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The general solution of a quadratic functional equation and Ulam stability

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Abstract

In this paper, we investigate the general solution of a new quadratic functional equation. We prove that a function admits, in appropriate conditions, a unique quadratic mapping satisfying the corresponding functional equation. Finally, we discuss the Ulam stability of that functional equation by using the directed method and fixed point method, respectively.

Keywords: functional equation, Ulam stability, quadratic mapping.
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1. Introduction and Preliminaries

In 1940, an important talk presented by S. M. Ulam has led to intense work on the stability problem of functional equations [21]. Ulam posed the problem, in short, ”Give condition in order for a linear mapping near an approximately linear mapping to exist.” In the following year, Hyers gave an partial answer to the problem [6]. Since then, various generalizations of Ulam's problem and Hyers' theorem have been extensively studied and many elegant results have been obtained [1, 18, 14, 19, 13, 15, 9, 11, 2]. The theory of nonlinear analysis has become a fast developing field during the past decades. Functional equations have substantially grown to become an important branch of this field. In [7], the authors deal with a comprehensive illustration of the stability of functional equations, and then, the further research has been presented [3]. Very recently, most classical results on the Hyers-Ulam-Rassias stability have been offered in an integrated and self-contained version in [8]. It is worth noting that among the stability problem of functional equations, the study of the Ulam stability of different types of quadratic functional equations is an important and interesting topic, and it has attracted many scholars [20, 12, 10, 4, 16, 17].

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The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y),$$

is called a *quadratic functional equation.* Every solution of the quadratic functional equation is a *quadratic mapping.* A mapping $B : X \times X \to Y$ is called *biadditive* if

$$B(x_1+x_2,x_3)=B(x_1,x_3)+B(x_2,x_3),$$

and

$$B(x_1,x_2+x_3)=B(x_1,x_2)+B(x_1,x_3)$$

for all $x_1,x_2,x_3 \in X$. If $B(x_1,x_2)=B(x_2,x_1)$, then we say that B is *symmetric.*

Now we recall a fundamental result in fixed point theory.

Theorem 1.1 ([5]). *Let (X,d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each $x \in X$, either*

$$d(J^n x, J^{n+1}x)=\infty$$

for all $n \geq 0$ or, there exists an n_0 such that

(i) $d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$.

(ii) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;

(iii) y^* is the unique fixed point of J in the set $Y=\{y \in X \mid d(J^{n_0}x,y)<\infty\}$;

(iv) $d(y,y*) \leq \frac{1}{1-L}d(y,Jy)$ for all $y \in Y$.

In this paper, we introduce a new functional equation:

$$f(x+y-z)+f(x+z-y)+f(y+z-x)=f(x-y)+f(x-z)+f(z-y)+f(x)+f(y)+f(z). \quad (1.1)$$

The aim of this paper is to discuss the general solution and then establish the Ulam stability of (1.1). More precisely, we discuss the Ulam stability of (1.1) by applying the direct method and the fixed point method, respectively.

Throughout this paper, let X and Y be a real vector space and a Banach space, respectively.

2. General solution of Eq.(1.1)

In this section, we discuss the general solution of (1.1) in a real vector space.

Lemma 2.1. *Let $f : X \to Y$ be a mapping. If f satisfies* (1.1) *for all $x,y,z \in X$, then f is a quadratic mapping.*

Proof. Let $x=y=z=0$ in (1.1), we obtain $f(0)=0$. Plug $x=y, z=0$ in (1.1), we get

$$f(2x)=3f(x)+f(-x) \quad (2.1)$$

Plug $y=2x, z=x$ in (1.1), we get

$$f(2x)=2f(x)+2f(-x) \quad (2.2)$$

Subtracting (2.2) from (2.1) gives $f(x)=f(-x)$, which means f is an even mapping. Thus by (2.1), we have $f(2x)=4f(x)$.

Plug $z=0$ in (1.1), since f is even, we have

$$f(x+y)+f(x-y)=2f(x)+2f(y).$$

Hence, f is a quadratic mapping. □

Theorem 2.2. *Let $f : X \to Y$ be a mapping. Then f satisfy* (1.1) *if and only if there exists a symmetric biadditive mapping $B : X \times X \to Y$ such that, $f(x) = B(x, x)$, for all $x \in X$.*

Proof. ($\Leftarrow$) Assume there exists a symmetric biadditive mapping $B : X \times X \to Y$ such that $f(x) = B(x, x)$ for all $x \in X$. Now we show that B satisfies (1.1). Indeed,

$$\begin{aligned} B(x+y-z, x+y-z) &= B(x, x+y-z) + B(y, x+y-z) - B(z, x+y-z) \\ &= B(x, x) + B(y, y) + B(z, z) + 2B(x, y) - 2B(y, z) - 2B(x, z), \end{aligned}$$

$$\begin{aligned} B(x+z-y, x+z-y) &= B(x, x+z-y) + B(z, x+z-y) - B(y, x+z-y) \\ &= B(x, x) + B(y, y) + B(z, z) - 2B(x, y) - 2B(y, z) + 2B(x, z), \end{aligned}$$

$$\begin{aligned} B(y+z-x, y+z-x) &= B(y, y+z-x) + B(z, y+z-x) - B(x, y+z-x) \\ &= B(x, x) + B(y, y) + B(z, z) - 2B(x, y) + 2B(y, z) - 2B(x, z), \end{aligned}$$

$$\begin{aligned} B(x-y, x-y) &= B(x, x-y) - B(y, x-y) \\ &= B(x, x) + B(y, y) - 2B(x, y), \end{aligned}$$

$$\begin{aligned} B(x-z, x-z) &= B(x, x-z) - B(z, x-z) \\ &= B(x, x) + B(z, z) - 2B(x, z), \end{aligned}$$

$$\begin{aligned} B(z-y, z-y) &= B(z, z-y) - B(y, z-y) \\ &= B(y, y) + B(z, z) - 2B(y, z), \end{aligned}$$

Hence,

$$\begin{aligned} &B(x+y-z, x+y-z) + B(x+z-y, x+z-y) + B(y+z-x, y+z-x) = \\ &\qquad B(x-y, x-y) + B(x-z, x-z) + B(z-y, z-y) + B(x, x) + B(y, y) + B(z, z), \end{aligned}$$

which implies that B and thus f satisfy (1.1).

($\Rightarrow$) Let

$$f_e(x) = \frac{f(x) + f(-x)}{2}, \ f_o(x) = \frac{f(x) - f(-x)}{2}$$

for all $x \in X$. Then $f(x) = f_e(x) + f_o(x)$. Set $B(x, x) = f_e(x)$. Since f satisfies(1.1), it follows from Lemma 2.1 that f is even and a quadratic mapping. Therefore, we obtain a symmetric biadditive mapping B such that $f(x) = B(x, x)$. $\square$

3. Stability of Eq.(1.1) with direct method

In this section, we study the Ulam stability of (1.1) by employing the direct method. Define

$$\begin{aligned} D_q f(x, y, z) &= f(x+y-z) + f(x+z-y) + f(y+z-x) \\ &\quad - f(x-y) - f(x-z) - f(z-y) - f(x) - f(y) - f(z) \end{aligned} \tag{3.1}$$

Theorem 3.1. *Let $\varphi : X^3 \to [0, +\infty)$ be a function such that*

$$\Phi(x, y, z) = \sum_{k=0}^{\infty} 4^k \varphi(2^{-k}x, 2^{-k}y, 2^{-k}z) < \infty \tag{3.2}$$

for all $x, y, z \in X$. Assume that $f : X \to Y$ is a mapping satisfying

$$\| D_q f(x, y, z) \| \leq \varphi(x, y, z) \tag{3.3}$$

for all $x, y, z \in X$. Then, $Q(x) = \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$, exists for each $x \in X$ and defines a unique quadratic *mapping $Q : X \to Y$ such that*

$$\| f(x) - Q(x) \| \leq \Phi(\frac{x}{2}, \frac{x}{2}, 0) \tag{3.4}$$

for all $x \in X$.

Proof. Plug $x = y = z = 0$ in (3.3). Since $\Phi(0,0,0) = \sum_{k=0}^{\infty} 4^k \varphi(0,0,0) < \infty$ implies that $\varphi(0,0,0) = 0$, we get $f(0) = 0$, and then, plug $x = y, z = 0$ in (3.3), it follows that

$$\| f(2x) - 4f(x) \| \leq \varphi(x, x, 0) \tag{3.5}$$

for all $x \in X$. Replacing x by $\frac{x}{2}$ in (3.5), we obtain

$$\| f(x) - 4f(\frac{x}{2}) \| \leq \varphi(\frac{x}{2}, \frac{x}{2}, 0) \tag{3.6}$$

Replacing x by $\frac{x}{2^{n-1}}$ and multiplying both sides by 4^{n-1} in (3.6), we have

$$\| 4^{n-1} f(\frac{x}{2^{n-1}}) - 4^n f(\frac{x}{2^n}) \| \leq 4^{n-1} \varphi(\frac{x}{2^n}, \frac{x}{2^n}, 0) \tag{3.7}$$

for all $x \in X$ and $n \in \mathbb{N}$. Consequently (3.6) and (3.7) together give

$$\| f(x) - 4^n f(\frac{x}{2^n}) \| \leq \sum_{i=0}^{n-1} 4^i \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, 0) \tag{3.8}$$

for all $x \in X$ and any positive integer n. Hence, for any $k \in \mathbb{N}$, we have

$$\begin{aligned} \| 4^k f(\frac{x}{2^k}) - 4^{k+n} f(\frac{x}{2^{k+n}}) \| &= 4^k \| f(\frac{x}{2^k}) - 4^n f(\frac{x}{2^{k+n}}) \| \\ &\leq 4^k \sum_{i=0}^{n-1} 4^i \varphi(\frac{x}{2^{k+i+1}}, \frac{x}{2^{k+i+1}}, 0) \\ &= \frac{1}{4} \sum_{i=0}^{n-1} 4^{k+i+1} \varphi(\frac{x}{2^{k+i+1}}, \frac{x}{2^{k+i+1}}, 0). \end{aligned} \tag{3.9}$$

By condition (3.2) we obtain $\lim_{k\to\infty} \sum_{i=0}^{n-1} 4^{k+i+1} \varphi(\frac{x}{2^{k+i+1}}, \frac{x}{2^{k+i+1}}, 0) = 0$. Therefore, the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in Banach space Y. Thus one can set

$$Q(x) = \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$.

We want now to prove that Q is a solution of (1.1). Replacing x, y, z by $\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}$, in (3.3), respectively, and, multiplying both sides by 4^n, we get

$$\begin{aligned} 4^n \| f(\frac{x+y-z}{2^n}) \quad &+ f(\tfrac{x+z-y}{2^n}) + f(\tfrac{y+z-x}{2^n}) - f(\tfrac{x-y}{2^n}) - f(\tfrac{x-z}{2^n}) - f(\tfrac{z-y}{2^n}) - f(\tfrac{x}{2^n}) - f(\tfrac{y}{2^n}) - f(\tfrac{z}{2^n}) \| \\ &\leq 4^n \varphi(\tfrac{x}{2^n}, \tfrac{y}{2^n}, \tfrac{z}{2^n}). \end{aligned}$$

Since $\lim_{n\to\infty} 4^n \varphi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}) = 0$, the function Q satisfies (1.1). From (3.8) we obtain

$$\lim_{n\to\infty} \| f(x) - 4^n f(\frac{x}{2^n}) \| \leq \lim_{n\to\infty} \sum_{i=0}^{n-1} 4^i \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, 0),$$

or equivalently,

$$\| f(x) - Q(x) \| \leq \Phi(\frac{x}{2}, \frac{x}{2}, 0) \tag{3.10}$$

for all $x \in X$.Thus we have obtained (3.4).

To complete the proof, it remains to show the uniqueness of Q. Assume that there exists another one, denoted by $R : X \to Y$ such that $Q(x) \not\equiv R(x)$. Then

$$\begin{aligned} \| Q(x) - R(x) \| &= 4^n \| Q(\frac{x}{2^n}) - R(\frac{x}{2^n}) \| \\ &\leq 4^n(\| Q(\frac{x}{2^n}) - f(\frac{x}{2^n}) \| + \| f(\frac{x}{2^n}) - R(\frac{x}{2^n}) \|) \\ &\leq \frac{1}{8} 4^{n+1} \Phi(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}, 0). \end{aligned} \tag{3.11}$$

Since $\lim_{n\to\infty} 4^{n+1}\Phi(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}, 0) = 0$, we have $Q(x) \equiv R(x)$ for all $x \in X$. □

Corollary 3.2. *Let X be a real normed space, and let $p > 2$, $\theta > 0$. Assume $f : X \to Y$ is a mapping satisfying*

$$\| D_q f(x, y, z) \| \leq \theta(\| x \|^p + \| y \|^p + \| z \|^p)$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that both (1.1) *and*

$$\| f(x) - Q(x) \| \leq \frac{\theta}{2^{p-1} - 2} \| x \|^p$$

hold for all $x \in X$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi(x, y, z) = \theta(\| x \|^p + \| y \|^p + \| z \|^p)$$

for all $x, y, z \in X$. □

Theorem 3.3. *Let $\Psi : X^3 \to [0, +\infty)$ be a function such that*

$$\Psi(x, y, z) = \sum_{k=0}^{\infty} \frac{1}{4^k} \psi(2^k x, 2^k y, 2^k z) < \infty \tag{3.12}$$

for all $x, y, z \in X$. Assume that $f : X \to Y$ is a mapping with $f(0) = 0$ and satisfies

$$\| D_q f(x, y, z) \| \leq \psi(x, y, z) \tag{3.13}$$

for all $x, y, z \in X$. Then

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$$

exists for each $x \in X$ and defines a unique quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leq \frac{1}{4} \Psi(x, x, 0) \tag{3.14}$$

for all $x \in X$.

Proof. Plug $x = y, z = 0$ in (3.13), and, as $f(0) = 0$, we get

$$\| f(2x) - 4f(x) \| \leq \psi(x, x, 0)$$

or equivalently,

$$\| \frac{1}{4} f(2x) - f(x) \| \leq \frac{1}{4} \psi(x, x, 0) \tag{3.15}$$

for all $x \in X$.

Replacing x by $2^{n-1}x$ in and multiplying both sides by $\frac{1}{4^{n-1}}$ (3.15), we have

$$\| \frac{1}{4^n} f(2^n x) - \frac{1}{4^{n-1}} f(2^{n-1}x) \| \leq \frac{1}{4^n}\psi(2^{n-1}x, 2^{n-1}x, 0) \tag{3.16}$$

for all $x \in X$ and $n \in \mathbb{N}$. Thus it follows from (3.15) and (3.16) that

$$\begin{aligned} \| \frac{1}{4^n} f(2^n x) - f(x) \| &\leq \sum_{i=1}^{n} \frac{1}{4^i}\psi(2^{i-1}x, 2^{i-1}x, 0) \\ &= \frac{1}{4}\sum_{i=0}^{n-1} \frac{1}{4^i}\psi(2^i x, 2^i x, 0) \end{aligned} \tag{3.17}$$

for all $x \in X$ and any positive integer n.

We now prove the sequence $\{\frac{f(2^n x)}{4^n}\}$ is a Cauchy sequence. For any $k \in \mathbb{N}$, by (3.17), we have

$$\begin{aligned} \| \frac{1}{4^{n+k}} f(2^{n+k}x) - \frac{1}{4^k} f(2^k x) \| &= \frac{1}{4^k} \| \frac{1}{4^n} f(2^{n+k}x) - f(2^k x) \| \\ &\leq \frac{1}{4^{k+1}} \sum_{i=0}^{n-1} \frac{1}{4^i}\psi(2^{i+k}x, 2^{i+k}x, 0) \\ &= \frac{1}{4}\sum_{i=0}^{n-1} \frac{1}{4^{i+k}}\psi(2^{i+k}x, 2^{i+k}x, 0). \end{aligned} \tag{3.18}$$

It follows from (3.12) that $\lim_{k\to\infty} \frac{1}{4^k}\psi(2^k x, 2^k y, 2^k z) = 0$, and, the last expression of (3.18) tends to zero as $k \to \infty$. Consequently, the sequence $\{\frac{f(2^n x)}{4^n}\}$ is Cauchy and hence converges, since the completeness of Y. Thus we define

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$$

for all $x \in X$.

We want now to prove that Q satisfies 1.1. Replacing x, y, z by $2^n x, 2^n y, 2^n z$, in (3.13), respectively, and, dividing both side by 4^n, we get

$$\begin{aligned} &\frac{1}{4^n} \| f[2^n(x+y-z)] + f[2^n(x+z-y)] + f[2^n(y+z-x)] - f[2^n(x-y)] \\ &- f[2^n(x-z)] - f[2^n(z-y)] - f(2^n x) - f(2^n y) - f(2^n z) \| \leq \frac{1}{4^n}\psi(2^n x, 2^n y, 2^n z). \end{aligned}$$

Since $\lim_{n\to\infty} 4^n \frac{1}{4^n}\psi(2^n x, 2^n y, 2^n z) = 0$, the function Q is a solution of 1.1.

From (3.17) we have

$$\lim_{n\to\infty} \| \frac{1}{4^n} f(2^n x) - f(x) \| \leq \lim_{n\to\infty} \frac{1}{4}\sum_{i=0}^{n-1} \frac{1}{4^i}\psi(2^i x, 2^i x, 0),$$

that is,

$$\| Q(x) - f(x) \| \leq \frac{1}{4}\Psi(x, x, 0)$$

for all $x \in X$. Thus (3.14) holds.

Finally, we need to show that Q is unique. Suppose $Q' : X \to Y$ is another different solution of (1.1). Thus

$$\begin{aligned} \| Q(x) - Q'(x) \| &= \frac{1}{4^n} \| Q(2^n x) - Q'(2^n x) \| \\ &\leq \frac{1}{4^n}(\| Q(2^n x) - f(2^n x) \| + \| f(2^n x) - Q'(2^n x) \|) \leq \frac{1}{2}\frac{1}{4^n}\Psi(2^n x, 2^n x, 0). \end{aligned} \tag{3.19}$$

Since $\lim_{n\to\infty} \frac{1}{4^n}\Psi(2^n x, 2^n x, 0) = 0$, we have $Q(x) \equiv Q'(x)$ for all $x \in X$. $\square$

Corollary 3.4. *Let X be a real normed space, and let $0 < p < 2$, $\theta > 0$. Assume $f : X \to Y$ is a mapping with $f(0) = 0$ satisfying*

$$\| D_q f(x,y,z) \| \leq \theta(\| x \|^p + \| y \|^p + \| z \|^p)$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that both (1.1) and

$$\| f(x) - Q(x) \| \leq \frac{2\theta}{1 - 2^{p-2}} \| x \|^p$$

hold for all $x \in X$.

Proof. The proof follows from Theorem 3.3 by taking

$$\varphi(x,y,z) = \theta(\| x \|^p + \| y \|^p + \| z \|^p)$$

for all $x, y, z \in X$. $\square$

4. Stability of Eq.(1.1)with fixed point method

Using the fixed point method, the Ulam stability of (1.1) have been investigated in this section.

Theorem 4.1. *Let $\varphi : X^3 \to [0, +\infty)$ be a function with Lipschitz constant $L < 1$ such that*

$$\varphi(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}) \leq \frac{L}{4}\varphi(x,y,z) \tag{4.1}$$

for all $x, y, z \in X$. Assume that $f : X \to Y$ is a mapping satisfying

$$\| D_q f(x,y,z) \| \leq \varphi(x,y,z) \tag{4.2}$$

for all $x, y, z \in X$. Then

$$Q(x) = \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

exists for each $x \in X$ and defines a unique quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leq \frac{L}{4(1-L)}\varphi(x,x,0) \tag{4.3}$$

for all $x \in X$.

Proof. Set $S = \{g \mid g : X \to Y, g(0) = 0\}$. Define $d(g_1, g_2) = \inf\{C > 0 \mid \| g_1(x) - g_2(x) \| \leq C\varphi(x,x,0)\}$ for all $x \in X$, where $\inf \emptyset = +\infty$.

Claim that (S, d) is complete. Suppose $\{g_n\}$ is a Cauchy sequence in (S, d), then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m > N$, we have $d(g_n, g_m) < \varepsilon$, thus for each $x \in X$, we get

$$\| g_n(x) - g_m(x) \| \leq \varepsilon\varphi(x,x,0). \tag{4.4}$$

Fix $x_0 \in X$, we obtain $\{g_n(x_0)\}$ converges, which means that for each $x \in X$, $\{g_n(x)\}$ converges. Set

$$\lim_{n\to\infty} g_n(x) = g(x),$$

where $g : X \to Y$.

We want now to prove that $\{g_n\}$ converges to g in (S, d). Note that

$$\| g_n(x) - g(x) \| = \lim_{m\to\infty} \| g_n(x) - g_m(x) \| \leq \varepsilon\varphi(x,x,0) \quad (n > N)$$

for all $x \in X$. Therefore, $\{g_n\}$ uniformly converges to g and $g \in S$. Hence (S, d) is complete.

Consider a linear mapping $T : S \to S$ with

$$Tg(x) = 4g(\frac{x}{2})$$

for all $x \in X$. Let $g_1, g_2 \in S$ such that $d(g_1, g_2) = \varepsilon$. Then

$$\begin{aligned} \| Tg_1(x) - Tg_2(x) \| &= 4 \| g_1(\frac{x}{2}) - g_2(\frac{x}{2}) \| \\ &\le 4\varepsilon\varphi(\frac{x}{2}, \frac{x}{2}, 0) \\ &\le L\varepsilon\varphi(x, x, 0), \end{aligned}$$

which implies that $d(Tg_1, Tg_2) \le L\varepsilon = Ld(g_1, g_2)$. Note that from (3.6) and (4.1), we have

$$\begin{aligned} \| f(x) - Tf(x) \| &= \| f(x) - 4f(\frac{x}{2}) \| \\ &\le \varphi(\frac{x}{2}, \frac{x}{2}, 0) \\ &\le \frac{L}{4}\varphi(x, x, 0). \end{aligned}$$

Therefore, $d(f, Tf) \le \frac{L}{4}$. By Theorem 1.1 , there exists a mapping $Q : X \to Y$ satisfying the following:

(1) There exists a unique fixed point Q of T, i.e., $Q(x) = 4Q(\frac{x}{2})$.

(2) $d(T^n f, f) \to 0$ as $n \to \infty$. Thus we define $Q(x) = \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ for all $x \in X$.

(3) $d(f, Q) \le \frac{1}{1-L} d(f, Tf)$. Since $d(f, Tf) \le \frac{L}{4}$, we get $d(f, Q) \le \frac{L}{4(1-L)}$ and (4.3) holds.

Finally, we want to prove that Q is a quadratic mapping. Replacing x, y, z by $\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}$ in (4.2), respectively.

$$\begin{aligned} &\| f(\tfrac{x+y-z}{2^n}) + f(\tfrac{x+z-y}{2^n}) + f(\tfrac{y+z-x}{2^n}) - f(\tfrac{x-y}{2^n}) - f(\tfrac{x-z}{2^n}) - f(\tfrac{z-y}{2^n}) - f(\tfrac{x}{2^n}) - f(\tfrac{y}{2^n}) - f(\tfrac{z}{2^n}) \| \\ &\le \varphi(\tfrac{x}{2^n}, \tfrac{y}{2^n}, \tfrac{z}{2^n}). \end{aligned}$$

Then

$$\begin{aligned} &4^n \| f(\tfrac{x+y-z}{2^n}) + f(\tfrac{x+z-y}{2^n}) + f(\tfrac{y+z-x}{2^n}) - f(\tfrac{x-y}{2^n}) - f(\tfrac{x-z}{2^n}) - f(\tfrac{z-y}{2^n}) - f(\tfrac{x}{2^n}) - f(\tfrac{y}{2^n}) - f(\tfrac{z}{2^n}) \| \\ &\le 4^n\varphi(\tfrac{x}{2^n}, \tfrac{y}{2^n}, \tfrac{z}{2^n}). \end{aligned}$$

Since $\lim_{n\to\infty} 4^n\varphi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}) = 0$, the function Q satisfies (1.1) and hence is quadratic. This completes the proof.

□

Theorem 4.2. *Let $\varphi : X^3 \to [0, +\infty)$ be a function with Lipschitz constant $L < 1$ such that*

$$\psi(x, y, z) \le 4L\psi(\frac{x}{2}, \frac{y}{2}, \frac{z}{2})$$

for all $x, y, z \in X$. Assume that $f : X \to Y$ is a mapping satisfying

$$\| D_q f(x, y, z) \| \le \psi(x, y, z)$$

for all $x, y, z \in X$. Then

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$$

exists for each $x \in X$ and defines a unique quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \le \frac{L}{4(1-L)}\psi(x, x, 0)$$

for all $x \in X$.

Proof. Define

$$S = \{g \mid g : X \to Y, g(0) = 0\},$$

and introduce

$$d(g_1, g_2) = \inf\{C > 0 \mid \| g_1(x) - g_2(x) \| \leq C\psi(x, x, 0)\}$$

for all $x \in X$, where $\inf \emptyset = +\infty$.

Consider a linear mapping $T : S \to S$ with

$$Tg(x) = \frac{1}{4} g(2x)$$

for all $x \in X$.

The rest of proof is similar to the proof of Theorem 4.1.

□

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