 + P \cap \overline{B}(\theta, q\|x_1 - x_0\|).$$

Then we choose an element $x_2 \in Tx_1$, so that

$$\|x_1 - x_2\| \le q\|x_1 - x_0\|, x_2 \le x_1.$$

From the condition (3), we have

$$Tx_1 \subset Tx_2 - P \cap \overline{B}(\theta, q\|x_1 - x_2\|).$$

Then we choose an element $x_3 \in Tx_2$, so that

$$\|x_3 - x_2\| \le q\|x_2 - x_1\| \le q^2\|x_1 - x_0\|, x_2 \le x_3.$$

Repeating arguments above for the pair x_1, x_2 in place x_0, x_1 and so on, we can construct an sequence $\{x_n\}$ satisfying

$$\|x_n - x_{n-1}\| \le q^{n-1}\|x_1 - x_0\|, x_n \in Tx_{n-1}.$$

The sequence $\{x_n\}$ is a Cauchy sequence, let $x_n \to x^*$. From the condition (1) and $x_n \to x^*, x_{n-1} \to x^*, x_n \in Tx_{n-1}$. we have $x^* \in Tx^*$. □

Theorem 3.6. *Let the Banach space E be ordered by a normal cone P, and $T:P \to 2^P\setminus\{\phi\}$ be a multi-valued decreasing operator, satisfying* (1) *and* (2) *in Theorem 3.5. Assume in addition the following hypothesis*

(3) *There exists a linear operator $L : E \to E$ with spectral radius $r(L) < 1, L(P) \subset P$ and satisfying for $x \leq y$ that*

$$Ty \subset Tx - [\theta, L(y-x)],$$

$$Tx \subset Ty + [\theta, L(y-x)].$$

Then T has a fixed point in P.

Proof. Since $\lim_{n\to\infty}(\|L^n\|)^{\frac{1}{n}} = r(L)$, we have $\|L^n\| \leq q^n$ for some $q \in (0,1)$. From the hypothesis (2) we can find an element $x_1 \in Tx_0$ such that $x_0 \leq x_1$. From the hypothesis (3), we have

$$Tx_0 \subset Tx_1 + [\theta, L(x_1 - x_0)].$$

Then we choose an element $x_2 \in Tx_1$, such that

$$0 \leq x_1 - x_2 \leq L(x_1 - x_0),$$

which implies

$$\|x_1 - x_2\| \leq N.\|L\|.\|x_1 - x_0\|.$$

From the condition (3), we have

$$Tx_1 \subset Tx_2 - [\theta, L(x_1 - x_2)].$$

Then we choose an element $x_3 \in Tx_2$, such that

$$0 \leq x_3 - x_2 \leq L(x_1 - x_0).$$

which implies

$$\|x_3 - x_2\| \leq N.\|L\|.\|x_2 - x_1\| \leq N.\|L^2\|.\|x_1 - x_0\|.$$

Repeating arguments above for the pair x_1, x_2 in place x_0, x_1 and so on, we can construct an sequence $\{x_n\}$ satisfying

$$\|x_n - x_{n-1}\| \leq N.\|L^n\|.\|x_1 - x_0\|, \quad x_n \in Tx_{n-1}.$$

The sequence $\{x_n\}$ is a Cauchy sequence, let $x_n \to x^*$. From condition (1) and $x_n \to x^*, x_{n-1} \to x^*, x_n \in Tx_{n-1}$, we have $x^* \in Tx^*$. □

Remark 3.7. In Theorem 3.5, Theorem 3.6, the assumption (2) $Tx_0 \prec_2 \{x_0\}$, for some $x_0 \in P$ can be substituted by (2′) $\{x_0\} \prec_2 Tx_0$, for some $x_0 \in P$.

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