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Fractional differential equations with integral boundary conditions

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Abstract

In this paper, the existence of solutions of fractional differential equations with integral boundary conditions is investigated. The upper and lower solutions combined with monotone iterative technique is applied. Problems of existence and unique solutions are discussed.

Keywords: Fractional differential equations, upper and lower solutions, monotone iterative, convergence, integral boundary conditions.
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1. Introduction

We consider the following integral boundary value problem for nonlinear fractional differential equation:

$$\begin{cases} D^q x(t) = f(t, x(t)), & t \in J = [0, T], T > 0, \\ x(0) = \lambda \int_0^T x(s)ds + d, & d \in R, \end{cases} \tag{1.1}$$

where $f \in C(J \times R, R)$, $\lambda \geq 0$ and $0 < q < 1$. The integral boundary conditions $\lambda = 1$ or -1, which have been considered by authors ([14, 18]).

Recently, the fractional differential equations have been of great interest and development. It is caused both of the theory of fractional calculus itself and by the applications of such constructions in various sciences such as physics, mechanics, chemistry, engineering, etc. For details, see [1]-[22].

In order to obtain the solutions of fractional differential equations, the monotone iterative technique have been given extensive attention in recent years (see [2, 9, 13, 17, 21]). On the other hand, the method of upper and lower solutions is an interesting and powerful tools to deal with existence results for differential

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equations problem. So, many authors developed the upper and lower solutions methods to solve fractional differential equations (see [6, 10, 12, 16, 22]). Based on above methods of the application in the fractional differential equations, we used the upper and lower solutions combined with monotone iterative technique treatment of fractional differential equations.

2. Preliminaries

Definition 2.1. The Riemann-Liouville fractional integral defined as follow

$$I^q u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s) ds,$$

where Γ denotes the Gamma function.

Definition 2.2. The Riemann-Liouville fractional derivative defined as follow

$$D^q u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dx} \int_0^t (t-s)^{-q} u(s) ds,$$

where Γ denotes the Gamma function.

To study the problem (1.1), we first consider the following problem:

$$\begin{cases} D^q u(t) = \delta(t), t \in J, \\ u(0) = \lambda \int_0^T u(s) ds + d, \end{cases} \tag{2.1}$$

where $\delta \in C(J, R)$.

Lemma 2.3. *$u(t) \in C^1(J,R)$ is a solution of (2.1) if and only if $u(t) \in C^1(J,R)$ is a solution of the following integral equation*

$$u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \delta(s) ds + \lambda \int_0^T u(s) ds + d.$$

Proof. The proof is easy, so we omit it. □

Lemma 2.4. *If $\lambda < \frac{\Gamma(q+1) - T^q}{T\Gamma(q+1)}$, then (2.1) has a unique solution $u \in C(J, R)$.*

Proof. Define an operator $A : C(J, R) \to C(J, R)$ as

$$Au(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \delta(s) ds + \lambda \int_0^T u(s) ds + d,$$

For any $u, v \in C(J, R)$, we have

$$\begin{aligned} |Au(t) - Av(t)| &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |u(s) - v(s)| ds + \lambda \int_0^T |u(s) - v(s)| ds \\ &\le [\frac{T^q}{\Gamma(q+1)} + \lambda T] |u(s) - v(s)|. \end{aligned}$$

Therefore, $\|Au - Av\| < \|u - v\|$, we know that A is a contraction operator on $C(J, R)$. Consequently, by the contraction mapping theorem, A has a unique fixed point u, i.e. $u(t)$ is a unique solution of (2.1). □

Definition 2.5. A function $u \in C^1(J, R)$ is said to an upper solution of problem (1.1) for $\lambda \geq 0$ on J. If

$$\begin{cases} D^q u(t) \geq f(t, u(t)), t \in J, \\ u(0) \geq \lambda \int_0^T u(s)ds + d, \end{cases} \tag{2.2}$$

and a lower solution of (1.1) if the inequalities are reversed.

Let $\Omega = \{u : y_0(t) \leq u \leq z_0(t)\}$ and $\mathcal{D} = \{w \in C^1(J, R) : y_0(t) \leq w(t) \leq z_0(t), t \in J\}$ be nonempty sets and $M = \sup_{t \in J} M(t)$.

Lemma 2.6 ([9]). *Let $m : R^+ \to R$ be locally Hölder continuous such that for any $t \in (0, \infty)$, we have $m(t_1) = 0$ and $m(t) \leq 0$ for $0 \leq t \leq t_1$. Then it follows that $D^q m(t_1) \geq 0$.*

Lemma 2.7 (Comparison Result [17]). *Let $p : C_{1-q}([0, T]) \to R$ be locally Hölder continuous, and p satisfies*

$$\begin{cases} D^q p(t) \geq -M(t)p(t), \\ t^{1-q} p(t) \mid_{t=0} \geq 0. \end{cases} \tag{2.3}$$

If $MT^q \Gamma(1 - q) < 1$, then, $p(t) \geq 0, \forall t \in J$.

3. Main results

In this section, we mainly investigate the existence of extremal solutions of problem (1.1) by the method of upper and lower solutions combined with monotone iterative technique.

Theorem 3.1. *Assume that (H_1): $f \in C(J \times \Omega, R)$, ($H_2$): there exists $M > 0$ such that $f(t, v) - f(t, u) \leq M[u - v]$ if $v \leq u, u, v \in \Omega, t \in J$, and $u, v \in \mathcal{D}$ are upper and lower solutions of problem (1), respectively, and $v(t) \leq u(t)$ on J. If*

$D^q y(t) = f(t, u(t)) - M[y(t) - u(t)], t \in J, \quad y(0) = \lambda \int_0^T u(s)ds + d,$
$D^q z(t) = f(t, v(t)) - M[z(t) - v(t)], t \in J, \quad z(0) = \lambda \int_0^T v(s)ds + d,$ *then*
$v(t) \leq z(t) \leq y(t) \leq u(t), t \in J,$

and y, z are upper and lower solutions of problem (1.1), respectively.

Proof. Note that there exist unique solutions for z and y. Put $q = u - y, p = z - v$, so

$$D^q p(t) = D^q z(t) - D^q v(t) \geq f(t, v(t)) - M[z(t) - v(t)] - f(t, v(t)) = -Mp(t), t \in J,$$

$$p(0) \geq \lambda \int_0^T v(s)ds - \lambda \int_0^T v(s)ds = 0,$$

and

$$D^q q(t) = D^q u(t) - D^q y(t) \geq f(t, u(t)) - M[y(t) - u(t)] - f(t, u(t)) = -Mq(t), t \in J.$$

$$q(0) \geq \lambda \int_0^T u(s)ds - \lambda \int_0^T u(s)ds = 0.$$

By Lemma 2.7, we have $p(t) \geq 0$, $q(t) \geq 0$, $t \in J$, showing that $z(t) \geq v(t)$,$u(t) \geq y(t)$, $t \in J$. Now let $m = y - z$. Assumption (H_2) yields

$$\begin{aligned} D^q m(t) = D^q y(t) - D^q z(t) &= f(t, u(t)) - M[y(t) - u(t)] - f(t, v(t)) - M[z(t) - v(t)] \\ &= f(t, u(t)) - f(t, v(t)) - M[y(t) - u(t) - z(t) + v(t)] \\ &\geq -M[u(t) - v(t)] + M(u(t) - v(t)) - M(y(t) - z(t)) = -Mm(t), \quad t \in J. \end{aligned}$$

$$m(0) \geq \lambda \int_0^T u(s)ds - \lambda \int_0^T v(s)ds = 0.$$

Hence $m(t) \geq 0, t \in J$ showing that $z(t) \leq y(t), t \in J$. So $v(t) \leq z(t) \leq y(t) \leq u(t), t \in J$.
Now, we need to show that y, z are upper and lower solutions of problem (1), respectively. Using Assumption H_2, we have

$$\begin{aligned} D^q y(t) &= f(t, u(t)) - M[y(t) - u(t)] \\ &= f(t, u(t)) - M[y(t) - u(t)] - f(t, y(t)) + f(t, y(t)) \\ &\geq f(t, y(t)) - M[y(t) - u(t)] + M[y(t) - u(t)] = f(t, y(t)) \end{aligned}$$

$$y(0) = \lambda \int_0^T u(s)ds + d \geq \lambda \int_0^T y(s)ds + d.$$

Similarly, we can prove that

$$D^q z(t) \leq f(t, z(t))$$

$$z(0) \leq \lambda \int_0^T z(s)ds + d.$$

So, y, z are upper and lower solutions of (1), respectively. □

Theorem 3.2. *Assume that the conditions (H_1), (H_2) and (H_3): $y_0, z_0 \in C^1(J, R)$ are upper and lower solutions of (1), respectively, and such that $y_0(t) \geq z_0(t), t \in J$ are satisfied. Then there exist monotone sequences $\{z_n, y_n\}$ such that $z_n(t) \to z(t)$, $y_n(t) \to y(t)$, $t \in J$ as $n \to \infty$ and this convergence is uniformly and monotonically on J. Moreover, z, y are extremal solutions of (1.1) in $\mathcal{D}$.*

Proof. For $n = 1, 2, \cdots$, we suppose that

$$D^q z_{n+1}(t) = f(t, z_n(t)) - M[z_{n+1}(t) - z_n(t)], t \in J, \quad z_{n+1}(0) = \lambda \int_0^T z_n(s)ds + d.$$

$$D^q y_{n+1}(t) = f(t, y_n(t)) - M[y_{n+1}(t) - y_n(t)], t \in J, \quad y_{n+1}(0) = \lambda \int_0^T y_n(s)ds + d,$$

obviously, by Theorem 3.1, we have that $z_0(t) \leq z_1(t) \leq y_1(t) \leq y_0(t), t \in J$, and y_1, z_1 are upper and lower solutions of (1), respectively.

Assume that

$$z_0(t) \leq z_1(t) \leq \cdots \leq z_k(t) \leq y_k(t) \leq \cdots \leq y_1(t) \leq y_0(t), t \in J,$$

for some $k \geq 1$ and let y_k, z_k be upper and lower solutions of (1), respectively. Then, using again Theorem 3.1, we get $z_k(t) \leq z_{k+1}(t) \leq y_{k+1}(t) \leq y_k(t), t \in J$. By induction, we have that

$$z_0(t) \leq z_1(t) \leq \cdots \leq z_n(t) \leq y_n(t) \leq \cdots \leq y_1(t) \leq y_0(t), t \in J.$$

Obviously, the sequences $\{y_n\}, \{z_n\}$ are uniformly bounded and equicontinuous, applying the standard arguments, we have

$$\lim_{n\to\infty} y_n = y(t), \qquad \lim_{n\to\infty} z_n = z(t)$$

uniformly on J. indeed, y and z are extremal generalized solutions of (1). To prove that y, z are extremal generalized solutions of (1), Assume that for some k, $z_k(t) \leq w(t) \leq y_k(t), t \in J$. Put $p = w - z_{k+1}$, $q = y_{k+1} - w$. Then

$$\begin{aligned} D^q p(t) &= f(t, w(t)) - f(t, z_k(t)) + M[z_{k+1}(t) - z_k(t)] \\ &\geq -M[w(t) - z_k(t)] - M[z_k(t)] - z_{k+1}(t) = -Mp(t), \end{aligned}$$

$$p(0) \geq \lambda \int_0^T [w(s) - z_k(s)]ds \geq 0,$$

and

$$\begin{aligned}D^q q(t) &= f(t, y_k(t)) - f(t, w(t)) - M[y_{k+1}(t) - y_k(t)] \\ &\geq -M[y_k(t) - w(t)] - M[y_{k+1}(t) - y_k(t)] = -Mq(t),\end{aligned}$$

$$q(0) \geq \lambda \int_0^T [y_k(s) - w(s)]ds \geq 0,$$

By Lemma 2.7, we have $z_{k+1}(t) \leq w(t) \leq y_{k+1}(t), t \in J$. It proves, by induction, that $z_n(t) \leq w(t) \leq y_n(t), t \in J$, for all n. Taking the limit $n \to \infty$, we get $z(t) \leq w(t) \leq y(t), t \in J$. □

Example 3.3. Consider the following integral boundary problem:

$$\begin{cases} D^q u(t) = e^{tsin^2 u(t)}, t \in J = [0, ln2], \\ u(0) = \lambda \int_0^T u(s)ds, \end{cases} \tag{3.1}$$

where D^q is Riemann-Liouville fractional derivative of order $0 < q < 1$. In fact, $0 \leq D^q u(t) = e^{tsin^2 u(t)} \leq e^t, t \in J, x \in R$. Take $y_0(t) = e^t, z_0(t) = 0$ on J are upper and lower solutions of problem (3.1), respectively. By Theorem 3.2, problem (3.1) has extremal solutions in the segment $[z_0, y_0]$.

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