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Stability in nonlinear delay Volterra integro-differential systems

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Abstract

We employ Lyapunov functionals to the system of Volterra integro-differential equations of the form

$$x'(t) = Px(t) - \int_{t-r}^{t} C(t,s)g(x(s))ds,$$

and obtain conditions for the stability of the of the zero solution. In addition, we will furnish an example as an application.

Keywords: Volterra integro-differential equations, zero solution, stability, Lyapunov functional.
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1. Introduction

In this report, we explore the use of Lyapunov functionals and obtain conditions for the zero solution of the nonlinear delay Volterra integro-differential system

$$x'(t) = Px(t) - \int_{t-r}^{t} C(t,s)g(x(s))ds, \tag{1.1}$$

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where $r > 0$ is a constant, P is a constant $n \times n$ matrix and C is an $n \times n$ matrix of functions that are continuous on $-r \le t \le s < \infty$. The function $g(x) : \mathbb{R}^n \to \mathbb{R}^n$ and is continuous in x. Throughout this paper it is understood that if $x \in \mathbb{R}^n$, then $|x|$ is taken to be the Euclidean norm. In the case $P = 0$ and (1.1) is scalar, (1.1) has its roots in the study of reactors, by Brownwell and Ergen [1] in 1954. Later on, the same study was revisited by Nohel [8], in 1960 and then by Levin and Nohel [7], in 1964.

Recently, in [2] and [3], Burton used the notion of fixed point theory to alleviate some of the difficulties that arise from the use of Lyapunov functionals and obtained results concerning the stability and asymptotic stability of the zero solution of (1.1) when it is scalar with P being identically zero matrix. For more reading on the use of fixed point theory in the study of functional differential equations, we refer the reader to [4], [6], [8] and the references there in. One of the major difficulties that we encountered was relating back the solution $x(t)$ to the Lyapunov functional so that some inequality can be obtained.

The use of Lyapunov method allowed us to deduce inequalities that all solutions must satisfy and from which we deduce the exponential stability and instability.
In [10], Wang used Lyapunov functionals and obtained inequalities from which exponential stability was deduced on the zero solution of the constant delay equation

$$x'(t) = a(t)x(t) + b(t)x(t-h)$$

provided that

$$-\frac{1}{2h} \le a(t) + b(t+h) \le -hb^2(t+h)$$

hold. In [5], the first author extended [10] to the multiple delays differential equation

$$x'(t) = a(t)x(t) + \sum_{i=1}^{n} b_i(t)x(t-h_i), \text{ where } h_i > 0, i = 1, 2, \cdot, \cdot, \cdot n.$$

Later on, in [9], the first author used Lyapunov functional and obtained results concerning the exponential stability of the zero solution of the scalar highly nonlinear Volterra integro-differential equation

$$x'(t) = -\int_{t-r}^{t} C(t,s)g(x(s))ds. \tag{1.2}$$

However, the extension of [9] to (1.1) is impossible since we are dealing with system.

Let $x \in \mathbb{R}^n$ and $U = (u)_{ij}$ be an $n \times n$ matrix. Then we define the norms $|x|$ to be the Euclidean norm and

$$|U| = \max_{1 \le j \le n} \sum_{i=1}^{n} |u_{ij}|.$$

It should cause no confusion to denote the norm of a continuous function $\varphi : [-r, \infty) \to \mathbb{R}^n$ with

$$||\varphi|| = \sup_{-r \le s < \infty} |\varphi(s)|.$$

The notation x_t means that $x_t(\tau) = x(t+\tau), \tau \in [-r, 0]$ as long as $x(t+\tau)$ is defined. Thus, x_t is a function mapping an interval $[-r, 0]$ into $\mathbb{R}^n$. We say $x(t) \equiv x(t, t_0, \psi)$ is a solution of (1.1) if $x(t)$ satisfies (1.1) for $t \ge t_0$ and $x_{t_0} = x(t_0 + s) = \psi(s)$, $s \in [-r, 0]$. Throughout this paper it is to be understood that when a function is written without its argument, then the argument is t. We begin with stability definitions. For $t_0 \ge 0$ we define

$$E_{t_0} = [-r, t_0].$$

Let $C(t)$ denote the set of continuous functions $\phi : [-r, \infty) \to \mathbb{R}^n$ and $\|\phi\| = \sup\{|\phi(s)| : -r \le s \le t\}$.

Definition 1.1. The zero solution of (1.1) is stable if for each $\varepsilon > 0$ and each $t_0 \ge -r$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $[\phi \in E_{t_0} \to \mathbb{R}^n, \phi \in C(t) : \|\phi\| < \delta]$ implies $|x(t, t_0, \phi)| < \varepsilon$ for all $t_0 \ge 0$.

2. Main Results

Now we turn our attention to the linear equation (1.1). We will construct a Lyapunov functional; $V(t,x) := V(t)$ and show that for some positive α and under suitable conditions, $V'(t) \leq -\alpha|x|^2$ along the solutions of (1.1). For the sake of rewriting (1.1) so that a suitable Lyapunov functional can be displayed, we let

$$A(t,s) := \int_{t-s}^{r} C(u+s,s)du, \ t,s \geq 0.$$

Let D be a positive definite symmetric and constant $n \times n$ matrix. Assume the existence of positive constants λ, μ_1 and μ_2 we have that

$$P^T D + DP = -\mu_1 I, \tag{2.1}$$

$$x^T DA(t,t)g(x) \geq \mu_2 |x|^2 \text{ if } x \neq 0, \tag{2.2}$$

$$|g(x)| \leq \lambda |x| \tag{2.3}$$

and

$$\frac{\partial |A(t,s)|}{\partial t} \leq 0, \ \text{ for all } (t,s) \in [0,\infty) \times [t-r,t]. \tag{2.4}$$

It is clear that conditions (2.2) and (2.3) imply that $g(0) = 0$. Since D is a positive definite symmetric matrix, then there exists a positive constant k such that

$$k|x|^2 \leq x^T Dx, \ \text{ for all } x. \tag{2.5}$$

In order to construct a suitable Lyapunov functional we put (1.1) in the form

$$x'(t) = Px(t) - A(t,t)g(x(t)) + \frac{d}{dt}\int_{t-r}^{t} A(t,s)g(x(s))ds. \tag{2.6}$$

Theorem 2.1. *Let* (2.1)- (2.4) *hold, and suppose there are constants* $\gamma > 0$ *and* $\alpha > 0$ *so that*

$$-\mu_1 - 2\mu_2 + \gamma r \lambda^2 |A(t,t)| + \big(\lambda |A^T(t,t)D| + |P^T D|\big)\int_{t-r}^{t} |A(t,s)|ds \leq -\alpha, \tag{2.7}$$

$$-\gamma + \lambda |A^T(t,t)D| + |P^T D| \leq 0, \tag{2.8}$$

and

$$1 - \lambda \int_{t-r}^{t} |A(t,s)|ds > 0 \tag{2.9}$$

then, the zero solution of (1.1) *is stable.*

Proof. Define the Lyapunov functional $V = V(t,x)$ by

$$\begin{aligned} V(t) \ &= \ \Big(x - \int_{t-r}^{t} A(t,s)g(x(s))ds\Big)^T D\Big(x - \int_{t-r}^{t} A(t,s)g(x(s))ds\Big) \\ &+\gamma \int_{-r}^{0}\int_{t+s}^{t} |A(t,z)||g(x(z))|^2 dz\, ds. \end{aligned} \tag{2.10}$$

First we note that the right side of (2.10) is scalar. Let $x(t) = x(t,t_0,\psi)$ be a solution of (1.1) and define $V(t)$ by (2.10). Then along solutions of (1.1) we have

$$V'(t) \ = \ \Big(x - \int_{t-r}^{t} A(t,s)g(x(s))ds\Big)^T D[Px - A(t,t)g(x)]$$

$$
\begin{aligned}
&+ \quad [Px - A(t,t)g(x)]^T D\Big(x(t) - \int_{t-r}^{t} A(t,s)g(x(s))ds\Big) \\
&+ \quad \gamma r|A(t,t)||g(x)|^2 - \gamma \int_{-r}^{0} |A(t,t+s)|g^2(x(t+s))ds \\
&+ \quad \gamma \int_{-r}^{0}\int_{t+s}^{t} \frac{\partial |A(t,z)|}{\partial t} g^2(x(z))dz\, ds \\
&\leq \quad x^T\big(P^T D + DP\big)x - x^T DAg(x) - g^T(x)A^T(t,t)Dx \\
&- \quad x^T P^T D \int_{t-r}^{t} A(t,s)g(x(s))ds - \big(\int_{t-r}^{t} A(t,s)g(x(s))ds\big)^T DPx \\
&+ \quad \big(\int_{t-r}^{t} A(t,s)g(x(s))ds\big)^T DA(t,t)g(x) + g^T(x)A^T(t,t)D\int_{t-r}^{t} A(t,s)g(x(s))ds \\
&+ \quad \gamma r|A(t,t)||g(x)|^2 - \gamma \int_{-r}^{0} |A(t,t+s)||g(x(t+s))|^2 ds \qquad (2.11)
\end{aligned}
$$

In what to follow we perform some calculations to simplify (2.11). First, if we let $u = t+s$, then

$$
-\int_{-r}^{0} |A(t,t+s)||g(x(t+s))|^2 ds = -\int_{t-r}^{t} |A(t,s)||g(x(s))|^2 ds. \qquad (2.12)
$$

Since,

$$
g^T(x)A^T(t,t)Dx = \big(x^T DAg(x)\big)^T,
$$

we have that

$$
-\big(g^T(x)A^T(t,t)Dx + x^T DAg(x)\big) \leq -2\mu_2|x|^2 \text{ by (2.2).} \qquad (2.13)
$$

Also,

$$
\begin{aligned}
& -x^T P^T D \int_{t-r}^{t} A(t,s)g(x(s))ds - \big(\int_{t-r}^{t} A(t,s)g(x(s))ds\big)^T DPx \\
= \quad & -x^T P^T D \int_{t-r}^{t} A(t,s)g(x(s))ds - \Big[\big(\int_{t-r}^{t} A(t,s)g(x(s))ds\big)^T DPx\Big]^T \\
= \quad & -2x^T P^T D \int_{t-r}^{t} A(t,s)g(x(s))ds \\
\leq \quad & 2|x^T||P^T D| \int_{t-r}^{t} |A(t,s)||g(x(s))|ds \\
\leq \quad & 2|x^T||P^T D| \int_{t-r}^{t} |A(t,s)||g(x(s))|ds \\
\leq \quad & |P^T D| \int_{t-r}^{t} |A(t,s)|(|x|^2 + |g(x(s))|^2)ds. \qquad (2.14)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
& \big(\int_{t-r}^{t} A(t,s)g(x(s))ds\big)^T DA(t,t)g(x) + g^T(x)A^T(t,t)D\int_{t-r}^{t} A(t,s)g(x(s))ds \\
= \quad & 2g^T(x)A^T(t,t)D\int_{t-r}^{t} A(t,s)g(x(s))ds
\end{aligned}
$$

$$\begin{aligned}
&\le 2\lambda|x||A^T(t,t)D|\int_{t-r}^{t}|A(t,s)||g(x(s))|ds \\
&\le \lambda|A^T(t,t)D|\int_{t-r}^{t}|A(t,s)|\big(|x|^2+|g(x(s))|^2\big)ds. \qquad (2.15)
\end{aligned}$$

By substituting expressions (2.12)-(2.15) into (2.11) yields

$$\begin{aligned}
V'(t) &\le \Big[-\mu_1-2\mu_2+\gamma r\lambda^2|A(t,t)|+\big(\lambda|A^T(t,t)D|+|P^TD|\big)\int_{t-r}^{t}|A(t,s)|ds\Big]|x|^2 \\
&+ \Big[-\gamma+\lambda|A^T(t,t)D|+|P^TD|\Big]\int_{t-r}^{t}|A(t,s)||g(x(s))|^2ds \\
&\le -\alpha|x|^2. \qquad (2.16)
\end{aligned}$$

Let $\varepsilon>0$ be given, we will find $\delta>0$ so that $|x(t,t_0,\phi)|<\varepsilon$ as long as $[\phi\in E_{t_0}\to\mathbb{R}:\|\phi\|<\delta]$. Let

$$L^2=|D|\Big(1+\int_0^{t_0}|A(t_0,s)|ds\Big)^2+\lambda^2\nu\int_{-r}^{0}\int_{t_0+s}^{t_0}|A(t_0,z)|dz\,ds$$

By (2.16) we have V is decreasing and hence for $t\ge t_0\ge 0$ we have that

$$\begin{aligned}
V(t,x) &\le V(t_0,\phi) \\
&\le \big(\phi(t_0)-\int_{t_0-r}^{t_0}A(t_0,s)\phi(s)ds\big)^2+\nu\lambda\int_{-r}^{0}\int_{t_0+s}^{t_0}|A(t_0,z)||\phi(z)|^2dz\,ds \\
&= \delta^2\Big(1+\int_{t_0-r}^{t_0}|A(t_0,s)|\Delta s\Big)^2+\nu\lambda\,\delta^2\int_{-r}^{0}\int_{t_0+s}^{t_0}|A(t_0,z)|dz\,ds \\
&\le \delta^2L^2. \qquad (2.17)
\end{aligned}$$

By (2.10), we have

$$\begin{aligned}
V(t,x) &\ge \Big(x-\int_{t-r}^{t}A(t,s)g(x(s))ds\Big)^TD\Big(x-\int_{t-r}^{t}A(t,s)g(x(s))ds\Big) \\
&\ge k^2\Big(|x|-\big|\int_{t-r}^{t}A(t,s)g(x(s))ds\big|\Big)^2.
\end{aligned}$$

Combining the above two inequalities leads to

$$|x(t)|\le\frac{\delta L}{k}+\lambda\int_{t-r}^{t}|A(t,s)||x(s)|ds.$$

So as long as $|x(t)|<\varepsilon$, we have

$$|x(t)|<\frac{\delta L}{k}+\varepsilon\lambda\int_{t-r}^{t}|A(t,s)|ds,\ \text{for all } t\ge t_0.$$

Thus, we have from the above inequality that
$|x(t)|<\varepsilon$ for $\delta<\frac{k}{L}(1-\lambda\int_{t-r}^{t}|A(t,s)|ds)\varepsilon$. Note that by (2.9), the above inequality regarding δ is valid. □

We have the following corollary.

Corollary 2.2. *Assume all the conditions of Theorem 2.1 hold. Let $x(t)$ be any solution of* (1.1)*. Then satisfies* $|x(t)|^2\in L_{[t_0,\infty)},\ t_0\ge 0$.

Proof. We know from Theorem 2.1 that the zero solution is stable. Thus, for the same δ of stability, we take $x(t,t_0,\phi)| < 1$. Since V is decreasing, we have by integrating (2.16) from t_0 to t and using (2.17) that,

$$V(t,x) \le V(t_0,\phi) \le \delta^2 L^2 - \alpha \int_{t_0}^{t} |x(s)|^2 ds.$$

Since,

$$V(t,x) \ge \Big(x - \int_{t-r}^{t} A(t,s)g(x(s))ds\Big)^T D\Big(x - \int_{t-r}^{t} A(t,s)g(x(s))ds\Big),$$

we have that

$$\Big(x - \int_{t-r}^{t} A(t,s)g(x(s))ds\Big)^T D\Big(x - \int_{t-r}^{t} A(t,s)g(x(s))ds\Big) \tag{2.18}$$
$$\le \quad \delta^2 L^2 - \alpha \int_{t_0}^{t} |x(s)|^2 ds. \tag{2.19}$$

Also, using Schwarz inequality one obtains

$$\begin{aligned}\Big(\int_{t-r}^{t} |A(t,s)||g(x(s))|ds\Big)^2 &= \Big(\int_{t-r}^{t} |A(t,s)|^{1/2}|A(t,s)|^{1/2}|g(x(s))|ds\Big)^2 \\ &\le \lambda^2 \int_{t-r}^{t} |A(t,s)|ds \int_{t-r}^{t} |A(t,s)||x(s)|^2 ds.\end{aligned}$$

As $\int_{t-r}^{t} |A(t,s)|ds$ is bounded by (2.9) and $|x|^2 < 1$, we have $\int_{t-r}^{t} |A(t,s)||x(s)|^2 ds$ is bounded and hence $\int_{t-r}^{t} |A(t,s)||g(x(s))|ds$ is bounded. Therefore, from (2.18), we arrive at

$$\begin{aligned}\alpha \int_{t_0}^{t} |x(s)|^2 ds &\le \delta^2 L^2 - \Big(x - \int_{t-r}^{t} A(t,s)g(x(s))ds\Big)^T D\Big(x - \int_{t-r}^{t} A(t,s)g(x(s))ds\Big) \\ &\le \delta^2 L^2 + |D|\Big(|x| + |\int_{t-r}^{t} A(t,s)g(x(s))ds|\Big) \le K,\end{aligned}$$

from which we deduce that $|x(t)|^2 \in L_{[t_0,\infty)},\ t_0 \ge 0$. □

It is straight forward to extend the results of this paper to (1.2).

Theorem 2.3. *Let* (2.2)- (2.4) *hold, and suppose there are constants* $\gamma > 0$ *and* $\alpha > 0$ *so that*

$$-2\mu_2 + \gamma r \lambda^2 |A(t,t)| + \lambda |A^T(t,t)D| \int_{t-r}^{t} |A(t,s)|ds \le -\alpha, \tag{2.20}$$

$$-\gamma + \lambda |A^T(t,t)D| \le 0, \tag{2.21}$$

and

$$1 - \lambda \int_{t-r}^{t} |A(t,s)|ds > 0 \tag{2.22}$$

then, the zero solution of (1.2) *is stable and* $|x(t)|^2 \in L_{[t_0,\infty)},\ t_0 \ge 0$.

Proof. The proof is immediate consequence of Theorem 2.1 and Corollary 2.1. □

Next, we display an example in which we show that zero solution is stable.

Let $P = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$ and $C(t,s) = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix}$, then

$A(t,s) = \begin{pmatrix} \frac{1}{3}(r-t+s) & 0 \\ 0 & \frac{1}{3}(r-t+s) \end{pmatrix}$ and $A(t,t) = \begin{pmatrix} \frac{1}{3}r & 0 \\ 0 & \frac{1}{3}r \end{pmatrix}$.

From $P^T D + DP = -\mu_1 I$, we obtain

$D = \begin{pmatrix} -\mu_1 & 0 \\ 0 & -\mu_1 \end{pmatrix}$. Let $g(x) = \begin{pmatrix} \frac{3\mu_2}{\mu_1 r}x_1 \\ \frac{3\mu_2}{\mu_1 r}x_2 \end{pmatrix}$. Then $X^T DA(t,t)g(x) = \mu_2\left(x_1^2 + x_2^2\right)$. By letting $\frac{3\mu_2}{|\mu_1|}r \le \lambda < \frac{3}{r^2}$ we have that $|g(x)| \le \lambda\,|x|$. Now $|A(t,s)| \le \left|\frac{1}{3}(r-t+s)\right| \le \frac{r}{3}$ for all $s \in [t-r,t]$. Thus $\frac{\partial |A(t,s)|}{\partial t} \le 0$. Hence the conditions (2.1)- (2.4) are satisfied. We can easily see that condition (2.5) is satisfied for $0 < k < -\mu_1$, $\mu_1 < 0$. Left to verify conditions (2.7)– (2.9).

$\int_{t-r}^{t} |A(t,s)|\, ds \le \int_{t-r}^{t} \frac{r}{3} ds = \frac{r^2}{3}$. Hence from condition (2.9) we will have $\lambda \frac{r^2}{3} < 1$. Let $\gamma > 0$ such that

$$-\gamma + \lambda \frac{|\mu_1|\, r}{3} + \frac{|\mu_1|}{2} \le 0,$$

from which we conclude that condition (2.8) is satisfied. Choose μ_1, μ_2 and r in a such way that

$$-\mu_1 - 2\mu_2 + \gamma\frac{3}{r^2} + \frac{|\mu_1|\, r}{3} + \frac{|\mu_1|}{2} \le -\alpha, \text{ for some } \alpha > 0.$$

Then condition (2.7) holds. Thus we have shown that the zero solution of

$$x'(t) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} x(t) - \int_{t-r}^{t} \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} \frac{3\mu_2}{\mu_1 r}x_1 \\ \frac{3\mu_2}{\mu_1 r}x_2 \end{pmatrix} ds$$

is stable.

Open Problem In light of this work and [9], what can be said about the exponential stability of the zero solution of (1.1)?

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