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A variant form of Korpelevich's algorithm and its convergence analysis

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Abstract

A variant form of Korpelevich's algorithm is presented for solving the generalized variational inequality in Banach spaces. It is shown that the presented algorithm converges strongly to a special solution of the generalized variational inequality.

Keywords: Korpelevich's algorithm, variational inequalities, accretive mappings, Banach spaces.
2010 MSC: 47H05, 47J25.

1. Introduction

Let H be a real Hilbert space and $\emptyset \neq C \subset H$ a closed convex set. Let $A : C \to H$ be a nonlinear mapping. The variational inequality is to find a point $x^\dagger \in C$ such that

$$\langle Ax^\dagger, x^\ddagger - x^\dagger\rangle \geq 0,\ \forall x^\ddagger \in C, \tag{1.1}$$

which was introduced and studied by Stampacchia [9]. Variational inequalities are being used as mathematical programming tools and models to study a wide class of unrelated problems arising in mathematical, physical, regional, engineering, and nonlinear optimization sciences. For example, in [16, 19, 20], the solutions of the variational inequalities are being used as the mathematical programming tools related to some fixed points problems. For some related works, we refer the reader to [2, 5–7, 13, 18]. Especially, Korpelevich

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[8] introduced the following Korpelevich's algorithm to solve (1.1). For given $x_0 \in C$, define a sequence $\{x_n\}$ by the following form

$$\begin{aligned} y_n &= P_C(x_n - \tau A x_n), \\ x_{n+1} &= P_C(x_n - \tau A y_n), \quad n \geq 0, \end{aligned} \tag{1.2}$$

where P_C is the metric projection from $\mathbb{R}^n$ onto its subset C, $\tau \in (0, 1/\kappa)$ and $A : C \to \mathbb{R}^n$ is a monotone operator.

Remark 1.1. Korpelevich's algorithm (1.2) fails, in general, to converge strongly in the setting of infinite-dimensional Hilbert spaces.

In order to obtain the strong convergence, Yao et al. [15] presented the following modified Korpelevich's algorithm. For given $x_0 \in C$, let $\{x_n\}$ be a sequence defined by

$$\begin{aligned} y_n &= P_C[x_n - \tau A x_n - \alpha_n x_n], \\ x_{n+1} &= P_C[x_n - \tau A y_n + \mu(y_n - x_n)], \quad n \geq 0. \end{aligned} \tag{1.3}$$

Consequently, Yao et al. proved that the sequence $\{x_n\}$ generated by (1.3) converges strongly to the solution of (1.1).

In [14, 17], the authors suggested some iterative algorithms for finding the minimum-norm solution of the variational inequalities.

On the other hand, in [1], Aoyama et al. extended the variational inequality (1.1) to the generated variational inequality under the setting of Banach spaces which is to find a point $x^\dagger \in C$ such that

$$\langle A x^\dagger, J(x^\ddagger - x^\dagger)\rangle \geq 0, \ \forall x^\ddagger \in C, \tag{1.4}$$

where C is a nonempty closed convex subset of a real Banach space E. We use $S(C, A)$ to denote the solution set of (1.4).

Note that the generalized variational inequality (1.4) is connected with the fixed point problem for nonlinear mappings. To solve (1.4), Aoyama et al. [12] introduced an iterative algorithm. For given $x_0 \in C$, let $\{x_n\}$ be a sequence defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C[x_n - \lambda_n A x_n], \quad n \geq 0, \tag{1.5}$$

where Q_C is a sunny nonexpansive retraction from E onto C, and $\{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset (0, \infty)$ are two real number sequences. We also note that the sequence $\{x_n\}$ generated by (1.5) has only weak convergence in the setting of infinite-dimensional Banach spaces.

The main purpose of this paper is to solve problem (1.4). Motivated by the above algorithm (1.3), we suggest a variant form of Korpelevich's algorithm by replacing the metric projection with the sunny nonexpansive retraction. It is shown that the presented algorithm converges strongly to a special solution of the variational inequality (1.4).

2. Preliminaries

Let E be a real Banach space and $\emptyset \neq C \subset E$ a closed convex set.

Definition 2.1. A mapping $A : C \to E$ is said to be accretive if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq 0$$

for all $x, y \in C$.

Definition 2.2. A mapping $A : C \to E$ is said to be α-strongly accretive if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C,$$

where $\alpha > 0$ is a positive constant.

Definition 2.3. A mapping A of C into E is said to be α-inverse-strongly accretive if, for $\alpha > 0$,

$$\langle Ax - Ay, j(x-y)\rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in C$.

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \left\| \frac{x+y}{2} \right\| \leq 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space E is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in U$. It is also said to be uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in U$. The norm of E is said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. And we define a function $\rho : [0, \infty) \to [0, \infty)$ called the modulus of smoothness of E as follows:

$$\rho(\tau) = \sup \left\{ \frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau \right\}.$$

It is known that E is uniformly smooth if and only if $\lim_{\tau \to 0} \rho(\tau)/\tau = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space E is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$.

Lemma 2.4 ([11])**.** *Let q be a given real number with $1 < q \leq 2$ and let E be a q-uniformly smooth Banach space. Then*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + 2\|Ky\|^q$$

for all $x, y \in E$, where K is the q-uniformly smoothness constant of E and J_q is the generalized duality mapping from E into 2^{E^*} defined by

$$J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}, \quad \forall x \in E.$$

Let D be a subset of C and let Q be a mapping of C into D. Then Q is said to be *sunny* if

$$Q(Qx + t(x - Qx)) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping Q of C into itself is called a *retraction* if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Qz = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of Q. A subset D of C is called a *sunny nonexpansive* retract of C if there exists a sunny nonexpansive retraction from C onto D. We know the following lemma concerning with the sunny nonexpansive retraction.

Lemma 2.5 ([4])**.** *Let C be a closed convex subset of a smooth Banach space E, let D be a nonempty subset of C and Q a retraction from C onto D. Then Q is sunny and nonexpansive if and only if*

$$\langle u - Qu, j(y - Qu)\rangle \leq 0$$

for all $u \in C$ and $y \in D$.

Lemma 2.6 ([1])**.** *Let C be a nonempty closed convex subset of a smooth Banach space E. Let $Q_C : E \to C$ be a sunny nonexpansive retraction and let $A : C \to E$ be an accretive operator. Then for all $\lambda > 0$,*

$$S(C, A) = F(Q_C(I - \lambda A)),$$

where $S(C, A) = \{x^* \in C : \langle Ax^*, J(x - x^*)\rangle \geq 0, \quad \forall x \in C\}$.

Lemma 2.7 ([10])**.** *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E. Let the mapping $A : C \to E$ be α-inverse-strongly accretive. Then,*

$$\|(I-\lambda A)x-(I-\lambda A)y\|^2 \le \|x-y\|^2+2\lambda(K^2\lambda-\alpha)\|Ax-Ay\|^2.$$

In particular, if $0 \le \lambda \le \frac{\alpha}{K^2}$, then $I-\lambda A$ is nonexpansive.

Lemma 2.8 ([3])**.** *Let E be a uniformly convex Banach space and $\emptyset \neq C \subset E$ be a bounded closed convex set. Let $T : C \to C$ be a nonexpansive mapping. If $\{x_n\}$ is a sequence of C such that $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then x is a fixed point of T.*

Lemma 2.9 ([12])**.** *Let $\{a_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be three real number sequences satisfying*

(i) $\{a_n\} \subset [0,\infty)$, $\{\gamma_n\} \subset (0,1)$, *and* $\sum_{n=0}^{\infty}\gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=0}^{\infty}|\delta_n| < \infty$;
(iii) $a_{n+1} \le (1-\gamma_n)a_n + \delta_n, n \ge 0$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

In this section, we present our algorithm based on Korpelevich's algorithm and consequently, we will show its strong convergence.

In the sequel, we assume that E is a uniformly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping. Let $\emptyset \neq C \subset E$ be a closed convex set. Let $A : C \to E$ be an α-strongly accretive and L-Lipschitz continuous mapping. Let $Q_C : E \to C$ be a sunny nonexpansive retraction.

Algorithm 3.1. For given $x_0 \in C$, define a sequence $\{x_n\}$ iteratively by

$$\begin{aligned} y_n &= Q_C[x_n - \lambda_n A x_n + \alpha_n(u_n - x_n)], \\ x_{n+1} &= Q_C[x_n - \mu_n A y_n + \delta_n(y_n - x_n)], n \ge 0, \end{aligned} \tag{3.1}$$

where $\{u_n\} \subset C$ is a sequence and $\{\lambda_n\} \subset (0,2\alpha)$, $\{\alpha_n\} \subset [0,1]$, $\{\mu_n\}$, and $\{\delta_n\} \subset [0,1]$ are four real number sequences.

Theorem 3.2. *Suppose that $S(C,A) \neq \emptyset$. Assume the following conditions are satisfied:*

(C1): $\lim_{n\to\infty} u_n = u \in C$;
(C2): $\lambda_n \in [a,b] \subset (0, \frac{\alpha}{K^2L^2})$;
(C3): $\frac{\mu_n}{\delta_n} < \frac{\alpha}{K^2L^2}$ $(\forall n \ge 0)$, *where K is the smooth constant of E;*
(C4): $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, *and* $\lim_{n\to\infty}\frac{\alpha_n}{\alpha_{n-1}} = 1$;
(C5): $\lim_{n\to\infty}\frac{\delta_n-\delta_{n-1}}{\alpha_n} = 0$, $\lim_{n\to\infty}\frac{\mu_n-\mu_{n-1}}{\alpha_n} = 0$, *and* $\lim_{n\to\infty}\frac{\lambda_n-\lambda_{n-1}}{\alpha_n} = 0$.

Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $Q'(u)$, where Q' is a sunny nonexpansive retraction of E onto $S(C,A)$.

Proof. Let $p \in S(C,A)$. Since $\lim_{n\to\infty} u_n = u \in C$, we can choose a constant $M > 0$ such that $\|u_n - p\| \le M$ for all $n \ge 0$. First, from Lemma 2.6, we have $p = Q_C[p - \nu Ap]$ for all $\nu > 0$. In particular, $p = Q_C[p - \lambda_n Ap] = Q_C[\alpha_n p + (1-\alpha_n)(p - \frac{\lambda_n}{1-\alpha_n}Ap)]$ for all $n \ge 0$.

Since $A : C \to E$ is α-strongly accretive and L-Lipschitzian, it must be $\frac{\alpha}{L^2}$-inverse-strongly accretive mapping. Thus, by Lemma 2.7, we have

$$\|(I-\lambda_n A)x-(I-\lambda_n A)y\|^2 \le \|x-y\|^2+2\lambda_n\Big(K^2\lambda_n-\frac{\alpha}{L^2}\Big)\|Ax-Ay\|^2.$$

Since $\alpha_n \to 0$ and $\lambda_n \in [a,b] \subset (0, \frac{\alpha}{K^2L^2})$, we get $\alpha_n < 1 - \frac{K^2L^2\lambda_n}{\alpha}$ for enough large n. Without loss of generality, we may assume that, for all $n \in \mathbb{N}$, $\alpha_n < 1 - \frac{K^2L^2\lambda_n}{\alpha}$, i.e., $\frac{\lambda_n}{1-\alpha_n} \in (0, \frac{\alpha}{K^2L^2})$. Hence, $I - \frac{\lambda_n}{1-\alpha_n}A$ is nonexpansive.

From (3.1), we have

$$\begin{aligned}
\|y_n - p\| &= \|Q_C[x_n - \lambda_n Ax_n + \alpha_n(u_n - x_n)] - Q_C[p]\| \\
&= \|Q_C[\alpha_n u_n + (1-\alpha_n)(x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n)] - Q_C[\alpha_n p + (1-\alpha_n)(p - \frac{\lambda_n}{1-\alpha_n}Ap)]\| \\
&\le \|\alpha_n(u_n - p) + (1-\alpha_n)[(x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n) - (p - \frac{\lambda_n}{1-\alpha_n}Ap)]\| \\
&\le \alpha_n\|u_n - p\| + (1-\alpha_n)\|(I - \frac{\lambda_n}{1-\alpha_n}A)x_n - (I - \frac{\lambda_n}{1-\alpha_n}A)p\| \\
&\le \alpha_n\|u_n - p\| + (1-\alpha_n)\|x_n - p\|.
\end{aligned} \tag{3.2}$$

By (3.1) and (3.2), we get

$$\begin{aligned}
\|x_{n+1} - p\| &= \|Q_C[x_n - \mu_n Ay_n + \delta_n(y_n - x_n)] - Q_C[p - \mu_n Ap]\| \\
&= \|Q_C[(1-\delta_n)x_n + \delta_n(y_n - \frac{\mu_n}{\delta_n}Ay_n)] - Q_C[(1-\delta_n)p + \delta_n(p - \frac{\mu_n}{\delta_n}Ap)]\| \\
&\le \|(1-\delta_n)(x_n - p) + \delta_n[(y_n - \frac{\mu_n}{\delta_n}Ay_n) - (p - \frac{\mu_n}{\delta_n}Ap)]\| \\
&\le (1-\delta_n)\|x_n - p\| + \delta_n\|(y_n - \frac{\mu_n}{\delta_n}Ay_n) - (p - \frac{\mu_n}{\delta_n}Ap)\| \\
&\le (1-\delta_n)\|x_n - p\| + \delta_n\|y_n - p\| \\
&\le (1-\delta_n)\|x_n - p\| + \delta_n\alpha_n\|u_n - p\| + \delta_n(1-\alpha_n)\|x_n - p\| \\
&= (1-\delta_n\alpha_n)\|x_n - p\| + \delta_n\alpha_n\|u_n - p\| \\
&\le \max\{\|x_n - p\|, \|u_n - p\|\}.
\end{aligned} \tag{3.3}$$

By the induction, we obtain $\|x_{n+1} - p\| \le \max\{\|x_0 - p\|, M\}$. So, $\{x_n\}$ is bounded. We compute (3.1) to get

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \|Q_C[x_n - \lambda_n Ax_n + \alpha_n(u_n - x_n)] - Q_C[x_{n-1} - \lambda_{n-1}Ax_{n-1} + \alpha_{n-1}(u_{n-1} - x_{n-1})]\| \\
&\le \|(1-\alpha_n)(x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n) - (1-\alpha_{n-1})(x_{n-1} - \frac{\lambda_{n-1}}{1-\alpha_{n-1}}Ax_{n-1}) + \alpha_n u_n - \alpha_{n-1}u_{n-1}\| \\
&= \|(1-\alpha_n)[(x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n) - (x_{n-1} - \frac{\lambda_n}{1-\alpha_n}Ax_{n-1})] + (\alpha_{n-1} - \alpha_n)x_{n-1} \\
&\quad + (\lambda_{n-1} - \lambda_n)Ax_{n-1} + \alpha_n u_n - \alpha_{n-1}u_{n-1}\| \\
&\le (1-\alpha_n)\|(x_n - \frac{\lambda}{1-\alpha_n}Ax_n) - (x_{n-1} - \frac{\lambda_n}{1-\alpha_n}Ax_{n-1})\| \\
&\quad + |\alpha_n - \alpha_{n-1}|(\|x_{n-1}\| + \|u_n\|) + |\lambda_n - \lambda_{n-1}|\|Ax_{n-1}\| + \alpha_{n-1}\|u_n - u_{n-1}\| \\
&\le (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\|x_{n-1}\| + \|u_n\|) \\
&\quad + |\lambda_n - \lambda_{n-1}|\|Ax_{n-1}\| + \alpha_{n-1}\|u_n - u_{n-1}\|,
\end{aligned}$$

and thus

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|Q_C[x_n - \mu_n Ay_n + \delta_n(y_n - x_n)] - Q_C[x_{n-1} - \mu_{n-1}Ay_{n-1} + \delta_{n-1}(y_{n-1} - x_{n-1})]\| \\
&\le \|[x_n - \mu_n Ay_n + \delta_n(y_n - x_n)] - [x_{n-1} - \mu_{n-1}Ay_{n-1} + \delta_{n-1}(y_{n-1} - x_{n-1})]\| \\
&= \|[(1-\delta_n)x_n + \delta_n(y_n - \frac{\mu_n}{\delta_n}Ay_n)] - [(1-\delta_{n-1})x_{n-1} + \delta_{n-1}(y_{n-1} - \frac{\mu_{n-1}}{\delta_{n-1}}Ay_{n-1})]\|
\end{aligned}$$

$$
\begin{aligned}
&\le (1-\delta_n)\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\|x_{n-1}\| \\
&\quad + \delta_n\|(y_n - \frac{\mu_n}{\delta_n}Ay_n) - (y_{n-1} - \frac{\mu_n}{\delta_n}Ay_{n-1})\| + |\mu_n - \mu_{n-1}|\|Ay_{n-1}\| + |\delta_n - \delta_{n-1}|\|y_{n-1}\| \\
&\le (1-\delta_n)\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|(\|x_{n-1}\| + \|y_{n-1}\|) \\
&\quad + \delta_n\|y_n - y_{n-1}\| + |\mu_n - \mu_{n-1}|\|Ay_{n-1}\| \\
&\le (1-\delta_n\alpha_n)\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|(\|x_{n-1}\| + \|y_{n-1}\|) + |\mu_n - \mu_{n-1}|\|Ay_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|\delta_n(\|x_{n-1}\| + \|u_n\|) + \delta_n|\lambda_n - \lambda_{n-1}|\|Ax_{n-1}\| + \alpha_{n-1}\delta_n\|u_n - u_{n-1}\|.
\end{aligned}
$$

This together with conditions (C1), (C4), (C5), and Lemma 2.9 imply that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

From (3.2), we have

$$
\begin{aligned}
\|y_n - p\|^2 &\le \|\alpha_n(u_n - p) + (1-\alpha_n)[(x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n) - (p - \frac{\lambda_n}{1-\alpha_n}Ap)]\|^2 \\
&\le \alpha_n\|u_n - p\|^2 + (1-\alpha_n)\|(x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n) - (p - \frac{\lambda_n}{1-\alpha_n}Ap)\|^2 \\
&\le \alpha_n\|u_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 + 2\lambda_n\Big(\frac{K^2\lambda_n}{1-\alpha_n} - \frac{\alpha}{L^2}\Big)\|Ax_n - Ap\|^2.
\end{aligned}
\tag{3.4}
$$

By (3.1), (3.3), and (3.4), we obtain

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le \|(1-\delta_n)(x_n - p) + \delta_n[(y_n - \frac{\mu_n}{\delta_n}Ay_n) - (p - \frac{\mu_n}{\delta_n}Ap)]\|^2 \\
&\le (1-\delta_n)\|x_n - p\|^2 + \delta_n\|(y_n - \frac{\mu_n}{\delta_n}Ay_n) - (p - \frac{\mu_n}{\delta_n}Ap)\|^2 \\
&\le (1-\delta_n)\|x_n - p\|^2 + \delta_n[\|y_n - p\|^2 + \frac{2\mu_n}{\delta_n}(\frac{K^2\mu_n}{\delta_n} - \frac{\alpha}{L^2})\|Ay_n - Ap\|^2] \\
&\le \delta_n[\alpha_n\|u_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 + 2\lambda_n(\frac{K^2\lambda_n}{1-\alpha_n} - \frac{\alpha}{L^2})\|Ax_n - Ap\|^2] \\
&\quad + (1-\delta_n)\|x_n - p\|^2 + 2\mu_n(\frac{K^2\mu_n}{\delta_n} - \frac{\alpha}{L^2})\|Ay_n - Ap\|^2 \\
&= \alpha_n\delta_n\|u_n - p\|^2 + (1-\delta_n\alpha_n)\|x_n - p\|^2 + 2\lambda_n\delta_n(\frac{K^2\lambda_n}{1-\alpha_n} - \frac{\alpha}{L^2})\|Ax_n - Ap\|^2 \\
&\quad + 2\mu_n(\frac{K^2\mu_n}{\delta_n} - \frac{\alpha}{L^2})\|Ay_n - Ap\|^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
0 &\le -2\lambda_n\delta_n(\frac{K^2\lambda_n}{1-\alpha_n} - \frac{\alpha}{L^2})\|Ax_n - Ap\|^2 - 2\mu_n(\frac{K^2\mu_n}{\delta_n} - \frac{\alpha}{L^2})\|Ay_n - Ap\|^2 \\
&\le \alpha_n\delta_n\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&= \alpha_n\delta_n\|u_n - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)(\|x_n - p\| - \|x_{n+1} - p\|) \\
&\le \alpha_n\delta_n\|u_n - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\|.
\end{aligned}
$$

Since $\alpha_n \to 0$ and $\|x_n - x_{n+1}\| \to 0$, we derive

$$\lim_{n\to\infty}\|Ax_n - Ap\| = \lim_{n\to\infty}\|Ay_n - Ap\| = 0.$$

It follows that

$$\lim_{n\to\infty}\|Ay_n - Ax_n\| = 0.$$

Noting that A is α-strongly accretive, we deduce

$$\|Ay_n - Ax_n\| \geq \alpha\|y_n - x_n\|,$$

which implies that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0,$$

that is,

$$\lim_{n\to\infty} \|Q_C[x_n - \lambda_n Ax_n + \alpha_n(u_n - x_n)] - x_n\| = 0.$$

It follows that

$$\lim_{n\to\infty} \|Q_C[x_n - \lambda_n Ax_n] - x_n\| = 0.$$

Next, we show that

$$\limsup_{n\to\infty}\langle Q'(u), j(x_n - Q'(u))\rangle \geq 0. \tag{3.5}$$

To prove (3.5), since $\{x_n\}$ is bounded, we can choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to z and

$$\limsup_{n\to\infty}\langle Q'(u), j(x_n - Q'(u))\rangle = \lim_{i\to\infty}\langle Q'(u), j(x_{n_i} - Q'(u))\rangle. \tag{3.6}$$

Next, we first prove $z \in S(C, A)$. Since λ_{n_i} is bounded, there exists a subsequence $\lambda_{n_{i_j}}$ such that $\lambda_{n_{i_j}} \to \tilde{\lambda}$. It follows that

$$\lim_{j\to\infty} \|Q_C(I - \lambda_{n_{i_j}} A)x_{n_{i_j}} - x_{n_{i_j}}\| = 0. \tag{3.7}$$

By Lemma 2.8 and (3.7), we have $z \in F(Q_C(I - \tilde{\lambda}A))$, it follows from Lemma 2.6 that $z \in S(C, A)$.

Now, from (3.6) and Lemma 2.5, we have

$$\limsup_{n\to\infty}\langle u - Q'(u), j(x_n - Q'(u))\rangle = \lim_{j\to\infty}\langle u - Q'(u), j(x_{n_{i_j}} - Q'(u))\rangle = \langle u - Q'(u), j(z - Q'(u))\rangle \leq 0.$$

Noticing that $\|x_n - y_n\| \to 0$, we deduce

$$\limsup_{n\to\infty}\langle u - Q'(u), j(y_n - Q'(u))\rangle \leq 0.$$

Since $u_n \to u$, we get

$$\limsup_{n\to\infty}\langle u_n - Q'(u), j(y_n - Q'(u))\rangle \leq 0.$$

Using Lemma 2.5, we obtain

$$\langle Q_C[\alpha_n u_n + (1-\alpha_n)(x_n - \frac{\lambda}{1-\alpha_n}Ax_n)] - [\alpha_n u_n + (1-\alpha_n)(x_n - \frac{\lambda}{1-\alpha_n}Ax_n)], j(y_n - Q'(u))\rangle \leq 0$$

and

$$\begin{aligned}\langle [\alpha_n Q'(u) + (1-\alpha_n)(Q'(u) - \frac{\lambda_n}{1-\alpha_n}AQ'(u))] - Q_C[\alpha_n Q'(u) + (1-\alpha_n)(Q'(u) \\ - \frac{\lambda_n}{1-\alpha_n}AQ'(u))], j(y_n - Q'(u))\rangle \leq 0.\end{aligned}$$

So,

$$\begin{aligned}\|y_n - Q'(u)\|^2 = \|Q_C[\alpha_n u_n + (1-\alpha_n)(x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n)] \\ - Q_C[\alpha_n Q'(u) + (1-\alpha_n)(Q'(u) - \frac{\lambda_n}{1-\alpha_n}AQ'(u))]\|^2\end{aligned}$$

$$
\begin{aligned}
&\leq \langle \alpha_n(u_n - Q'(u)) + (1-\alpha_n)[(x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n) - (Q'(u) - \frac{\lambda_n}{1-\alpha_n}AQ'(u))], j(y_n - Q'(u))\rangle \\
&\leq \alpha_n\langle u_n - Q'(u), j(y_n - Q'(u))\rangle + (1-\alpha_n)\|(x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n) \\
&\quad - (Q'(u) - \frac{\lambda_n}{1-\alpha_n}AQ'(u))\|\|y_n - Q'(u)\| \\
&\leq \alpha_n\langle u_n - Q'(u), j(y_n - Q'(u))\rangle + (1-\alpha_n)\|x_n - Q'(u)\|\|y_n - Q'(u)\| \\
&\leq \alpha_n\langle u_n - Q'(u), j(y_n - Q'(u))\rangle + \frac{1-\alpha_n}{2}(\|x_n - Q'(u)\|^2 + \|y_n - Q'(u)\|^2),
\end{aligned}
$$

which implies that

$$
\|y_n - Q'(u)\|^2 \leq (1-\alpha_n)\|x_n - Q'(u)\|^2 + 2\alpha_n\langle u_n - Q'(u), j(y_n - Q'(u))\rangle. \tag{3.8}
$$

Finally, we prove that the sequence $x_n \to Q'(u)$. As a matter of fact, from (3.1) and (3.8), we have

$$
\begin{aligned}
\|x_{n+1} - Q'(u)\|^2 &\leq (1-\delta_n)\|x_n - Q'(u)\|^2 + \delta_n\|y_n - Q'(u)\|^2 \\
&\leq (1-\delta_n\alpha_n)\|x_n - Q'(u)\|^2 + 2\delta_n\alpha_n\langle u_n - Q'(u), j(y_n - Q'(u))\rangle.
\end{aligned}
$$

Applying Lemma 2.9 to the last inequality, we conclude that x_n converges strongly to $Q'(u)$. This completes the proof. □

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