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A new kind of repairable system with repairman vacations

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Abstract

In this paper, a new kind of series repairable system with repairman vacation is discussed, in which the failure rate functions of all the units and the delayed vacation rate function of the repairman are related to the working time of the system. The system model of a group of integro-differential equations is established by using probability analysis method, which then is translated into an initial value problem of a class of abstract semi-linear evolution equation in a suitable Banach space for further study. Then the conditions of the existence and uniqueness of the system solution is analyzed by using C_0-semigroup theory. And by using Laplace transform method, some steady-state reliability indexes, such as system availability, failure frequency, and the probability that the repairman is on vacation, are discussed. Numerical examples are also presented at the end of the paper. ©2015 All rights reserved.

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1. Introduction

A repairable system with repairman vacation is widely presented in repairable systems and queueing systems. It is related to a variety of areas, such as aviation, aerospace, defense, finance and network communications. The well understanding of this type of systems is of both theoretical significance and real applications. The models with repairman vacation were originally presented in queueing systems. In the early twentieth century, researchers, such as Jain, Rakhee & Singh [3], Ke & Wang [5], Liu, Tang & Luo [11], et al., introduced the concept of repairman vacation into repairable systems, including delayed

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vacation, single vacation and multiple vacations. To the authors' best knowledge, most of available references related to repairman vacation are interested in the system steady-state behaviors (for examples, please refer to [4, 15, 14]). Guo, Xu, Gao & Zhu [2] discussed the well-posedness and stability of a series repairable system with a repairman following delayed-multiple vacations policy, and analyzed the sensitivity analysis of system parameters. However, the failure rates in all available references are either constants or at most, are related to the age of a system (for example, please see [9, 10]), but not related to the working time of a system. But in practice, the failure rates of a unit is generally dependent on the working time of the system. For example, the failure rate can be increased with the system running. Then it will be decreased with the measures such as preventive maintenance, periodic detection and periodic maintenance. For this reason, we are dedicated to studying a repairable system with repairman vacation in which the failure rate of all the units are related to the working time of the system.

The rest of present study is arranged as follows. In the following section, the system model is established by using probability analysis method and then is translated into an initial value problem of a class of abstract semi-linear evolution equation in a Banach space. In Section 3, some properties of the system operator are discussed, thereby the existence and uniqueness, and the continuous dependence of the system solution for the initial value are derived by using C_0-semigroup theory. In Section 4, we derive some reliability indexes, such as steady-state availability, steady-state failure frequency, and steady-state probability of repairman on vacation by using Laplace transform method. Section 5 does numerical analysis. And a brief conclusion is presented in Section 6 at the end of the paper.

2. System formulation

The system model considered in this paper is an n-unit series repairable system with a repairman following delayed-multiple vacations policy. The system is described specifically as follows: at the initial time $t = 0$, all the units are new, the system begins to work and the repairman is preparing for the vacation. If there is a unit failed in the delayed-vacation period, the repairman deals with it immediately and the delayed vacation is terminated. Otherwise, he leaves for a vacation after the delayed-vacation period is end. Whenever the repairman returns from a vacation, he either prepares for the next vacation if no units are failed in the system or deals with the failed units immediately. The repair facility is neither failed nor deteriorated during the whole process. The system is repaired as good as new.

Set all possible states of the system at time t as follows. 00: The system is working and the repairman is preparing for the vacation. 01: The system is working and the repairman is on vacation. $1i$: The unit i is failed and the repairman is on vacation, and $2i$: The repairman is dealing with the failed unit i, $i = 1, 2, \cdots, n$. Then with the probability analysis method, the system model can be described as below.

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} + \varepsilon(t) + \Lambda(t)\right] P_{00}(t) = \int_0^\infty r(x)P_{01}(t,x)\mathrm{d}x + \sum_{i=1}^{n}\int_0^\infty \mu_i(y)P_{2i}(t,y)\mathrm{d}y \tag{2.1}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \Lambda(t) + r(x)\right] P_{01}(t,x) = 0 \tag{2.2}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + r(x)\right] P_{1i}(t,x) = \lambda_i(t)P_{01}(t,x) \tag{2.3}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \mu_i(y)\right] P_{2i}(t,y) = 0. \tag{2.4}$$

The boundary conditions are

$$P_{01}(t,0) = \varepsilon(t)P_{00}(t) \tag{2.5}$$

$$P_{1i}(t,0) = 0 \tag{2.6}$$

$$P_{2i}(t,0) = \lambda_i(t)P_{00}(t) + \int_0^\infty r(x)P_{1i}(t,x)\mathrm{d}x, \tag{2.7}$$

where $i = 1, 2, \cdots, n$, $\Lambda(t) = \sum_{i=1}^{n} \lambda_i(t)$. The initial conditions are

$$P_{00}(0) = 1, \text{ the others equal to } 0. \tag{2.8}$$

Here $P_{00}(t)$ represents the probability that the system is in state 00 at time t; $P_{01}(t,x)\,\mathrm{d}x$ represents the probability that the system is in state 01 with elapsed vacation time lying in $[x,\ x+\mathrm{d}x)$ at time t; $P_{1i}(t,x)\,\mathrm{d}x$ represents the probability that the system is in state $1i$ with elapsed vacation time lying in $[x,\ x+\mathrm{d}x)$ at time t; $P_{2i}(t,y)\,\mathrm{d}y$ represents the probability that the system is in state $2i$ with elapsed repair time lying in $[y,\ y+\mathrm{d}y)$ at time t. $\varepsilon(t)$ denotes the delayed vacation rate function, $r(x)$ denotes the vacation rate function, $\lambda_i(t)$ denotes the failure rate function of unit i, $\mu_i(y)$ denotes the repair rate function of unit i, $i = 1, 2, \cdots, n$.

Concerning the practical background, we can assume that $\varepsilon(t)$, $\lambda_i(t)$, $r(x)$, $\mu_i(y)$ are all nonnegative bounded functions satisfying $\varepsilon(t) \to \hat{\varepsilon} \geq 0$, $\lambda_i(t) \to \hat{\lambda_i} \geq 0\,(t \to \infty)$, $r(x)$, $\mu_i(y) \in L[0,T]\,(0 < T < \infty)$ and $\int_0^\infty r(x)\mathrm{d}x = \int_0^\infty \mu_i(y)\mathrm{d}y = \infty$, $i = 1, 2, \cdots, n$.

For further study, we will translate the system (2.1)-(2.8) into an initial value problem of a class of abstract semi-linear evolution system in a Banach space.

Choose the state space X as below.

$$X = \{P = (P_{00}, P_{01}, P_{11}, \cdots, P_{1n}, P_{21}, \cdots, P_{2n})^{\mathrm{T}} | P_{00} \in \mathbb{R}, P_j \in L^1(\mathbb{R}_+),$$
$$\|P\| = |P_{00}| + \sum_j \|P_j\| < \infty,\ j = 01, 1i, 2i, i = 1, 2, \cdots, n\},$$

where $\mathbb{R}_+$ represents the set of nonnegative real numbers. Obviously, X is a Banach space.

Define operator $A : D(A) \subset X \to X$ as follows.

$$\begin{aligned}
&A(P_{00}, P_{01}(x), P_{11}(x), \cdots, P_{1n}(x), P_{21}(y), \cdots, P_{2n}(y))^{\mathrm{T}} \\
&= \Bigg(\int_0^\infty r(x)P_{01}(x)\mathrm{d}x + \sum_{i=1}^{n}\int_0^\infty \mu_i(y)P_{2i}(y)\mathrm{d}y, -P_{01}^{'}(x) - r(x)P_{01}(x), \\
&\quad - P_{11}^{'}(x) - r(x)P_{11}(x), \cdots, -P_{1n}^{'}(x) - r(x)P_{1n}(x), \\
&\quad -P_{21}^{'}(y) - \mu_1(y)P_{21}(y), \cdots, -P_{2n}^{'}(y) - \mu_n(y)P_{2n}(y)\Bigg)^{\mathrm{T}},
\end{aligned}$$

with

$$D(A) = \left(\begin{array}{c} P = (P_{00}, P_{01}, P_{11}, \cdots, P_{1n}, P_{21}, \cdots, P_{2n})^{\mathrm{T}} \in X \,| \\ P_j \text{ are differentiable in } \mathbb{R}_+ \text{ and } P_j' \in L^1(\mathbb{R}_+), \\ j = 01, 1i, 2i, i = 1, 2, \cdots, n. \end{array}\right)$$

Let $f(t,P) : [0,\infty) \times X \to X$ be

$$\begin{aligned}
f(t,P) = (&-(\varepsilon(t) + \Lambda(t))P_{00}(t), -\Lambda(t)P_{01}(t,x), \\
&\lambda_1(t)P_{01}(t,x), \cdots, \lambda_n(t)P_{01}(t,x), 0, \cdots, 0)^{\mathrm{T}}.
\end{aligned}$$

Then the system (2.1)-(2.8) can be translated into an initial value problem of a class of abstract semi-linear evolution system in Banach space X:

$$\begin{cases}
\dfrac{\mathrm{d}P(t,\cdot)}{\mathrm{d}t} = AP(t,\cdot) + f(t, P(t,\cdot)) \quad t \geq 0 \\
P(t,\cdot) = (P_{00}(t), P_{01}(t,x), P_{11}(t,x), \cdots, P_{1n}(t,x), P_{21}(t,y), \cdots, P_{2n}(t,y))^{\mathrm{T}} \\
P(0,\cdot) = P_0 = (1, 0, 0, \cdots, 0)^{\mathrm{T}}_{1,2n+2}.
\end{cases} \tag{2.9}$$

3. Existence and uniqueness of system solution

The unique existence of the solution of the initial value problem of abstract semi-linear evolution equations, as so far, has been focused on by few researchers (such as Li [6, 7, 8] and Wang [13]). In this section, we only discuss the unique existence of the mild solution of system (2.9) because of its limitation of the physical condition, by using C_0-semigroup theory. Some properties of the system operator A will be presented first.

Lemma 3.1. *The system operator A is densely defined in X.*

Proof. For any $F = (f_{00}, f_{01}, f_{11}, \cdots, f_{1n}, f_{21}, \cdots, f_{2n})^{\mathrm{T}} \in X$, then $f_j \in L^1(\mathbb{R}_+), j = 01, 1i, 2i,$ $i = 1, \cdots, n$. Thus for any $\eta > 0$, there exist positive numbers G_j and δ_j such that $\int_{G_{01}}^{\infty} |f_{01}(x)|\mathrm{d}x < \frac{\eta}{9}$, $\int_0^{\delta_{01}} |f_{01}(x)|\mathrm{d}x < \frac{\eta}{18}$ and $\int_{G_k}^{\infty} |f_k(\xi)|\mathrm{d}\xi < \frac{\eta}{9n}$, $\int_0^{\delta_k} |f_k(\xi)|\mathrm{d}\xi < \frac{\eta}{18n}$, $k = 1i, 2i, i = 1, \cdots, n$. Let

$$\delta = \min\left\{\delta_{01}, \delta_{1i}, \delta_{2i}, \frac{3\eta}{2\{\eta r + 9[(\varepsilon+\Lambda)|f_{00}| + \int_0^{\infty} r(x)|f_{1i}(x)|\mathrm{d}x]\}}\right\}.$$

Take $P_{00} = f_{00}$ and

$$P_{01}(x) = \begin{cases} \varepsilon P_{00}, & 0 \le x < \delta \\ g_{01}(x), & \delta \le x \le G_{01} \\ 0, & G_{01} < x < \infty \end{cases} \qquad P_{1i}(x) = \begin{cases} 0, & 0 \le x < \delta \\ g_{1i}(x), & \delta \le x \le G_{1i} \\ 0, & G_{1i} < x < \infty \end{cases}$$

$$P_{2i}(y) = \begin{cases} \lambda_i P_{00} + \int_0^{\infty} r(x)P_{1i}(x)\mathrm{d}x, & 0 \le y < \delta \\ g_{2i}(y), & \delta \le y \le G_{2i} \\ 0, & G_{2i} < y < \infty. \end{cases}$$

Here, g_j are continuously differentiable functions satisfying $g_j(G_j) = 0$, $g_{01}(\delta) = \varepsilon P_{00}$, $g_{1i}(\delta) = 0$, $g_{2i}(\delta) = \lambda_i P_{00} + \int_0^{\infty} r(x)P_{1i}(x)\mathrm{d}x$ and $\int_{\delta}^{G_{01}} |f_{01}(x) - P_{01}(x)|\mathrm{d}x < \frac{\eta}{9}$, $\int_{\delta}^{G_k} |f_k(\xi) - P_k(\xi)|\mathrm{d}\xi < \frac{\eta}{9n}$, $j = 01, k; k = 1i, 2i; i = 1, 2, \cdots, n$. Then P_j are continuously differentiable functions and $P_j' \in L^1(\mathbb{R}_+)$. Thus $P = (P_{00}, P_{01}, P_{11}, \cdots, P_{1n}, P_{21}, \cdots, P_{2n})^{\mathrm{T}} \in D(A)$. Furthermore, it is not difficult to prove that $\|F - P\| < \eta$. Therefore, $D(A)$ is dense in X. □

Lemma 3.2. *$\{\xi | \xi > \varepsilon + \Lambda\} \subset \rho(A)$, where $\rho(A)$ is the resolvent set of system operator A. And there exists a constant $W > 0$, such that for any $\xi > W$,*

$$\|R(\xi; A)\| \le \frac{1}{\xi - W}$$

where $R(\xi; A) = (\xi I - A)^{-1}$.

Proof. For any $F = (f_{00}, f_{01}, f_{11}, \cdots, f_{1n}, f_{21}, \cdots, f_{2n})^{\mathrm{T}} \in X$, consider the operator equation $(\xi I - A)P = F$. That is

$$\xi P_{00} - \int_0^{\infty} r(x)P_{01}(x)\mathrm{d}x - \sum_{i=1}^{n} \int_0^{\infty} \mu_i(y)P_{2i}(y)\mathrm{d}y = f_{00} \tag{3.1}$$

$$P_{01}'(x) + [\xi + r(x)]P_{01}(x) = f_{01}(x) \tag{3.2}$$

$$P_{1i}'(x) + [\xi + r(x)]P_{1i}(x) = f_{1i}(x) \tag{3.3}$$

$$P_{2i}'(y) + [\xi + \mu_i(y)]P_{2i}(y) = f_{2i}(y) \tag{3.4}$$

$$P_{01}(0) = \varepsilon P_{00} \tag{3.5}$$

$$P_{1i}(0) = 0 \tag{3.6}$$

$$P_{2i}(0) = \lambda_i P_{00} + \int_0^{\infty} r(x)P_{1i}(x)\mathrm{d}x \tag{3.7}$$

where $i = 1, 2, \cdots, n$. Solving equations (3.2)-(3.4) with the help of (3.5)-(3.7) derives

$$\begin{aligned} P_{01}(x) =& \varepsilon P_{00} e^{-\int_0^x [\xi + r(s)]\mathrm{d}s} + \int_0^x f_{01}(s) e^{-\int_s^x [\xi + r(\tau)]\mathrm{d}\tau} \mathrm{d}s \\ \triangleq& \varepsilon P_{00} e^{-\int_0^x [\xi + r(s)]\mathrm{d}s} + Y_{01}(x) \end{aligned} \tag{3.8}$$

$$P_{1i}(x) = \int_0^x f_{1i}(s) e^{-\int_s^x [\xi + r(\tau)]\mathrm{d}\tau} \mathrm{d}s \triangleq Y_{1i}(x) \tag{3.9}$$

$$\begin{aligned} P_{2i}(y) =& \left[\lambda_i P_{00} + \int_0^\infty r(x) P_{1i}(x)\mathrm{d}x\right] e^{-\int_0^y [\xi + \mu_i(s)]\mathrm{d}s} + \int_0^y f_{2i}(s) e^{-\int_s^y [\xi + \mu_i(\tau)]\mathrm{d}\tau} \mathrm{d}s \\ \triangleq& \lambda_i P_{00} e^{-\int_0^y [\xi + \mu_i(s)]\mathrm{d}s} + Y_{2i}(y). \end{aligned} \tag{3.10}$$

Substituting (3.8) and (3.10) into (3.1) yields

$$\begin{aligned} &(\xi - \varepsilon M - \sum_{i=1}^n \lambda_i N_i) P_{00} \\ =& f_{00} + \int_0^\infty r(x) Y_{01}(x)\mathrm{d}x + \sum_{i=1}^n \int_0^\infty \mu_i(y) Y_{2i}(y)\mathrm{d}y, \end{aligned}$$

where $M = \int_0^\infty r(x) e^{-\int_0^x [\xi + r(s)]\mathrm{d}s}\mathrm{d}x$, $N_i = \int_0^\infty \mu_i(y) e^{-\int_0^y [\xi + \mu_i(s)]\mathrm{d}s}\mathrm{d}y$, $i = 1, 2, \cdots, n$. It is not hard to prove that for any $\xi > 0$ and $t \geq 0$, $\int_t^\infty r(x) e^{-\int_t^x [\xi + r(s)]\mathrm{d}s}\mathrm{d}x < 1$, $\int_t^\infty \mu_i(y) e^{-\int_t^y [\xi + \mu_i(s)]\mathrm{d}s}\mathrm{d}y$ < 1, $i = 1, 2, \cdots, n$. Then $\xi - \varepsilon M - \sum_{i=1}^n \lambda_i N_i > \xi - \varepsilon - \Lambda > 0$, for any $\xi > \varepsilon + \Lambda$. Thus

$$P_{00} = \frac{f_{00} + \int_0^\infty r(x) Y_{01}(x)\mathrm{d}x + \sum_{i=1}^n \int_0^\infty \mu_i(y) Y_{2i}(y)\mathrm{d}y}{\xi - \varepsilon M - \sum_{i=1}^n \lambda_i N_i}. \tag{3.11}$$

Moreover, according to [1], we can deduce that there exists a constant $S > 0$, such that $\forall t \geq 0$, $\int_t^\infty e^{-\int_t^x r(\tau)\mathrm{d}\tau}\mathrm{d}x \leq S$, $\int_t^\infty e^{-\int_t^y \mu_i(\tau)\mathrm{d}\tau}\mathrm{d}y \leq S$, $i = 1, 2, \cdots, n$. Then from (3.8)-(3.11), it can be seen that for any $\xi > \varepsilon + \Lambda$, the equations (3.1)-(3.7) have a unique solution $P = (P_{00}, P_{01}, P_{11}, \cdots, P_{1n}, P_{21}, \cdots, P_{2n})^{\mathrm{T}} \in D(A)$. This means that $(\xi I - A)$ is surjective. Because $(\xi I - A)$ is closed and $D(A)$ is dense in X, then $(\xi I - A)^{-1}$ exists and is bounded by Inverse Operator Theorem, for any $\xi > \varepsilon + \Lambda$.

Furthermore, the following estimations can be obtained with the equations (3.8)-(3.11).

$$|P_{00}| < \frac{\|F\|}{\xi - \varepsilon - \Lambda} \tag{3.12}$$

$$\int_0^\infty |P_{01}(x)|\mathrm{d}x < \frac{\varepsilon |P_{00}|}{\xi} + \frac{\|f_{01}\|}{\xi} \tag{3.13}$$

$$\int_0^\infty |P_{1i}(x)| < \frac{\|f_{1i}\|}{\xi} \tag{3.14}$$

$$\int_0^\infty |P_{2i}(y)| < \frac{\lambda_i |P_{00}|}{\xi} + \frac{\|f_{1i}\|}{\xi} + \frac{\|f_{2i}\|}{\xi}. \tag{3.15}$$

Thus equations (3.12)-(3.15) follows the estimation:

$$\begin{aligned} \|P\| =& |P_{00}| + \|P_{01}\| + \sum_{i=1}^n \|P_{1i}\| + \sum_{i=1}^n \|P_{2i}\| \\ <& \frac{3\xi - \varepsilon - \Lambda}{\xi(\xi - \varepsilon - \Lambda)} \|F\| < \frac{1}{\xi - W} \|F\| \end{aligned}$$

where $0 < W < 8(\varepsilon + \Lambda)/9$. This means that for any $\xi > W$, $(\xi I - A)^{-1}$ exits and $\|(\xi I - A)^{-1}\| < \frac{1}{\xi - W}$. $\square$

Lemma 3.3 ([12]). *Let $f : [t_0, T] \times X \to X$ be continuous about t on $[t_0, T]$ and uniformly Lipschitz continuous (with constant L) on X, if $-A$ is the infinitesimal generator of a C_0 semigroup $T(t)$, $t \geq 0$, on X, then for every $u_0 \in X$ the initial value problem*

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} + Au(t) = f(t, u(t)), \quad t > t_0 \\ u(t_0) = u_0 \end{cases}$$

has a unique mild solution $u \in C([t_0, T] : X)$. Moreover, the mapping $u_0 \to u$ is Lipschitz continuous from X into $C([t_0, T] : X)$.

According to Hille-Yosida theorem [12] with Lemmas 3.1 and 3.2, the following result is obvious.

Theorem 3.4. *The system operator A generates a C_0 semigroup $T(t)$.*

Theorem 3.5. *For any $T > 0$, assume $\varepsilon(t)$ and $\lambda_i(t)\,(i = 1, 2, \cdots, n)$ are continuous on $[0, T]$. Then for any $P \in X$, if $P_{01}(t, \cdot) \in C([0, T] : L^1(\mathbb{R}_+))$, where P_{01} is the second component of P, the semi-linear evolution system* (2.9) *has a unique mild solution $P \in C([0, T] : X)$. Moreover, the mapping $P_0 \to P$ is Lipschitz continuous from X into $C([0, T] : X)$.*

Proof. For any $T > 0$, with the assumptions of the theorem, it is obvious that $f(t, P)$ is continuous about t on $[0, T]$. Furthermore, for any $t \in [0, T]$ and P, $Q \in X$, it is easy to know that

$$\|f(t, P) - f(t, Q)\| \leq L\|P - Q\|$$

where P_{00}, P_{01} and Q_{00}, Q_{01} are respectively the first and the second components of P and Q, and $L = 2\max\left\{\sup\limits_{t\in[0,T]} \varepsilon(t), \sup\limits_{t\in[0,T]} \Lambda(t)\right\}$. Therefore, the result of theorem 3.5 is obvious by using Lemma 3.3. □

4. Reliability indexes

In this section, we substitute the limit values $\hat{\varepsilon}$ and $\hat{\lambda}_i$ respectively for the delayed vacation rate $\varepsilon(t)$ of repairman and the failure rate $\lambda_i(t)$ of each unit i, $i = 1, 2, \cdots, n$ in system (2.1)-(2.8), which can be found in Section 2, and for simplicity, we will replace them with ε and λ_i in the following. Thus system is a special case of [2] and then is stable. So in this section we can study steady-state reliability indexes of the system with the method of Laplace transformation because the premise of Laplace transformation needs the condition that the system solution is unique existed and stable.

Applying the Laplace transformation to equations (2.1)-(2.8), we can obtain the following equations.

$$(s + \Lambda + \varepsilon)P^*_{00}(s) = 1 + \int_0^\infty r(x)P^*_{01}(s, x)\mathrm{d}x + \sum_{i=1}^{n} \int_0^\infty \mu_i(y)P^*_{2i}(s, y)\mathrm{d}y \tag{4.1}$$

$$\frac{\mathrm{d}P^*_{01}(s, x)}{\mathrm{d}x} + [s + \Lambda + r(x)]P^*_{01}(s, x) = 0 \tag{4.2}$$

$$\frac{\mathrm{d}P^*_{1i}(s, x)}{\mathrm{d}x} + [s + r(x)]P^*_{1i}(s, x) = \lambda_i P^*_{01}(s, x) \tag{4.3}$$

$$\frac{\mathrm{d}P^*_{2i}(s, y)}{\mathrm{d}y} + [s + \mu_i(y)]P^*_{2i}(s, y) = 0 \tag{4.4}$$

$$P^*_{01}(s, 0) = \varepsilon P^*_{00}(s) \tag{4.5}$$

$$P^*_{1i}(s, 0) = 0 \tag{4.6}$$

$$P^*_{2i}(s, 0) = \lambda_i P^*_{00}(s) + \int_0^\infty r(x)P^*_{1i}(s, x)\mathrm{d}x \tag{4.7}$$

where $i = 1, 2, \cdots, n$. Solving equations (4.1)-(4.4) with the help of (4.5)-(4.7) follows

$$P_{00}^*(s) = \frac{1}{s\Big[1 + \varepsilon g(s) + (1 + \frac{\varepsilon}{\Lambda}((s+\Lambda)f(s) - sg(s)))\sum_{i=1}^{n}\lambda_i h_i(s)\Big]}$$
$$P_{01}^*(s,x) = \varepsilon P_{00}^*(s)e^{-\int_0^x[s+\Lambda+r(\tau)]\mathrm{d}\tau}$$
$$P_{1i}^*(s,x) = \frac{\varepsilon\lambda_i}{\Lambda}P_{00}^*(s)e^{-\int_0^x[s+r(\tau)]\mathrm{d}\tau}(1 - e^{-\Lambda x})$$
$$P_{2i}^*(s,y) = \lambda_i P_{00}^*(s)\Big[1 + \frac{\varepsilon}{\Lambda}((s+\Lambda)f(s) - sg(s))\Big]e^{-\int_0^y[s+\mu_i(\tau)]\mathrm{d}\tau}$$

where

$$f(s) = \int_0^\infty e^{-\int_0^x(s+\Lambda+r(\tau))\mathrm{d}\tau}\mathrm{d}x, \quad g(s) = \int_0^\infty e^{-\int_0^x(s+r(\tau))\mathrm{d}\tau}\mathrm{d}x$$
$$h_i(s) = \int_0^\infty e^{-\int_0^y(s+\mu_i(\tau))\mathrm{d}\tau}\mathrm{d}y, \ i = 1, 2, \cdots, n.$$

For the preparation, some steady state indexes of the system (2.1)-(2.8) can be derived as follows.

Theorem 4.1. *The stead-state availability of the system is*

$$A_v = \frac{1 + \varepsilon f}{1 + \varepsilon g + (1 + \varepsilon f)\sum_{i=1}^{n}\lambda_i h_i}$$

where $f = \int_0^\infty e^{-\int_0^x[\Lambda+r(\tau)]\mathrm{d}\tau}\mathrm{d}x$, $g = \int_0^\infty e^{-\int_0^x r(\tau)\mathrm{d}\tau}\mathrm{d}x$, $h_i = \int_0^\infty e^{-\int_0^y \mu_i(\tau)\mathrm{d}\tau}\mathrm{d}y$, $i = 1, 2, \cdots, n.$

Proof. The instantaneous availability of the system at time t is $A_v(t) = P_{00}(t) + \int_0^\infty P_{01}(t,x)\mathrm{d}x$. According to Taubert theorem, the stead-state availability of the system can be obtained readily. That is

$$A_v = \lim_{t\to\infty} A_v(t) = \lim_{s\to 0} sA_v^*(s) = \lim_{s\to 0} s\left[P_{00}^*(s) + \int_0^\infty P_{01}^*(s,x)\mathrm{d}x\right]$$
$$= \frac{1 + \varepsilon f}{1 + \varepsilon g + (1 + \varepsilon f)\sum_{i=1}^{n}\lambda_i h_i}.$$

□

Theorem 4.2. *The stead-state probability of the repairman on vacation is*

$$P_v = \frac{\varepsilon g}{1 + \varepsilon g + (1 + \varepsilon f)\sum_{i=1}^{n}\lambda_i h_i}.$$

Proof. The probability that the repairman is on vacation at time t is $P_v(t) = \int_0^\infty P_{01}(t,x)\mathrm{d}x + \sum_{i=1}^{n}\int_0^\infty P_{1i}(t,x)\mathrm{d}x$. Then the stead-state probability of the repairman on vacation can be yielded by using the limit theorem as below.

$$P_v = \lim_{t\to\infty} P_v(t) = \lim_{s\to 0} sP_v^*(s)$$
$$= \lim_{s\to 0} s\left[\int_0^\infty P_{01}^*(s,x)\mathrm{d}x + \sum_{i=1}^{n}\int_0^\infty P_{1i}^*(s,x)\mathrm{d}x\right]$$
$$= \frac{\varepsilon g}{1 + \varepsilon g + (1 + \varepsilon f)\sum_{i=1}^{n}\lambda_i h_i}$$

where g and h_i $(i = 1, 2, \cdots, n)$ is defined in Theorem 4.1. □

Theorem 4.3. *The stead-state failure frequency of the system is*

$$W_f = \Lambda A_v$$

Proof. According to Ref. [1] the instantaneous failure frequency of the system at time t is $W_f(t) = \Lambda\Big[P_{00}(t) + \int_0^\infty P_{01}(t,x)\mathrm{d}x\Big]$. Applying the limit theorem and noting the result of Theorem 4.1, the result of Theorem 4.3 can be yielded readily. □

5. Numerical analysis

In this section, we present some examples to illustrate the results obtained above. For simplicity, we consider the simple system (i.e.one-unit system) corresponding to system (2.1)-(2.8), in which the repair time of the unit and the vacation time of the repairman both follow exponential distributions. That is, $r(x) \equiv r$, $\mu(y) \equiv \mu$, where r and μ are nonnegative constants.

Furthermore, we assume $\varepsilon(t) = 0.1$, for $t \in [0,1)$; 0.5, for $t \in [1,3)$; 1, for $t \in [3,\infty)$ and $\lambda(t) = 1$ for $t \in [0,1)$; 0.5, for $t \in [1,3)$; 0.01, for $t \in [3,\infty)$. Figures 1-4 present the instantaneous availabilities, instantaneous probabilities of the repairman on vacation, instantaneous probabilities of the system in failure state and instantaneous probabilities of the system in good state with different repair rate μ and vacation rate r.

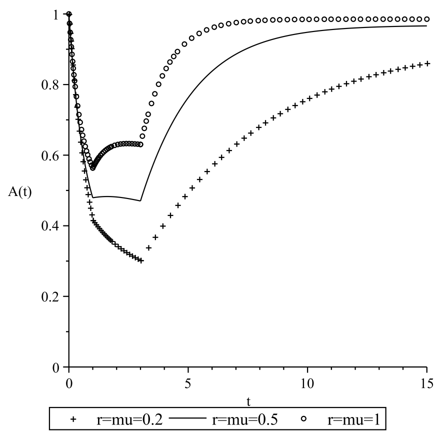

Figure 1: Transient availabilities with different repair and vacation rates

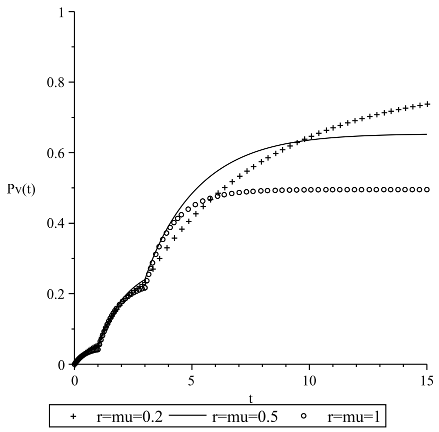

Figure 2: Transient vacation probabilities with different repair and vacation rates

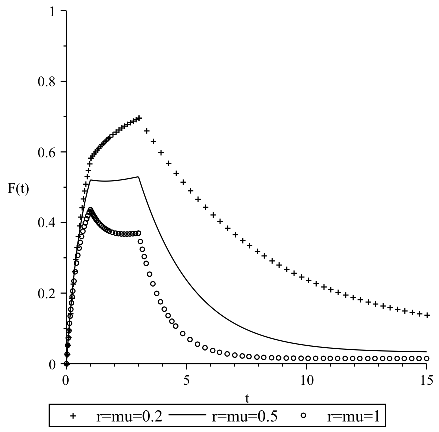

Figure 3: Transient failure probabilities with different repair and vacation rates

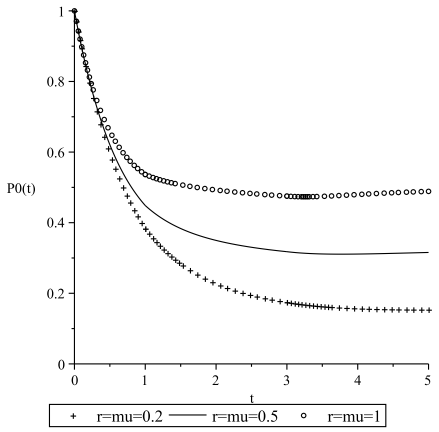

Figure 4: Transient good-state probabilities with different repair and vacation rates

From the figures, we can deduce the following conclusions.

(i) The instantaneous availability of the simple system first decreases then increases with the increasing of delay-vacation rate ε and decreasing of failure rate λ, and increases with the increasing of vacation rate r and repair rate μ.

(ii) The instantaneous probability of the repairman on vacation increases with the increasing of delay-vacation rate ε and decreasing of failure rate λ, and decreases with the increasing of vacation rate r and repair rate μ in steady state.

(iii) The instantaneous probability of the system in failure state first increases then decreases with the increasing of delay-vacation rate ε and decreasing of failure rate λ, and decreases with the increasing of vacation rate r and repair rate μ.

(iv) The instantaneous probability of system in good state (i.e. the probability of the repairman in delayed vacation stage) decreases with the increasing of delay-vacation rate ε and decreasing of failure rate λ, and increases with the increasing of vacation rate r and repair rate μ in steady state.

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