Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 7 (2014), 255–263
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

The existence of fixed and periodic point theorems in cone metric type spaces

Poom Kumam[a,*], Hamidreza Rahimi[b], Ghasem Soleimani Rad[b,c,*]

[a] *Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Bang Mod, Thrung Khru, Bangkok, 10140, Thailand.*

[b] *Department of Mathematics, Faculty of Science, Central Tehran Branch, Islamic Azad University, P.O. Box 13185/768, Tehran, Iran.*

[c] *Young Researchers and Elite Club, Central Tehran Branch, Islamic Azad University, Tehran, Iran*

Communicated by H. K. Nashine

Abstract

In this paper, we consider cone metric type spaces which are introduced as a generalization of symmetric and metric spaces by Khamsi and Hussain [M.A. Khamsi and N. Hussain, Nonlinear Anal. **73** (2010), 3123–3129]. Then we prove several fixed and periodic point theorems in cone metric type spaces.

Keywords: Metric type space, Fixed point, Periodic point, Property P, Property Q, Cone metric space.
2010 MSC: 47H10, 54H25, 47H09.

1. Introduction

Following Banach [3], if (X, d) is a complete metric space and T is a map of X satisfies $d(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in X$ where $\lambda \in [0, 1)$, then T has a unique fixed point. Afterward, several fixed point theorems were considered by other people [4, 7, 12, 14, 26]. The cone metric space was initiated in 2007 by Huang and Zhang [8] and several fixed and common fixed point results in cone metric spaces were introduced in [1, 9, 13, 17, 18, 19, 20, 21, 22, 23, 25, 27, 28].

The symmetric space, as metric-like spaces lacking the triangle inequality was introduced in 1931 by Wilson [29]. Recently, a new type of spaces which they called metric type spaces are defined by Khamsi

*Corresponding author
Email addresses: poom.kum@kmutt.ac.th (Poom Kumam), rahimi@iauctb.ac.ir (Hamidreza Rahimi), gha.soleimani.sci@iauctb.ac.ir (Ghasem Soleimani Rad)

Received 2013-11-22

and Hussain [15, 16]. Analogously with definition of metric type space, Ćvetković et al. [5] defined cone metric type space. On the other hand, several fixed point theorems in cone metric type spaces were proved by other researchers [5, 11, 24].

The purpose of this paper is to generalize and unify the fixed and periodic point theorems of Abbas and Jungck [1], Huang and Zhang [8], Rezapour and Hamlbarani [25], Abbas and Rhoades [2], Song et al. [27] on cone metric type spaces.

2. Preliminaries

Let us start by defining some important definitions.

Definition 2.1 ([29]). Let X be a nonempty set and the mapping $D : X \times X \to [0, \infty)$ satisfies

$(S1)$ $D(x, y) = 0 \Longleftrightarrow x = y$;

$(S2)$ $D(x, y) = D(y, x)$,

for all $x, y \in X$. Then D is called a symmetric on X and (X, D) is called a symmetric space.

Definition 2.2 ([6, 8]). Let E be a real Banach space and P be a subset of E. Then P is called a cone if and only if
(a) P is closed, non-empty and $P \neq \{0\}$;
(b) $a, b \in R, a, b \geq 0, x, y \in P$ imply that $ax + by \in P$;
(c) if $x \in P$ and $-x \in P$, then $x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by

$$x \leq y \Longleftrightarrow y - x \in P.$$

We shall write $x < y$ if $x \leq y$ and $x \neq y$. Also, we write $x \ll y$ if and only if $y - x \in intP$ (where $intP$ is the interior of P). The cone P is named normal if there is a number $k > 0$ such that for all $x, y \in E$, we have

$$0 \leq x \leq y \Longrightarrow \|x\| \leq k\|y\|.$$

The least positive number satisfying the above is called the normal constant of P.

Definition 2.3 ([8]). Let X be a nonempty set and the mapping $d : X \times X \to E$ satisfies
$(d1)$ $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
$(d2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;
$(d3)$ $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.
Then, d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.4 ([15, 16]). Let X be a nonempty set, and $K \geq 1$ be a real number. Suppose the mapping $D : X \times X \to [0, \infty)$ satisfies
$(D1)$ $D(x, y) = 0$ if and only if $x = y$;
$(D2)$ $D(x, y) = D(y, x)$ for all $x, y \in X$;
$(D3)$ $D(x, z) \leq K(D(x, y) + D(y, z))$ for all $x, y, z \in X$.
(X, D, K) is called metric type space. Obviously, for $K = 1$, metric type space is a metric space.

Example 2.5 ([16]). Let X be the set of Lebesgue measurable functions on $[0, 1]$ such that $\int_0^1 |f(x)|^2 dx < \infty$. Suppose $D : X \times X \to [0, \infty)$ is defined by $D(f, g) = \int_0^1 |f(x) - g(x)|^2 dx$ for all $f, g \in X$. Then (X, D) is a metric type space with $K = 2$.

Definition 2.6 ([5]). Let X be a nonempty set, $K \geq 1$ be a real number and E a real Banach space with cone P. Suppose that the mapping $d : X \times X \to E$ satisfies
($cd1$) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
($cd2$) $d(x, y) = d(y, x)$ for all $x, y \in X$;
($cd3$) $d(x, z) \leq K(d(x, y) + d(y, z))$ for all $x, y, z \in X$.
(X, d, K) is called cone metric type space. Obviously, for $K = 1$, cone metric type space is a cone metric space.

Example 2.7 ([5]). Let $B = \{e_i | i = 1, \cdots, n\}$ be orthonormal basis of $\mathbb{R}^n$ with inner product $(.,.)$ and $p > 0$. Define

$$X_p = \big\{[x] | x : [0,1] \to \mathbb{R}^n, \int_0^1 |(x(t), e_j)|^p dt \in \mathbb{R}, j = 1, 2, \cdots, n\big\},$$

where $[x]$ represents class of element x with respect to equivalence relation of functions equal almost everywhere. Let $E = \mathbb{R}^n$ and

$$P_B = \big\{y \in \mathbb{R}^n | (y, e_i) \geq 0, i = 1, 2, \cdots, n\big\}$$

be a solid cone. Define $d : X_p \times X_p \to P_B \subset \mathbb{R}^n$ by

$$d(f, g) = \sum_{i=1}^{n} e_i \int_0^1 |((f - g)(t), e_i)|^p dt, \qquad f, g \in X_p.$$

Then (X_p, d, K) is cone metric type space with $K = 2^{p-1}$.

Similarly, we define convergence in cone metric type spaces.

Definition 2.8 ([5]). Let (X, d, K) be a cone metric type space, $\{x_n\}$ a sequence in X and $x \in X$.
(i) $\{x_n\}$ converges to x if for every $c \in E$ with $0 \ll c$ there exist $n_0 \in \mathbf{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$, and we write $\lim_{n\to\infty} d(x_n, x) = 0$
(ii) $\{x_n\}$ is called a Cauchy sequence if for every $c \in E$ with $0 \ll c$ there exist $n_0 \in \mathbf{N}$ such that $d(x_n, x_m) \ll c$ for all $m, n > n_0$, and we write $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

Lemma 2.9 ([5]). *Let (X, d, K) be a cone metric type space over-ordered real Banach space E. Then the following properties are often used, particularly when dealing with cone metric type spaces in which the cone need not be normal.*
(P_1) *If $u \leq v$ and $v \ll w$, then $u \ll w$.*
(P_2) *If $0 \leq u \ll c$ for each $c \in intP$, then $u = 0$.*
(P_3) *If $u \leq \lambda u$ where $u \in P$ and $0 \leq \lambda < 1$, then $u = 0$.*
(P_4) *Let $x_n \to 0$ in E and $0 \ll c$. Then there exists positive integer n_0 such that $x_n \ll c$ for each $n > n_0$.*

3. Fixed point results

Theorem 3.1. *Let (X, d, K) be a complete cone metric type space with constant $K \geq 1$ and P be a solid cone. Suppose the mappings f and g are two self-maps of X satisfying*

$$d(fx, gy) \leq ad(x, y) + b[d(x, fx) + d(y, gy)] + c[d(x, gy) + d(y, fx)], \tag{3.1}$$

for all $x, y \in X$, where

$$a, b, c \geq 0 \qquad and \qquad Ka + (K + 1)b + (K^2 + K)c < 1. \tag{3.2}$$

Then f and g have a unique common fixed point in X. Also, any fixed point of f is a fixed point of g, and conversely.

Proof. Suppose x_0 is an arbitrary point of X, and define $\{x_n\}$ by
$x_1 = fx_0 \ , \ x_2 = gx_1 \ , \ \cdots \ , \ x_{2n+1} = fx_{2n} \ , \ x_{2n+2} = gx_{2n+1} \ \ for \ \ n = 0, 1, 2, ...$
Now,

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(fx_{2n}, gx_{2n+1}) \\ &\leq ad(x_{2n}, x_{2n+1}) + b[d(x_{2n}, fx_{2n}) + d(x_{2n+1}, gx_{2n+1})] \\ &\quad +c[d(x_{2n}, gx_{2n+1}) + d(x_{2n+1}, fx_{2n})] \\ &= ad(x_{2n}, x_{2n+1}) + b[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] \\ &\quad +c[d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})] \\ &\leq (a+b)d(x_{2n}, x_{2n+1}) + bd(x_{2n+1}, x_{2n+2}) \\ &\quad +cK[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})], \end{aligned}$$

which implies that $d(x_{2n+1}, x_{2n+2}) \leq \lambda d(x_{2n}, x_{2n+1})$, where $\lambda = \frac{a+b+cK}{1-b-cK} < \frac{1}{K}$.
Similarly, we have $d(x_{2n+3}, x_{2n+2}) \leq \lambda d(x_{2n+2}, x_{2n+1})$, where $\lambda = \frac{a+b+cK}{1-b-cK} < \frac{1}{K}$.
Thus for all n,

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n) \leq \lambda^2 d(x_{n-2}, x_{n-1}) \leq \cdots \leq \lambda^n d(x_0, x_1). \tag{3.3}$$

Now for any $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq K[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &\leq Kd(x_n, x_{n+1}) + K^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ &\leq \cdots \leq Kd(x_n, x_{n+1}) + K^2 d(x_{n+1}, x_{n+2}) + \cdots \\ &\quad +K^{m-n-1} d(x_{m-2}, x_{m-1}) + K^{m-n} d(x_{m-1}, x_m). \end{aligned}$$

Now, by (3.3) and $\lambda < \frac{1}{K}$, we have

$$\begin{aligned} d(x_n, x_m) &\leq K(\lambda^n d(x_0, x_1)) + K^2(\lambda^{n+1} d(x_0, x_1)) + \cdots + K^{m-n}(\lambda^{m-1} d(x_0, x_1)) \\ &= (K\lambda^n + K^2\lambda^{n+1} + \cdots + K^{m-n}\lambda^{m-1}) d(x_0, x_1) \\ &= K\lambda^n(1 + K\lambda + \cdots + (K\lambda)^{m-n-1}) d(x_0, x_1) \\ &\leq \frac{K\lambda^n}{1-K\lambda} d(x_0, x_1) \to 0 \qquad when \qquad n \to \infty. \end{aligned}$$

Now, by (P_1) and (P_4), it follows that for every $c \in intP$ there exist positive integer N such that $d(x_n, x_m) \ll c$ for every $m > n > N$, so $\{x_n\}$ is a Cauchy sequence. Since cone metric type space X is complete, so there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. We show that $gz = fz = z$. Using (3.1) and (3.2), we have

$$\begin{aligned} d(z, gz) &\leq K[d(z, x_{2n+1}) + d(x_{2n+1}, gz)] = Kd(z, x_{2n+1}) + Kd(fx_{2n}, gz) \\ &\leq Kd(z, x_{2n+1}) + K\big(ad(x_{2n}, z) + b[d(x_{2n}, fx_{2n}) + d(z, gz)] \\ &\quad +c[d(x_{2n}, gz) + d(z, fx_{2n})]\big) \\ &\leq Kd(z, x_{2n+1}) + Kad(x_{2n}, z) + Kb[d(x_{2n}, x_{2n+1}) \\ &\quad +d(z, gz)] + Kc[K[d(x_{2n}, z) + d(z, gz)] + d(z, fx_{2n})] \\ &= K(1+c)d(z, x_{2n+1}) + K(a+cK)d(x_{2n}, z) + bKd(x_{2n}, x_{2n+1}) \\ &\quad +K(b+cK)d(z, gz). \end{aligned}$$

The sequence $\{x_n\}$ converges to z, so for every $c \in intP$ there exists $n_0 \in \mathbb{N}$ such that for any $n > n_0$

$$\begin{aligned} d(z, gz) &\leq \frac{K(1+c)}{1-K(b+cK)} d(z, x_{2n+1}) + \frac{K(a+cK)}{1-K(b+cK)} d(x_{2n}, z) \\ &\quad + \frac{bK}{1-K(b+cK)} d(x_{2n}, x_{2n+1}) \end{aligned}$$

$$\begin{aligned} \ll\ & \frac{K(1+c)}{1-K(b+cK)} \cdot \frac{1-K(b+cK)}{K(1+c)} \cdot \frac{c}{3} \\ & + \frac{K(a+cK)}{1-K(b+cK)} \cdot \frac{1-K(b+cK)}{K(a+cK)} \cdot \frac{c}{3} \\ & + \frac{bK}{1-K(b+cK)} \cdot \frac{1-K(b+cK)}{bK} \cdot \frac{c}{3} \end{aligned}$$

It follows that $d(z, gz) \ll c$ for every $c \in intP$, and by (P_2) we have $d(z, gz) = 0$, that is, $gz = z$. Now,

$$\begin{aligned} d(fz, z) &= d(fz, gz) \\ &\leq ad(z, z) + b[d(z, fz) + d(z, gz)] + c[d(z, gz) + d(z, fz)] \\ &= (b + c)d(fz, z). \end{aligned}$$

It follows that $d(fz, z) = 0$ by (P_3). Therefore, $gz = fz = z$. On the other hand if z_1 is another fixed point of f, then $fz_1 = gz_1 = z_1$ and

$$\begin{aligned} d(z, z_1) &= d(fz, gz_1) \\ &\leq ad(z, z_1) + b[d(z, fz) + d(z_1, gz_1)] + c[d(z, gz_1) + d(z_1, fz)] \\ &= (a + 2c)d(z, z_1), \end{aligned}$$

which is possible only if $z = z_1$ (by relation (3.2) and (P_3)). □

Corollary 3.2. *Let (X, d, K) be a complete cone metric type space with constant $K \geq 1$ and P be a solid cone. Suppose a self-map f of X satisfies*

$$d(f^p x, f^q y) \leq ad(x, y) + b[d(x, f^p x) + d(y, f^q y)] + c[d(x, f^q y) + d(y, f^p x)], \tag{3.4}$$

for all $x, y \in X$, where

$$a, b, c \geq 0 \qquad \text{and} \qquad Ka + (K + 1)b + (K^2 + K)c < 1, \tag{3.5}$$

and p and q are fixed positive integers. Then f has a unique fixed point in X.

Proof. Set $f \equiv f^p$ and $g \equiv f^q$ in inequality (3.1) and use the Theorem 3.1. □

Corollary 3.3. *Let (X, d, K) be a complete cone metric type space with constant $K \geq 1$ and P be a solid cone. Suppose a self-map f of X satisfies*

$$d(fx, fy) \leq ad(x, y) + b[d(x, fx) + d(y, fy)] + c[d(x, fy) + d(y, fx)], \tag{3.6}$$

for all $x, y \in X$, where

$$a, b, c \geq 0$$

and

$$Ka + (K + 1)b + (K^2 + K)c < 1. \tag{3.7}$$

Then f has a unique fixed point in X.

Proof. In Corollary 3.2, set $p = q = 1$. □

Remark 3.4. In Theorem 3.1 and Corollaries 3.2 and 3.3, if we suppose (X, d) is a cone metric space and P is a normal cone with normal constant k. Then the same assertions of Theorem 3.1, Corollaries 3.2 and 3.3 are true that were given in [2].

Following results is obtained from Corollary 3.3.

Corollary 3.5. *Let (X, d, K) be a complete cone metric type space with constant $K \geq 1$ and P be a solid cone. Suppose a self-map f of X satisfies*

$$d(fx, fy) \leq ad(x, y), \tag{3.8}$$

for all $x, y \in X$, where $a \in [0, \frac{1}{K}[$. Then f has a unique fixed point in X.

Remark 3.6. Corollary 3.5 is the Banach-type version of a fixed point results for contractive mappings in a metric type space. This Corollary was proved by Jovanović et al in [11].

Corollary 3.7. *Let (X, d, K) be a complete cone metric type space with constant $K \geq 1$ and P be a solid cone. Suppose a self-map f of X satisfies*

$$d(fx, fy) \leq b[d(x, fx) + d(y, fy)], \tag{3.9}$$

for all $x, y \in X$, where $b \in [0, \frac{1}{K+1}[$. Then f has a unique fixed point in X.

Corollary 3.8. *Let (X, d, K) be a complete cone metric type space with constant $K \geq 1$ and P be a solid cone. Suppose a self-map f of X satisfies*

$$d(fx, fy) \leq c[d(x, fy) + d(y, fx)], \tag{3.10}$$

for all $x, y \in X$, where $c \in [0, \frac{1}{K^2+K}[$. Then f has a unique fixed point in X.

Remark 3.9. In Corollaries 3.5, 3.7 and 3.8, suppose that (X, d) is a cone metric space, $K = 1$ and P is a normal cone with normal constant k. Then we obtain the Theorems 1, 2 and 3 that were given by Huang and Zhang in [8]. Also, if we delete normality condition of P, then we obtain Theorems 2.3, 2.6 and 2.7 that were given by Rezapour and Hamlbarani in [25].

Corollary 3.10. *Let (X, d, K) be a complete cone metric type space with constant $K \geq 1$, P be a solid cone and a self-map f of X satisfies*

$$d(fx, fy) \leq ad(x, y) + b[d(x, fx) + d(y, fy)], \tag{3.11}$$

for all $x, y \in X$, where

$$a, b \geq 0 \qquad \text{and} \qquad Ka + (K+1)b < 1. \tag{3.12}$$

Then f has a unique fixed point in X.

Corollary 3.11. *Let (X, d, K) be a complete cone metric type space with constant $K \geq 1$ and P be a solid cone. Suppose a self-map f of X satisfies*

$$d(fx, fy) \leq ad(x, y) + c[d(x, fy) + d(y, fx)], \tag{3.13}$$

for all $x, y \in X$, where

$$a, c \geq 0 \qquad \text{and} \qquad Ka + (K^2 + K)c < 1. \tag{3.14}$$

Then f has a unique fixed point in X.

Corollary 3.12. *Let (X, d, K) be a complete cone metric type space with constant $K \geq 1$ and P be a solid cone. Suppose a self-map f of X satisfies*

$$d(fx, fy) \leq \alpha_1 d(x, y) + \alpha_2 d(x, fx) + \alpha_3 d(y, fy) + \alpha_4 d(x, fy) + \alpha_5 d(y, fx), \tag{3.15}$$

for all $x, y \in X$, where

$$\alpha_i \geq 0 \ \text{for every} \ i \in \{1, 2, \cdots, 5\}$$

and

$$2K\alpha_1 + (K+1)(\alpha_2 + \alpha_3) + (K^2 + K)(\alpha_4 + \alpha_5) < 2. \tag{3.16}$$

Then f has a unique fixed point in X.

Proof. In (3.15) interchanging the roles of x and y, and adding the new inequality to (3.15), gives (3.6) with $a = \alpha_1$, $b = \frac{\alpha_2+\alpha_3}{2}$ and $c = \frac{\alpha_4+\alpha_5}{2}$. □

Remark 3.13. In Corollary 3.12, set $K = 1$. It reduces to the standard Hardy-Rogers condition [7] in cone metric spaces with $g = i_x$ (i_x is identity maps). Also, set $K = 1$ and let (X, d) be a cone metric space, P be a normal cone with normal constant k or non-normal cone. Then Theorem 2.1 and Corollary 2.1 of Song et al. in [27] are obtained.

Example 3.14. Let $X = E = \mathbb{R}$, $P = [0, \infty)$ and $d : X \times X \to [0, \infty)$ be defined by $d(x, y) = |x - y|^2$. Then (X, d) is a cone metric type space , but it is not a metric space since the triangle inequality is not satisfied. Starting with Minkowski inequality, we get $|x - z|^2 \leq 2(|x - y|^2 + |y - z|^2)$. Here $K = 2$.
Define the mapping $f : X \to X$ by $fx = M(x + b)$, where $x \in X$ and $M < \frac{1}{\sqrt{2}}$. Also, X is a complete space. Moreover, $d(fx, fy) = |M(x + b) - M(y + b)|^2 = M^2 d(x, y)$, that is, there exist $a = M^2 < \frac{1}{2} = \frac{1}{K}$ such that (3.8) is satisfied. According to Corollary 3.5, f has a unique fixed point.

4. Periodic point results

Recall if f is a map which has a fixed point z, then z is a fixed point of f^n for each $n \in \mathbb{N}$. However the converse is not true [2]. If a map $f : X \to X$ satisfies $Fix(f) = Fix(f^n)$ for each $n \in \mathbb{N}$, where $Fix(f)$ stands for the set of fixed points of f [10], then f is said to have property P. Furthermore recall that two mappings $f, g : X \to X$ is said to have property Q if $Fix(f) \bigcap Fix(g) = Fix(f^n) \bigcap Fix(g^n)$. The following results extend some theorems of [2].

Theorem 4.1. *Let (X, d, K) be a cone metric type space with constant $K \geq 1$ and P be a solid cone. suppose a self-map f of X satisfies*
(i) $d(fx, f^2x) \leq ad(x, fx)$ for all $x \in X$, where $a \in [0, \frac{1}{K}[$ and $K > 1$ or (ii) with strict inequality, $K = 1$ for all $x \in X$ with $x \neq fx$. If $Fix(f) \neq \emptyset$, then f has property P.

Proof. Proof is similar to the metric and cone metric spaces case. □

Theorem 4.2. *Let (X, d, K) be a complete cone metric type space with constant $K \geq 1$ and P be a solid cone. Suppose the mappings f and g are two self-maps of X satisfying (3.1) and (3.2) of Theorem 3.1. Then f and g have property Q.*

Proof. By Theorem 3.1, f and g have a unique common fixed point in X. Suppose $z \in Fix(f^n) \bigcap Fix(g^n)$, we have

$$\begin{aligned} d(z, gz) &= d(f(f^{n-1}z), g(g^n z)) \\ &\leq ad(f^{n-1}z, g^n z) + b[d(f^{n-1}z, f^n z) + d(g^n z, g^{n+1}z)] \\ &\quad + c[d(f^{n-1}z, g^{n+1}z) + d(g^n z, f^n z)] \\ &= ad(f^{n-1}z, z) + b[d(f^{n-1}z, z) + d(z, gz)] + cd(f^{n-1}z, gz), \end{aligned}$$

which implies that $d(z, gz) \leq \lambda d(f^{n-1}z, z)$, where $\lambda = \frac{a+b+cK}{1-b-cK} < \frac{1}{K}$ (by relation (3.2)), and we have

$$d(z, gz) = d(f^n z, g^{n+1}z) \leq \lambda d(f^{n-1}z, z) \leq \cdots \leq \lambda^n d(fz, z) \to 0 \quad as \quad n \to \infty.$$

Now, from (P_2) and (P_4), we have $d(z, gz) = 0$, and $gz = z$. Also, Theorem 3.1 implies that $fz = z$ and $z \in Fix(f) \bigcap Fix(g)$. □

Theorem 4.3. *Let (X, d, K) be a complete cone metric type space with constant $K \geq 1$ and P be a solid cone. Suppose a self-map f satisfies (3.6) of Corollary 3.3. Then f has property P.*

Proof. By Corollary 3.3, f has a unique fixed point in X. Suppose $z \in Fix(f^n)$, we have

$$\begin{aligned} d(z, fz) &= d(f(f^{n-1}z), f(f^n z)) \\ &\leq ad(f^{n-1}z, f^n z) + b[d(f^{n-1}z, f^n z) + d(f^n z, f^{n+1}z)] \\ &\quad + c[d(f^{n-1}z, f^{n+1}z) + d(f^n z, f^n z)] \\ &\leq ad(f^{n-1}z, z) + b[d(f^{n-1}z, z) + D(z, fz)] \\ &\quad + cK[d(f^{n-1}z, z) + d(z, fz)], \end{aligned}$$

which implies that
$d(z, fz) \leq \lambda d(f^{n-1}z, z)$ where $\lambda = \frac{a+b+cK}{1-b-cK} < \frac{1}{K}$, (by relation (3.2)). Hence,
$d(z, fz) = d(f^n z, f^{n+1}z) \leq \lambda d(f^{n-1}z, z) \leq \cdots \leq \lambda^n d(fz, z) \to 0$ when $n \to \infty$.
Now, from (P_2) and (P_4), we have $d(z, fz) = 0$, and $fz = z$. Hence $z \in Fix(f)$ and proof is complete. □

Corollary 4.4. *Let (X, d, K) be a complete cone metric type space with constant $K \geq 1$ and P be a solid cone. Suppose a self-map f satisfies* (3.15) *and* (3.16) *of Corollary 3.12. Then f has property P.*

Proof. See [11]. □

Corollary 4.5. *Let (X, d, K) be a complete cone metric type space with constant $K \geq 1$ and P be a solid cone. Suppose a self-map f satisfies any one of the inequalities (3.9), (3.10), (3.11-3.12) and (3.13-3.14). Then f has property P.*

Remark 4.6. Set $K = 1$, suppose (X, d) is a cone metric space and P be a normal cone, then we obtain Theorems 3.1, 3.2 and 3.3 of Abbas and Rhoades in [2].

Acknowledgements:

The first author was supported by the Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission (NRU No.57000621) and the second author was supported by Central Tehran Branch of Islamic Azad University and the Moreover, the third author would like to thank the Young Researchers and Elite club, Central Tehran Branch of Islamic Azad University. Also, the authors thank the anonymous referee for his/her valuable suggestions that helped to improve the final version of this paper.

References

[1] M. Abbas and G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl., **341** (2008), 416–420. 1

[2] M. Abbas and B. E. Rhoades, *Fixed and periodic point results in cone metric spaces*, Appl. Math. Lett., **22** (2009), 511–515. 1, 3.4, 4, 4.6

[3] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application auxéquations intégrales*, Fund. Math. J., **3** (1922), 133–181. 1

[4] L. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., **45** (1974), 267–273. 1

[5] A. S. Ćvetković, M. P. Stanić, S. Dimitrijević and S. Simić, *Common fixed point theorems for four mappings on cone metric type space*, Fixed Point Theory Appl., **2011**, (2011) 15 pages. 1, 2.6, 2.7, 2.8, 2.9

[6] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, (1985). 2.2

[7] G. E. Hardy and T. D. Rogers, *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull., **16** (1973), 201–206. 1, 3.13

[8] L. G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., **332** (2007), 1467–1475. 1, 2.2, 2.3, 3.9

[9] S. Janković, Z. Kadelburg, S. Radenović and B. E. Rhoades, *Assad-Kirk-type fixed point theorems for a pair of nonself mappings on cone metric spaces*, Fixed Point Theory Appl., **2009**, (2009) 16 pages. 1

[10] G. S. Jeong and B. E. Rhoades, *Maps for which $F(T) = F(T^n)$*, Fixed Point Theory Appl., **6** (2005), 87–131. 4

[11] M. Jovanović, Z. Kadelburg and S. Radenović, *Common fixed point results in metric-type spaces*, Fixed Point Theory Appl. **2010**, (2010) 15 pages. 1, 3.6, 4

[12] G. Jungck, *Commuting maps and fixed points*, Amer. Math. Monthly, **83** (1976), 261–263. 1

[13] Z. Kadelburg and S. Radenović, *Some common fixed point results in non-normal cone metric spaces*, J. Nonlinear Sci. Appl., **3** (2010), 193–202. 1

[14] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc., **10** (1968), 71–76. 1

[15] M. A. Khamsi, *Remarks on cone metric spaces and fixed point theorems of contractive mappings*, Fixed Point Theory Appl., **2010**, (2007) 7 pages. 1, 2.4

[16] M. A. Khamsi and N. Hussain, *KKM mappings in metric type spaces*, Nonlinear Anal., **73** (2010), 3123–3129. 1, 2.4, 2.5

[17] S. K. Mohanta and R. Maitra, *A characterization of completeness in cone metric spaces*, J. Nonlinear Sci. Appl., **6** (2013), 227–233. 1

[18] H. K. Nashine and M. Abbas, *Common fixed point of mappings satisfying implicit contractive conditions in TVS-valued ordered cone metric spaces*, J. Nonlinear Sci. Appl., **6** (2013), 205–215. 1

[19] S. Radenović, *Common fixed points under contractive conditions in cone metric spaces*, Comput. Math. Appl., **58** (2009), 1273–1278. 1

[20] S. Radojević, Lj. Paunović and S. Radenović, *Abstract metric spaces and Hardy-Rogers-type theorems*, Appl. Math. Lett., **24** (2011), 553–558. 1

[21] H. Rahimi, S. Radenović, G. Soleimani Rad and P. Kumam, *Quadrupled fixed point results in abstract metric spaces*, Comp. Appl. Math., **2013**, DOI 10.1007/s40314-013-0088-5. 1

[22] H. Rahimi, B.E. Rhoades, S. Radenović and G. Soleimani Rad, *Fixed and periodic point theorems for T-contractions on cone metric spaces*, Filomat. **27** (5) (2013), 881–888 (DOI 10.2298/FIL1305881R). 1

[23] H. Rahimi and G. Soleimani Rad, *Note on "Common fixed point results for noncommuting mappings without continuity in cone metric spaces"*, Thai. J. Math., **11** (3) (2013), 589–599. 1

[24] H. Rahimi, P. Vetro and G. Soleimani Rad, *Some common fixed point results for weakly compatible mappings in cone metric type space*, Miskolc. Math. Notes., **14** (1) (2013), 233–243. 1

[25] S. Rezapour and R. Hamlbarani, *Some note on the paper cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., **345** (2008), 719–724. 1, 3.9

[26] B.E. Rhoades, *A comparison of various definition of contractive mappings*, Trans. Amer. Math. Soc. **266** (1977), 257–290. 1

[27] G. Song, X. Sun, Y. Zhao and G. Wang, New common fixed point theorems for maps on cone metric spaces, Appl. Math. Lett., **23** (2010), 1033–1037. 1, 3.13

[28] S. Wang and B. Guo, *Distance in cone metric spaces and common fixed point theorems*, Appl. Math. Lett., **24** (2011), 1735–1739. 1

[29] W.A. Wilson, *On semi-metric spaces*, Amer. Jour. Math. **53** (1931), 361–373. 1, 2.1