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Quadruple fixed point theorems under (φ, ψ)-contractive conditions in partially ordered G-metric spaces with mixed g-monotone property

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Abstract

In this paper, we prove some quadruple coincidence and quadruple fixed point theorems for (φ, ψ)-contractive type mappings in partially ordered G-metric spaces with mixed g-monotone property. The results on fixed point theorems are generalizations of some results obtained by Mustafa [Z. Mustafa, Fixed Point Theory Appl., **2012** (2012), 22 pages]. We also give an example to support our results.

Keywords: partially ordered set, quadruple coincidence, quadruple fixed point, G-metric space, mixed g-monotone property.
2010 MSC: 47H10, 54H25.

1. Introduction and Preliminaries

Fixed point theory is one of the most powerful and fruitful tools in nonlinear analysis, differential equation, and economic theory and has been studied in many various metric spaces. Especially, in 2006, Mustafa and Sims [13] introduced a generalized metric spaces which are called G-metric space. Follow Mustafa and Sims' work, many authors developed and introduced various fixed point theorems in G-metric spaces (see [2,3,14,15,17,20]). Some authors have been interested in partially ordered G-metric spaces and prove some fixed point theorem. Simultaneously, fixed point theory has developed rapidly in partially ordered G-metric spaces (see [1,4,11,19]). In [5], the authors first introduced the concepts of mixed monotone property and quadruple fixed point for $F : X^4 \to X$ and several quadruple fixed point theorems have been

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proved in partially ordered metric spaces. Afterwards, a quadruple fixed point in partially ordered metric spaces is developed and related fixed points are obtained (see [6,7,8,9,10,16]). In [16], the authors first introduced the concepts of g-mixed monotone property and quadruple coincidence point for $F : X^4 \to X$ and $g : X \to X$ and several quadruple coincidence point theorems have been proved in partially ordered metric spaces. Then, in [18], Mustufa proved quadruple coincidence point in partially ordered G-metric spaces using $(\phi - \psi)$ contractions. In [12], Liu first proved quadruple coincidence point in partially ordered G-metric spaces with mixed g-monotone property.

Inspired by [2], in this paper, we prove some quadruple fixed point theorems for (φ, ψ)-contractive type mappings in partially ordered G-metric spaces with mixed g-monotone property. The results on fixed point theorems are generalizations of the results of Mustafa [18]. We also give an example to support our results.

Throughout this paper, let $\mathbf{N}$ denote the set of nonnegative integers, and $\mathbf{R}^+$ be the set of positive real numbers.

Before giving our main results, we need to recall some basic concepts and results in G-metric spaces.

Definition 1.1. ([13]) Let X be a non-empty set, $G : X \times X \times X \to \mathbf{R}^+$ be a function satisfying the following properties:
(G1) $G(x, y, z) = 0$ if $x = y = z$.
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$.
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$.
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables).
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).
Then the function G is called a generalized metric and the pair (X, G) is called a G-metric space.

Definition 1.2. ([13]) Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim\limits_{n,m\to\infty} G(x_n, x_m, x) = 0$, and one says the sequence $\{x_n\}$ is G-convergent to x.

Thus, if $x_n \to x$ in G-metric space (X, G), then, for any $\epsilon > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \epsilon$ for all $n, m > N$.

In [13], the authors have shown that the G-metric induces a Hausdorff topology, and the convergence described in the above definition is relative to this topology. The topology being Hausdorff, a sequence can converge at most to a point. Respectively, the authors achieve the following conclusions.

Definition 1.3. ([13]) Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if every $\epsilon > 0$, there exists a positive N such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l > N$, that is, if $G(x_n, x_m, x_l) \to 0$, as $n, m, l \to \infty$.

Lemma 1.4. ([13]) *If (X, G) is a G-metric space, then the following are equivalent.*
(1) $\{x_n\}$ is G-convergent to x.
(2) $G(x_n, x_n, x) \to 0$ as $n \to \infty$.
(3) $G(x_n, x, x) \to 0$ as $n \to \infty$.
(4) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

Lemma 1.5. ([13]) *If (X, G) is a G-metric space, then the following are equivalent.*
(1) The sequence $\{x_n\}$ is G-Cauchy.
(2) For every $\epsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_m) < \epsilon$ for all $n, m > N$.

Lemma 1.6. ([13]) *If (X, G) is a G-metric space, then $G(x, y, y) \leq 2G(y, x, x)$ for all $x, y \in X$.*

Lemma 1.7. ([13]) *If (X, G) is a G-metric space, then $G(x, x, y) \leq G(x, x, z) + G(z, z, y)$ for all $x, y, z \in X$.*

Definition 1.8. ([13]) Let (X, G), (X', G') be two G-metric spaces. Then a function $f : X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x; that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G'-convergent to $f(x)$.

Lemma 1.9. ([13]) *Let (X,G) be a G-metric spaces. Then the function $G(x,y,z)$ is jointly continuous in all three of its variables.*

Definition 1.10. ([13]) A G-metric space (X,G) is said to be G-complete (or a complete G-metric space) if every G-Cauchy sequence in (X,G) is convergent in X.

In [5], the authors introduced the following definitions.

Definition 1.11. ([5]) Let X be a nonempty set and $F : X^4 \to X$ be a given mapping. An element $(x,y,z,w) \in X^4$ is called a quadruple fixed point of F if

$$x = F(x,y,z,w), \quad y = F(y,z,w,x),$$

$$z = F(z,w,x,y) \text{ and } w = F(w,x,y,z).$$

Definition 1.12. ([5]) Let $(X,\preceq)$ be a partially ordered set and let $F : X^4 \to X$. The mapping F is said to have the mixed monotone property if $F(x,y,z,w)$ is monotone non-decreasing in x,z and is monotone non-increasing in y,w, that is, for any $x,y,z,w \in X$,

$$x_1,x_2 \in X, x_1 \preceq x_2 \Rightarrow F(x_1,y,z,w) \preceq F(x_2,y,z,w),$$

$$y_1,y_2 \in X, y_1 \preceq y_2 \Rightarrow F(x,y_1,z,w) \succeq F(x,y_2,z,w),$$

$$z_1,z_2 \in X, z_1 \preceq z_2 \Rightarrow F(x,y,z_1,w) \preceq F(x,y,z_2,w),$$

and

$$w_1,w_2 \in X, w_1 \preceq w_2 \Rightarrow F(x,y,z,w_1) \succeq F(x,y,z,w_2).$$

Definition 1.13. ([5]) Let X be a non-empty set. Then we say that the mappings $F : X^4 \to X$ and $g : X \to X$ are commutative if for all $x,y,z,w \in X$,

$$g(F(x,y,z,w)) = F(gx,gy,gz,gw).$$

In [16], the authors gave the following definitions.

Definition 1.14. ([16]) Let $(X,\preceq)$ be a partially ordered set and $F : X^4 \to X$ and $g : X \to X$ be two mappings. We say that F has the mixed-g-monotone property if $F(x,y)$ is g-monotone nondecreasing in x,z and it is g-monotone nonincreasing in y,w, that is, for any $x,y,z,w \in X$, we have:

$$x_1,x_2 \in X, \quad g(x_1) \preceq g(x_2) \Rightarrow F(x_1,y,z,w) \preceq F(x_2,y,z,w),$$

$$y_1,y_2 \in X, \quad g(y_1) \preceq g(y_2) \Rightarrow F(x,y_1,z,w) \succeq F(x,y_2,z,w),$$

$$z_1,z_2 \in X, \quad g(z_1) \preceq g(z_2) \Rightarrow F(x,y,z_1,w) \preceq F(x,y,z_2,w),$$

and

$$w_1,w_2 \in X, \quad g(w_1) \preceq g(w_2) \Rightarrow F(x,y,z,w_1) \succeq F(x,y,z,w_2).$$

Definition 1.15. ([16]) An element $(x,y,z,w) \in X^4$ is called a quadruple coincidence point of the mapping $F : X^4 \to X$ and $g : X \to X$ if

$$gx = F(x,y,z,w) \quad gy = F(y,z,w,x) \quad gz = F(z,w,x,y) \text{ and } gw = F(w,x,y,z).$$

(x,y,z,w) is said to be a quadruple point of coincidence of F and g.

Definition 1.16. ([16]) Let $F : X^4 \to X$ and $g : X \to X$. An element (x, y, z, w) is called a quadruple common fixed point of F and g if

$$F(x,y,z,w) = gx = x, \quad F(y,z,w,x) = gy = y,$$

$$F(z,w,x,y) = gz = z, \quad \textit{and} \quad F(w,x,y,z) = gw = w.$$

In [18], Mustafa considered the following class of functions. We denote by Φ the set of functions $\varphi : [0,+\infty) \to [0,+\infty)$ satisfying
(i_φ) φ is continuous and non-decreasing;
(ii_φ) $\varphi(t) = 0$ iff $t = 0$;
(iii_φ) $\varphi(s+t) \le \varphi(s) + \varphi(t)$ for all $s, t \ge 0$.

And let Ψ denote all functions $\psi : [0,+\infty) \to [0,+\infty)$ which satisfy
(i_ψ) $\lim\limits_{t\to r}\psi(t) > 0$ for all $r > 0$, and
(ii_ψ) $\lim\limits_{t\to 0^+}\psi(t) = 0$.
For example [18], the function $\varphi(t) = kt$, $k > 0$, $\varphi(t) = \frac{t}{1+t}$ are in Φ and $\psi_1(t) = kt$, $k > 0$, $\psi_2(t) = \frac{\ln(2k+1)}{2}$ are in Ψ.

Remark 1.17. ([18]) $\Phi \subseteq \Psi$.

Remark 1.18. ([18]) For all $t \in [0,+\infty)$, we have $\frac{1}{2}\varphi(t) \le \varphi(\frac{t}{2})$.

Mustafa [18] proved the following theorems.

Theorem 1.19. *Let $(X, \preceq)$ be a partially ordered set and (X, G) be a G-metric space. Let $F : X^4 \to X$ and $g : X \to X$ be such that F has the mixed g-monotone property. Assume that there exists $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$\begin{aligned}&\varphi\big(G(F(x,y,z,w),F(u,v,s,t),F(a,b,c,d))\big)\\ &\quad\le \frac{1}{4}\varphi\big(G(gx,gu,ga)+G(gy,gv,gb)+G(gz,gs,gc)+G(gw,gt,gd)\big)\\ &\quad\quad-\psi\left(\frac{G(gx,gu,ga)+G(gy,gv,gb)+G(gz,gs,gc)+G(gw,gt,gd)}{4}\right)\end{aligned} \tag{1.1}$$

for all $x,y,z,w,u,v,s,t,a,b,c,d \in X$ with $gx \succeq gu \succeq ga$, $gy \preceq gv \preceq gb$, $gz \succeq gs \succeq gc$ and $gw \preceq gt \preceq gd$. Suppose also that $F(X^4) \subseteq g(X)$ and g is continuous and commutes with F. If there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$gx_0 \preceq F(x_0,y_0,z_0,w_0), \quad gy_0 \succeq F(y_0,z_0,w_0,x_0),$$

$$gz_0 \preceq F(z_0,w_0,x_0,y_0), \quad \textit{and} \quad gw_0 \succeq F(w_0,x_0,y_0,z_0).$$

suppose either
(a) (X, G) is a complete G-metric space and F is continuous or
(b) $(g(X), G)$ is complete and $(X, G, \preceq)$ has the following property:

(i) if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,
(ii) if a non-increasing sequence $y_n \to y$, then $y \preceq y_n$ for all n,

then there exist $x, y, z, w \in X$ such that

$$F(x,y,z,w) = gx, \quad F(y,z,w,x) = gy, \quad F(z,w,x,y) = gz, \quad \textit{and} \quad F(w,x,y,z) = gw,$$

that is, F and g have a quadruple coincidence point.

Theorem 1.20. *In addition to the hypothesis of Theorem 1.19, suppose that for all $(x,y,z,w),(u,v,r,l) \in X^4$, there exists $(a,b,c,d) \in X^4$ such that $(F(a,b,c,d),F(b,c,d,a),F(c,d,a,b),F(d,a,b,c))$ is comparable to $(F(x,y,z,w),F(y,z,w,x),F(z,w,x,y),F(w,x,y,z))$ and $(F(u,v,r,l),F(v,r,l,u),F(r,l,u,v),F(l,u,v,r))$. Then F and g have a unique quadruple common fixed point (x,y,z,w) such that $x = gx = F(x,y,z,w)$, $y = gy = F(y,z,w,x)$, $z = gz = F(z,w,x,y)$, and $w = gw = F(w,x,y,z)$.*

2. Main results

In this section, we prove quadruple fixed point theorems for (φ, ψ)-contractive type mappings in partially ordered G-metric spaces with mixed g-monotone property.

Next, we prove our main results.

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set and (X, G) be a G-metric space. Let $F : X^4 \to X$ and $g : X \to X$ be such that F has the mixed-g-monotone property. Assume that there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$\begin{aligned}\varphi\Big(\frac{1}{4}\big[&G(F(x,y,z,w),F(u,v,s,t),F(a,b,c,d)) + G(F(y,z,w,x),F(v,s,t,u),F(b,c,d,a))\\ &+G(F(z,w,x,y,),F(s,t,u,v),F(c,d,a,b)) + G(F(w,x,y,z),F(t,u,v,s),F(d,a,b,c))\big]\Big)\\ \leq\ &\varphi\Big(\frac{G(gx,gu,ga)+G(gy,gv,gb)+G(gz,gs,gc)+G(gw,gt,gd)}{4}\Big)\\ &-\psi\Big(\frac{G(gx,gu,ga)+G(gy,gv,gb)+G(gz,gs,gc)+G(gw,gt,gd)}{4}\Big)\end{aligned} \tag{2.1}$$

for all $x, y, z, w, u, v, s, t, a, b, c, d \in X$ with $gw \succeq gu \succeq ga$, $gy \preceq gv \preceq gb$, $gz \succeq gs \succeq gc$ and $gw \preceq gt \preceq gd$. Suppose also that $F(X^4) \subseteq g(X)$ and g is continuous and commutes with F. If there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$gx_0 \preceq F(x_0, y_0, z_0, w_0), \quad gy_0 \succeq F(y_0, z_0, w_0, x_0),$$

$$gz_0 \preceq F(z_0, w_0, x_0, y_0), \quad \text{and} \quad gw_0 \succeq F(w_0, x_0, y_0, z_0).$$

suppose either
(a) (X, G) is a complete G-metric space and F is continuous or
(b) $(g(X), G)$ is complete and $(X, G, \preceq)$ has the following property:

(i) if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,
(ii) if a non-increasing sequence $y_n \to y$, then $y \preceq y_n$ for all n,

then there exist $x, y, z, w \in X$ such that

$$F(x,y,z,w) = gx, \quad F(y,z,w,x) = gy, \quad F(z,w,x,y) = gz, \quad \text{and} \quad F(w,x,y,z) = gw,$$

that is, F and g have a quadruple coincidence point.

Proof. Let $x_0, y_0, z_0, w_0 \in X$ be such that

$$gx_0 \preceq F(x_0, y_0, z_0, w_0), \quad gy_0 \succeq F(y_0, z_0, w_0, x_0),$$

$$gz_0 \preceq F(z_0, w_0, x_0, y_0), \quad \text{and} \quad gw_0 \succeq F(w_0, x_0, y_0, z_0).$$

Since $F(X^4) \subseteq g(X)$, we can choose $x_1, y_1, z_1, w_1 \in X$ such that

$$\begin{aligned} &gx_1 = F(x_0, y_0, z_0, w_0), \quad gy_1 = F(y_0, z_0, w_0, x_0),\\ &gz_1 = F(z_0, w_0, x_0, y_0), \quad \text{and} \quad gw_1 = F(w_0, x_0, y_0, z_0). \end{aligned} \tag{2.2}$$

Again since $F(X^4) \subseteq g(X)$, we can choose $x_2, y_2, z_2, w_2 \in X$ such that

$$\begin{aligned} &gx_2 = F(x_1, y_1, z_1, w_1), \quad gy_2 = F(y_1, z_1, w_1, x_1),\\ &gz_2 = F(z_1, w_1, x_1, y_1), \quad \text{and} \quad gw_2 = F(w_1, x_1, y_1, z_1). \end{aligned}$$

Continuing this process, we can construct sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ in X such that

$$\begin{aligned} &gx_{n+1} = F(x_n, y_n, z_n, w_n), \quad gy_{n+1} = F(y_n, z_n, w_n, x_n), \\ &gz_{n+1} = F(z_n, w_n, x_n, y_n), \quad and \quad gw_{n+1} = F(w_n, x_n, y_n, z_n). \end{aligned} \tag{2.3}$$

Next, we shall that

$$\begin{aligned} &gx_n \preceq gx_{n+1}, \quad gy_n \succeq gy_{n+1}, \\ &gz_n \preceq gz_{n+1}, \quad and \quad gw_n \succeq gw_{n+1} \quad for \quad n = 0, 1, 2, 3, \cdots . \end{aligned} \tag{2.4}$$

For this purpose, we use the mathematical induction. Since $gx_0 \preceq F(x_0, y_0, z_0, w_0)$, $gy_0 \succeq F(y_0, z_0, w_0, x_0)$, $gz_0 \preceq F(z_0, w_0, x_0, y_0)$, and $gw_0 \succeq F(w_0, x_0, y_0, z_0)$, then by (2.2), we get

$$gx_0 \preceq gx_1, \;\; gy_0 \succeq gy_1, \;\; gz_0 \preceq gz_1, \;\; and \;\; gw_0 \succeq gw_1,$$

that is, (2.4) holds for $n = 0$. We presume that (2.4) holds for some $n > 0$. As F has the mixed g-monotone property and $gx_n \preceq gx_{n+1}$, $gy_n \succeq gy_{n+1}$, $gz_n \preceq gz_{n+1}$, and $gw_n \succeq gw_{n+1}$, we obtain

$$\begin{aligned} gx_{n+1} &= F(x_n, y_n, z_n, w_n) \preceq F(x_{n+1}, y_n, z_n, w_n) \\ &\preceq F(x_{n+1}, y_{n+1}, z_n, w_n) \preceq F(x_{n+1}, y_{n+1}, z_{n+1}, w_n) \\ &\preceq F(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1}) = gx_{n+2}, \\ gy_{n+2} &= F(y_{n+1}, z_{n+1}, w_{n+1}, x_{n+1}) \preceq F(y_{n+1}, z_n, w_{n+1}, x_{n+1}) \\ &\preceq F(y_{n+1}, z_n, w_n, x_{n+1}) \preceq F(y_{n+1}, z_n, w_n, x_n) \\ &\preceq F(y_n, z_{n+1}, w_{n+1}, x_n) = gy_{n+1}, \end{aligned}$$

$$\begin{aligned} gz_{n+1} &= F(z_n, w_n, x_n, y_n) \preceq F(z_{n+1}, w_n, x_n, z_n) \\ &\preceq F(z_{n+1}, w_{n+1}, x_n, y_n) \preceq F(z_{n+1}, w_{n+1}, x_{n+1}, y_n) \\ &\preceq F(z_{n+1}, w_{n+1}, x_{n+1}, y_{n+1}) = gz_{n+2}, \end{aligned}$$

and

$$\begin{aligned} gw_{n+2} &= F(w_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}) \preceq F(w_{n+1}, x_n, y_{n+1}, z_{n+1}) \\ &\preceq F(w_{n+1}, x_n, y_n, z_{n+1}) \preceq F(w_{n+1}, x_n, y_n, z_n) \\ &\preceq F(w_n, x_{n+1}, y_{n+1}, z_n) = gw_{n+1}. \end{aligned}$$

Thus (2.4) holds for any $n \in \mathbf{N}$. Assume for some $n \in \mathbf{N}$,

$$gx_n = gx_{n+1}, \;\; gy_n = gy_{n+1}, \;\; gz_n = gz_{n+1} \;\; and \;\; gw_n = gw_{n+1},$$

then by (2.3), we have $gx_n = F(x_n, y_n.z_n, w_n)$, $gy_n = F(y_n, z_n, w_n, x_n)$, $gz_n = F(z_n, w_n, x_n, y_n)$, and $gw_n = F(w_n, x_n, y_n, z_n)$. It is clearly that (x_n, y_n, z_n, w_n) is a quadruple coincidence point of F and g. From now on, assume for any $n \in \mathbf{N}$ that at least

$$gx_n \neq gx_{n+1} \; or \; gy_n \neq gy_{n+1} \; or \; gz_n \neq gz_{n+1} \; or \; gw_n \neq gw_{n+1}. \tag{2.5}$$

Since $gx_n \preceq gx_{n+1}$, $gy_n \succeq gy_{n+1}$, $gz_n \preceq gz_{n+1}$, and $gw_n \succeq gw_{n+1}$, let

$$\begin{aligned} \delta_n = \frac{1}{4}\big[&G(gx_{n+1}, gx_{n+1}, gx_n) + G(gy_{n+1}, gy_{n+1}, gy_n) \\ &+ G(gz_{n+1}, gz_{n+1}, gz_n) + G(gw_{n+1}, gw_{n+1}, gw_n)\big], \end{aligned} \tag{2.6}$$

then from (2.1), (2.3) and (2.6), we have

$$\begin{aligned} \varphi(\delta_n) &= \varphi\bigg(\frac{1}{4}\big[G(gx_{n+1}, gx_{n+1}, gx_n) + G(gy_{n+1}, gy_{n+1}, gy_n) + G(gz_{n+1}, gz_{n+1}, gz_n) \\ &\qquad + G(gw_{n+1}, gw_{n+1}, gw_n)\big]\bigg) \\ &= \varphi\bigg(\frac{1}{4}\big[G(F(x_n, y_n, z_n, w_n), F(x_n, y_n, z_n, w_n), F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1})) \\ &\qquad + G(F(y_n, z_n, w_n, x_n), F(y_n, z_n, w_n, x_n), F(y_{n-1}, z_{n-1}, w_{n-1}, x_{n-1})) \\ &\qquad + G(F(z_n, w_n, x_n, y_n), F(z_n, w_n, x_n, y_n), F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1})) \\ &\qquad + G(F(w_n, x_n, y_n, z_n), F(w_n, x_n, y_n, z_n), F(w_{n-1}, x_{n-1}, y_{n-1}, z_{n-1}))\big]\bigg) \end{aligned}$$

$$\begin{aligned}
&\leq \varphi\bigg(\tfrac{1}{4}\big[G(gx_n, gx_n, gx_{n-1}) + G(gy_n, gy_n, gy_{n-1}) + G(gz_n, gz_n, gz_{n-1}) \\
&\qquad\qquad +G(gw_n, gw_n, gw_{n-1})\big]\bigg) \\
&\quad -\psi\bigg(\frac{1}{4}\big[G(gx_n, gx_n, gx_{n-1}) + G(gy_n, gy_n, gy_{n-1}) + G(gz_n, gz_n, gz_{n-1}) \\
&\qquad\qquad +G(gw_n, gw_n, gw_{n-1})\big]\bigg) \\
&= \varphi(\delta_{n-1}) - \psi(\delta_{n-1}).
\end{aligned} \tag{2.7}$$

Hence, $\varphi(\delta_n) \leq \varphi(\delta_{n-1})$. Using the fact that φ is nondecreasing, we get $\delta_n \leq \delta_{n-1}$. Thus, the sequence $\{\delta_n\}$ is decreasing, therefore, there is some $\delta \geq 0$ such that

$$\begin{aligned}
\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \frac{1}{4}\big[&G(gx_{n+1}, gx_{n+1}, gx_n) + G(gy_{n+1}, gy_{n+1}, gy_n) \\
&+G(gz_{n+1}, gz_{n+1}, gz_n) + G(gw_{n+1}, gw_{n+1}, gw_n)\big] = \delta.
\end{aligned} \tag{2.8}$$

We will show that $\delta = 0$. Suppose to the contrary that $\delta > 0$, taking the limit as $n \to \infty$ of both sides of (2.7) and using the fact that φ is continuous and $\lim\limits_{t\to r} \psi(t) > 0$ for $r > 0$, we have

$$\varphi(\delta) = \lim_{n\to\infty} \varphi(\delta_n) \leq \lim_{n\to\infty} \varphi(\delta_{n-1}) - \lim_{n\to\infty} \psi(\delta_{n-1}) = \varphi(\delta) - \lim_{n\to\infty} \psi(\delta_{n-1}) < \varphi(\delta),$$

which is a contradiction. Thus, $\delta = 0$, that is,

$$\begin{aligned}
\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \frac{1}{4}\big[&G(gx_{n+1}, gx_{n+1}, gx_n) + G(gy_{n+1}, gy_{n+1}, gy_n) \\
&+G(gz_{n+1}, gz_{n+1}, gz_n) + G(gw_{n+1}, gw_{n+1}, gw_n)\big] = 0.
\end{aligned} \tag{2.9}$$

Now we prove that (gx_n), (gy_n), (gz_n) and (gw_n) are G-Cauchy sequences in the G-metric space (X, G). Suppose on the contrary that at least one of (gx_n) (gy_n), (gz_n) and (gw_n) is not a G-Cauchy sequence in (X, G). Then there exists $\epsilon > 0$ and sequences of natural numbers $(m(k))$ and $(l(k))$ such that for every natural number k, $m(k) > l(k) \geq k$ and

$$\begin{aligned}
r_k = \frac{1}{4}[&G(gx_{m(k)}, gx_{m(k)}, gx_{l(k)}) + G(gy_{m(k)}, gy_{m(k)}, gy_{l(k)}) \\
&+G(gz_{m(k)}, gz_{m(k)}, gz_{l(k)}) + G(gw_{m(k)}, gw_{m(k)}, gw_{l(k)})] \geq \epsilon.
\end{aligned} \tag{2.10}$$

Now corresponding to $l(k)$ we choose $m(k)$ to be the smallest for which (2.10) holds. So

$$\begin{aligned}
\frac{1}{4}[&G(gx_{m(k)-1}, gx_{m(k)-1}, gx_{l(k)}) + G(gy_{m(k)-1}, gy_{m(k)-1}, gy_{l(k)}) \\
&+G(gz_{m(k)-1}, gz_{m(k)-1}, gz_{l(k)}) + G(gw_{m(k)-1}, gw_{m(k)-1}, gw_{l(k)})] < \epsilon.
\end{aligned} \tag{2.11}$$

Using the rectangle inequality and having in mind (2.10) and (2.11), we get

$$\begin{aligned}
\epsilon &\leq r_k \\
&= \frac{1}{4}[G(gx_{m(k)}, gx_{m(k)}, gx_{l(k)}) + G(gy_{m(k)}, gy_{m(k)}, gy_{l(k)}) + G(gz_{m(k)}, gz_{m(k)}, gz_{l(k)}) \\
&\qquad +G(gw_{m(k)}, gw_{m(k)}, gw_{l(k)})] \\
&\leq \frac{1}{4}[G(gx_{m(k)}, gx_{m(k)}, gx_{m(k)-1}) + G(gx_{m(k)-1}, gx_{m(k)-1}, gx_{l(k)}) + G(gy_{m(k)}, gy_{m(k)}, gy_{m(k)-1}) \\
&\qquad +G(gy_{m(k)-1}, gy_{m(k)-1}, gy_{l(k)}) + G(gz_{m(k)}, gz_{m(k)}, gz_{m(k)-1}) + G(gz_{m(k)-1}, gz_{m(k)-1}, gz_{l(k)}) \\
&\qquad +G(gw_{m(k)}, gw_{m(k)}, gw_{m(k)-1}) + G(gw_{m(k)-1}, gw_{m(k)-1}, gw_{l(k)})] \\
&< \delta_{m(k)-1} + \epsilon.
\end{aligned} \tag{2.12}$$

In (2.12), letting $n \to \infty$, we can get $\lim\limits_{n\to\infty} r_k = \epsilon^+$. Using the rectangle inequality, we get

$$\begin{aligned}
\epsilon &\le r_k \\
&= \frac{1}{4}\big[G(gx_{m(k)}, gx_{m(k)}, gx_{l(k)}) + G(gy_{m(k)}, gy_{m(k)}, gy_{l(k)}) + G(gz_{m(k)}, gz_{m(k)}, gz_{l(k)}) \\
&\qquad + G(gw_{m(k)}, gw_{m(k)}, gw_{l(k)})\big] \\
&\le \frac{1}{4}\big[G(gx_{m(k)}, gx_{m(k)}, gx_{m(k)+1}) + G(gx_{m(k)+1}, gx_{m(k)+1}, gx_{l(k)+1}) + G(gx_{l(k)+1}, gx_{l(k)+1}, gx_{l(k)}) \\
&\quad + G(gy_{m(k)}, gy_{m(k)}, gy_{m(k)+1}) + G(gy_{m(k)+1}, gy_{m(k)+1}, gy_{l(k)+1}) + G(gy_{l(k)+1}, gy_{l(k)+1}, gy_{l(k)}) \\
&\quad + G(gz_{m(k)}, gz_{m(k)}, gz_{m(k)+1}) + G(gz_{m(k)+1}, gz_{m(k)+1}, gz_{l(k)+1}) + G(gz_{l(k)+1}, gz_{l(k)+1}, gz_{l(k)}) \\
&\quad + G(gw_{m(k)}, gw_{m(k)}, gw_{m(k)+1}) + G(gw_{m(k)+1}, gw_{m(k)+1}, gw_{l(k)+1}) + G(gw_{l(k)+1}, gw_{l(k)+1}, gw_{l(k)})\big] \\
&= \delta_{l(k)} + \frac{1}{4}\big[G(gx_{m(k)}, gx_{m(k)}, gx_{m(k)+1}) + G(gy_{m(k)}, gy_{m(k)}, gy_{m(k)+1}) + G(gz_{m(k)}, gz_{m(k)}, gz_{m(k)+1}) \\
&\quad + G(gw_{m(k)}, gw_{m(k)}, gw_{m(k)+1})\big] + \frac{1}{4}\big[G(gx_{m(k)+1}, gx_{m(k)+1}, gx_{l(k)+1}) + G(gy_{m(k)+1}, gy_{m(k)+1}, \\
&\qquad gy_{l(k)+1}) + G(gz_{m(k)+1}, gz_{m(k)+1}, gz_{l(k)+1}) + G(gw_{m(k)+1}, gw_{m(k)+1}, gw_{l(k)+1})\big].
\end{aligned}$$

In the above of inequality, using that $G(x,x,y) \le 2G(x,y,y)$ for any $x,y \in X$, we obtain

$$\begin{aligned}
\epsilon &\le r_k \\
&\le \delta_{l(k)} + \frac{1}{2}\delta_{m(k)} + \frac{1}{4}\big[G(gx_{m(k)+1}, gx_{m(k)+1}, gx_{l(k)+1}) + G(gy_{m(k)+1}, gy_{m(k)+1}, gy_{l(k)+1}) \\
&\qquad + G(gz_{m(k)+1}, gz_{m(k)+1}, gz_{l(k)+1}) + G(gw_{m(k)+1}, gw_{m(k)+1}, gw_{l(k)+1})\big].
\end{aligned} \tag{2.13}$$

Now, using the property of φ, we have

$$\begin{aligned}
&\varphi\Big(\frac{1}{4}\big[G(gx_{m(k)+1}, gx_{m(k)+1}, gx_{l(k)+1}) + G(gy_{m(k)+1}, gy_{m(k)+1}, gy_{l(k)+1}) \\
&\qquad + G(gz_{m(k)+1}, gz_{m(k)+1}, gz_{l(k)+1}) + G(gw_{m(k)+1}, gw_{m(k)+1}, gw_{l(k)+1})\big]\Big) \\
&= \varphi\Big(\frac{1}{4}\big[G(F(x_{m(k)}, y_{m(k)}, z_{m(k)}, w_{m(k)}), F(x_{m(k)}, y_{m(k)}, z_{m(k)}, w_{m(k)}), F(x_{l(k)}, y_{l(k)}, z_{l(k)}, w_{l(k)}) \\
&\qquad + G(F(y_{m(k)}, z_{m(k)}, w_{m(k)}, x_{m(k)}), F(y_{m(k)}, z_{m(k)}, w_{m(k)}, x_{m(k)}), F(y_{l(k)}, z_{l(k)}, w_{l(k)}, x_{l(k)}) \\
&\qquad + G(F(z_{m(k)}, w_{m(k)}, x_{m(k)}, y_{m(k)}), F(z_{m(k)}, w_{m(k)}, x_{m(k)}, y_{m(k)}), F(z_{l(k)}, w_{l(k)}, x_{l(k)}, y_{l(k)}) \\
&\qquad + G(F(w_{m(k)}, x_{m(k)}, y_{m(k)}, z_{m(k)}), F(w_{m(k)}, x_{m(k)}, y_{m(k)}, z_{m(k)}), F(w_{l(k)}, x_{l(k)}, y_{l(k)}, z_{l(k)})\big]\Big) \\
&\le \varphi\Big(\frac{1}{4}[G(gx_{m(k)}, gx_{m(k)}, gx_{l(k)}) + G(gy_{m(k)}, gy_{m(k)}, gy_{l(k)}) + G(gz_{m(k)}, gz_{m(k)}, gz_{l(k)}) \\
&\qquad + G(gw_{m(k)}, gw_{m(k)}, gw_{l(k)})]\Big) - \psi\Big(\frac{1}{4}[G(gx_{m(k)}, gx_{m(k)}, gx_{l(k)}) + G(gy_{m(k)}, gy_{m(k)}, gy_{l(k)}) \\
&\qquad + G(gz_{m(k)}, gz_{m(k)}, gz_{l(k)}) + G(gw_{m(k)}, gw_{m(k)}, gw_{l(k)})]\Big) \\
&= \varphi(r_k) - \psi(r_k).
\end{aligned} \tag{2.14}$$

Combining (2.13), (2.14) and the the property of φ, we get

$$\begin{aligned}
\varphi(\epsilon) &\le \varphi(r_k) \\
&\le \varphi(\delta_{l(k)}) + \frac{1}{2}\varphi(\delta_{m(k)}) + \varphi\Big(\frac{1}{4}[G(gx_{m(k)+1}, gx_{m(k)+1}, gx_{l(k)+1}) + G(gy_{m(k)+1}, gy_{m(k)+1}, \\
&\qquad gy_{l(k)+1}) + G(gz_{m(k)+1}, gz_{m(k)+1}, gz_{l(k)+1}) + G(gw_{m(k)+1}, gw_{m(k)+1}, gw_{l(k)+1})]\Big) \\
&\le \varphi(\delta_{l(k)}) + \frac{1}{2}\varphi(\delta_{m(k)}) + \varphi(r_k) - \psi(r_k).
\end{aligned} \tag{2.15}$$

In (2.15), let $k \to \infty$, we have

$$\varphi(\epsilon) \le \lim_{k\to\infty} \varphi(r_k) \le \varphi(0) + \frac{1}{2}\varphi(0) + \varphi(\epsilon) - \lim_{k\to\infty} \varphi(r_k) < \varphi(\epsilon),$$

which is a contraction. This implies that (gx_n), (gy_n), (gz_n), and (gw_n) are G-Cauchy sequences in (X,G). Now suppose that the assumption (a) holds. Since X is a G-complete metric space, there exist $x,y,z,w \in X$

such that

$$\begin{aligned} \lim_{n\to\infty} g(x_n) = x, \qquad & \lim_{n\to\infty} g(y_n) = y, \\ \lim_{n\to\infty} g(z_n) = z, \ \ and \ \ & \lim_{n\to\infty} g(w_n) = w. \end{aligned} \tag{2.16}$$

From (2.16) and the continuity of g, we have

$$\begin{aligned} \lim_{n\to\infty} gg(x_n) = gx, \qquad & \lim_{n\to\infty} gg(y_n) = gy, \\ \lim_{n\to\infty} gg(z_n) = gz, \ \ and \ \ & \lim_{n\to\infty} gg(w_n) = gw. \end{aligned}$$

From the commutativity of F and g, we have

$$g(gx_{n+1}) = gF(x_n, y_n, z_n, w_n) = F(gx_n, gy_n, gz_n, gw_n), \tag{2.17}$$

$$g(gy_{n+1}) = gF(y_n, z_n, w_n, x_n) = F(gy_n, gz_n, gw_n, gx_n), \tag{2.18}$$

$$g(gz_{n+1}) = gF(z_n, w_n, x_n, y_n) = F(gz_n, gw_n, gx_n, gy_n), \tag{2.19}$$

and

$$g(gw_{n+1}) = gF(w_n, x_n, y_n, z_n) = F(gw_n, gx_n, gy_n, gz_n). \tag{2.20}$$

We shall show that $gx = F(x, y, z, w)$, $gy = F(y, z, w, x)$, $gz = F(z, w, x, y)$, and $gw = F(w, x, y, z)$. By letting $n \to \infty$ in (2.17)-(2.20) and using the continuity of F, we obtain

$$\begin{aligned} gx = \lim_{n\to\infty} g(gx_{n+1}) &= \lim_{n\to\infty} gF(x_n, y_n, z_n, w_n) = \lim_{n\to\infty} F(gx_n, gy_n, gz_n, gw_n) \\ &= F(x, y, z, w), \end{aligned}$$

$$\begin{aligned} gy = \lim_{n\to\infty} g(gy_{n+1}) &= \lim_{n\to\infty} gF(y_n, z_n, w_n, x_n) = \lim_{n\to\infty} F(gy_n, gz_n, gw_n, gx_n) \\ &= F(y, z, w, x), \end{aligned}$$

$$\begin{aligned} gz = \lim_{n\to\infty} g(gz_{n+1}) &= \lim_{n\to\infty} gF(z_n, w_n, x_n, y_n) = \lim_{n\to\infty} F(gz_n, gw_n, gx_n, gy_n) \\ &= F(z, w, x, y), \end{aligned}$$

and

$$\begin{aligned} gw = \lim_{n\to\infty} g(gw_{n+1}) &= \lim_{n\to\infty} gF(w_n, x_n, y_n, z_n) = \lim_{n\to\infty} F(gw_n, gx_n, gy_n, gz_n) \\ &= F(w, x, y, z). \end{aligned}$$

Hence, (x, y, z, w) is a coincidence point of F and g.

Now suppose that the assumption (b) holds. Since (gx_n), (gy_n), (gz_n), and (gw_n) are G-Cauchy sequences in the complete G-metric space $(g(X), G)$, then there exist $x, y, z, w \in X$ such that

$$gx_n \to gx, \ \ gy_n \to gy, \ \ gz_n \to gz, \ \ gw_n \to gw. \tag{2.21}$$

Since (gx_n), (gz_n) are non-decreasing and (gy_n), (gw_n) are non-increasing and since $(X, G, \leq)$ satisfies conditions (i) and (ii), we have

$$gx_n \preceq gx, \ \ gy_n \succeq gy, \ \ gz_n \preceq gz, \ \ and \ \ gw_n \succeq gw \ \ for \ \ all \ \ n \in \mathbf{N}.$$

If $gx_n = gx$, $gy_n = gy$, $gz_n = gz$, and $gw_n = gw$ for some $n \geq 0$, then $gx = gx_n \preceq gx_{n+1} \preceq gx = gx_n$, $gy \preceq gy_{n+1} \preceq gy_n = gy$, $gz = gz_n \preceq gz_{n+1} \preceq gz = gz_n$, and $gw \preceq gw_{n+1} \preceq gw_n = gw$, which implies that

$$gx_n = gx_{n+1} = F(x_n, y_n, z_n, w_n), \ \ gy_n = gy_{n+1} = F(y_n, z_n, w_n, x_n),$$

and

$$gz_n = gz_{n+1} = F(z_n, w_n, x_n, y_n), \ \ gw_n = gw_{n+1} = F(w_n, x_n, y_n, z_n),$$

that is, (x_n, y_n, z_n, w_n) is a quadruple coincidence point of F and g. Then, we suppose that $(gx_n, gy_n, gz_n, gw_n) \neq (gx, gy, gz, gw)$ for all $n \in \mathbf{N}$. By the rectangle inequality, consider now

$$\begin{aligned} G(gx, F(x,y,z,w), F(x,y,z,w)) &\leq G(gx, gx_{n+1}, gx_{n+1}) + G(gx_{n+1}, F(x,y,z,w), F(x,y,z,w)) \\ &= G(gx, gx_{n+1}, gx_{n+1}) + G(F(x_n, y_n, z_n, w_n), F(x,y,z,w), F(x,y,z,w)). \end{aligned}$$

It can conclude that

$$G(gx, F(x,y,z,w), F(x,y,z,w)) - G(gx, gx_{n+1}, gx_{n+1}) \leq G(F(x_n, y_n, z_n, w_n), F(x,y,z,w), F(x,y,z,w)). \tag{2.22}$$

Similarly, we can get

$$G(gy, F(y,z,w,x), F(y,z,w,x)) - G(gy, gy_{n+1}, gy_{n+1}) \leq G(F(y_n, z_n, w_n, x_n), F(y,z,w,x), F(y,z,w,x)), \tag{2.23}$$

$$G(gz, F(z,w,x,y)F(z,w,x,y)) - G(gz, gz_{n+1}, gz_{n+1}) \leq G(F(z_n, w_n, x_n, y_n), F(z,w,x,y), F(z,w,x,y)), \tag{2.24}$$

and

$$G(gw, F(w,x,y,z), F(w,x,y,z)) - G(gw, gw_{n+1}, gw_{n+1}) \leq G(F(w_n, x_n, y_n, z_n), F(w,x,y,z), F(w,x,y,z)). \tag{2.25}$$

By using (2.22)-(2.25), we have

$$\begin{aligned} &\frac{1}{4}[G(gx, F(x,y,z,w), F(x,y,z,w)) - G(gx, gx_{n+1}, gx_{n+1}) \\ &\quad + G(gy, F(y,z,w,x), F(y,z,w,x)) - G(gy, gy_{n+1}, gy_{n+1}) \\ &\quad + G(gz, F(z,w,x,y)F(z,w,x,y)) - G(gz, gz_{n+1}, gz_{n+1}) \\ &\quad + G(gw, F(w,x,y,z), F(w,x,y,z)) - G(gw, gw_{n+1}, gw_{n+1})] \\ &\leq \frac{1}{4}[G(F(x_n, y_n, z_n, w_n), F(x,y,z,w), F(x,y,z,w)) + G(F(y_n, z_n, w_n, x_n), F(y,z,w,x), F(y,z,w,x)) \\ &\quad + G(F(z_n, w_n, x_n, y_n), F(z,w,x,y), F(z,w,x,y)) + G(F(w_n, x_n, y_n, z_n), F(w,x,y,z), F(w,x,y,z))]. \end{aligned}$$

By the property of φ and (2.1), we can get

$$\begin{aligned} &\varphi\Big(\frac{1}{4}[G(gx, F(x,y,z,w), F(x,y,z,w)) - G(gx, gx_{n+1}, gx_{n+1}) \\ &\quad + G(gy, F(y,z,w,x), F(y,z,w,x)) - G(gy, gy_{n+1}, gy_{n+1}) \\ &\quad + G(gz, F(z,w,x,y)F(z,w,x,y)) - G(gz, gz_{n+1}, gz_{n+1}) \\ &\quad + G(gw, F(w,x,y,z), F(w,x,y,z)) - G(gw, gw_{n+1}, gw_{n+1})]\Big) \\ &\leq \varphi\Big(\frac{1}{4}[G(F(x_n, y_n, z_n, w_n), F(x,y,z,w), F(x,y,z,w)) + G(F(y_n, z_n, w_n, x_n), F(y,z,w,x), F(y,z,w,x)) \\ &\quad + G(F(z_n, w_n, x_n, y_n), F(z,w,x,y), F(z,w,x,y)) + G(F(w_n, x_n, y_n, z_n), F(w,x,y,z), F(w,x,y,z))]\Big) \\ &\leq \varphi\Big(\frac{1}{4}[G(gx_n, gx, gx) + G(gy_n, gy, gy) + G(gz_n, gz, gz) + G(gw_n, gw, gw)]\Big) - \psi\Big(\frac{1}{4}[G(gx_n, gx, gx) \\ &\quad + G(gy_n, gy, gy) + G(gz_n, gz, gz) + G(gw_n, gw, gw)]\Big). \end{aligned}$$

In the above inequality, let $n \to \infty$, using the property of ψ and (2.21), we have

$$\begin{aligned} &\varphi\Big(\frac{1}{4}[G(gx, F(x,y,z,w), F(x,y,z,w)) + G(gy, F(y,z,w,x), F(y,z,w,x)) \\ &\quad + G(gz, F(z,w,x,y), F(z,w,x,y)) + G(gw, F(w,x,y,z), F(w,x,y,z))]\Big) \\ &\leq \varphi(0) - 0 = 0. \end{aligned}$$

Hence, $G(gx, F(x, y, z, w), F(x, y, z, w)) = 0$, $G(gy, F(y, z, w, x), F(y, z, w, x)) = 0$, $G(gz, F(z, w, x, y), F(z, w, x, y)) = 0$, and $G(gw, F(w, x, y, z), F(w, x, y, z)) = 0$, that is, $gx = F(x, y, z, w)$, $gy = F(y, z, w, x)$, $gz = F(z, w, x, y)$ and $gw = F(w, x, y, z)$. The proof is completed. □

If we take $\varphi(t) = t$ in Theorem 2.1, we can get the following corollary.

Corollary 2.2. *Let $(X, \preceq)$ be a partially ordered set and (X, G) be a G-metric space. Let $F : X^4 \to X$ and $g : X \to X$ be such that F has the mixed g-monotone property. Assume that there exists $\psi \in \Psi$ such that*

$$\begin{aligned}&G(F(x,y,z,w),F(u,v,s,t),F(a,b,c,d))+G(F(y,z,w,x),F(v,s,t,u),F(b,c,d,a))\\&\qquad+G(F(z,w,x,y,),F(s,t,u,v),F(c,d,a,b))+G(F(w,x,y,z),F(t,u,v,s),F(d,a,b,c))\\&\le G(gx,gu,ga)+G(gy,gv,gb)+G(gz,gs,gc)+G(gw,gt,gd)\\&\qquad\qquad-4\psi\left(\frac{G(gx,gu,ga)+G(gy,gv,gb)+G(gz,gs,gc)+G(gw,gt,gd)}{4}\right)\end{aligned}$$

for all $x, y, z, w, u, v, s, t, a, b, c, d \in X$ with $gx \succeq gu \succeq ga$, $gy \preceq gv \preceq gb$, $gz \succeq gs \succeq gc$ and $gw \preceq gt \preceq gd$. Suppose also that $F(X^4) \subseteq g(X)$ and g is continuous and commutes with F. If there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$gx_0 \preceq F(x_0, y_0, z_0, w_0), \quad gy_0 \succeq F(y_0, z_0, w_0, x_0),$$
$$gz_0 \preceq F(z_0, w_0, x_0, y_0), \quad \textit{and} \quad gw_0 \succeq F(w_0, x_0, y_0, z_0).$$

suppose either
(a) (X, G) is a complete G-metric space and F is continuous or
(b) $(g(X), G)$ is complete and $(X, G, \preceq)$ has the following property:

(i) if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,
(ii) if a non-increasing sequence $y_n \to y$, then $y \preceq y_n$ for all n,

then there exist $x, y, z, w \in X$ such that

$$F(x, y, z, w) = gx, \quad F(y, z, w, x) = gy, \quad F(z, w, x, y) = gz, \quad \textit{and} \quad F(w, x, y, z) = gw,$$

that is, F and g have a quadruple coincidence point.

If we take $\psi(t) = (1 - k)t$ for all $k \in [0, 1)$ in Corollary 2.1, we can get the following corollary.

Corollary 2.3. *Let $(X, \preceq)$ be a partially ordered set and (X, G) be a G-metric space. Let $F : X^4 \to X$ and $g : X \to X$ be such that F has the mixed g-monotone property. Assume that there exists $k \in [0, 1)$ such that*

$$\begin{aligned}&G(F(x,y,z,w),F(u,v,s,t),F(a,b,c,d))+G(F(y,z,w,x),F(v,s,t,u),F(b,c,d,a))\\&\qquad+G(F(z,w,x,y,),F(s,t,u,v),F(c,d,a,b))+G(F(w,x,y,z),F(t,u,v,s),F(d,a,b,c))\\&\le k[G(gx,gu,ga)+G(gy,gv,gb)+G(gz,gs,gc)+G(gw,gt,gd)]\end{aligned}$$

for all $x, y, z, w, u, v, s, t, a, b, c, d \in X$ with $gx \succeq gu \succeq ga$, $gy \preceq gv \preceq gb$, $gz \succeq gs \succeq gc$ and $gw \preceq gt \preceq gd$. Suppose also that $F(X^4) \subseteq g(X)$ and g is continuous and commutes with F. If there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$gx_0 \preceq F(x_0, y_0, z_0, w_0), \quad gy_0 \succeq F(y_0, z_0, w_0, x_0),$$
$$gz_0 \preceq F(z_0, w_0, x_0, y_0), \quad \textit{and} \quad gw_0 \succeq F(w_0, x_0, y_0, z_0).$$

suppose either
(a) (X, G) is a complete G-metric space and F is continuous or
(b) $(g(X), G)$ is complete and $(X, G, \preceq)$ has the following property:

(i) if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,
(ii) if a non-increasing sequence $y_n \to y$, then $y \preceq y_n$ for all n,

then there exist $x, y, z, w \in X$ *such that*

$$F(x,y,z,w) = gx, \ \ F(y,z,w,x) = gy, \ \ F(z,w,x,y) = gz, \ \ \textit{and} \ \ F(w,x,y,z) = gw,$$

that is, F *and* g *have a quadruple coincidence point.*

Now, we shall prove the existence and uniqueness of a quadruple common fixed point. According to [18], for a product X^4 of a partially ordered set $(X, \preceq)$, we define a partial ordering in the following way. For all $(x,y,z,w), (u,v,r,h) \in X^4$,

$$(x,y,z,w) \preceq (u,v,r,h) \Leftrightarrow x \preceq u, y \succeq v, z \preceq r, \ \text{and} \ \ w \succeq l.$$

We say that (x,y,z,w) and (u,v,r,l) are comparable if

$$(x,y,z,w) \preceq (u,v,r,l) \ \text{ or } \ (u,v,r,l) \preceq (x,y,z,w).$$

Also, we say that (x,y,z,w) is equal to (u,v,r,l) if and only if $x = u$, $y = v$, $z = r$, $w = l$.

Theorem 2.4. *In addition to the hypothesis of Theorem 2.1, suppose that for all* $(x,y,z,w), (u,v,r,l) \in X^4$, *there exists* $(a,b,c,d) \in X^4$ *such that* $(F(a,b,c,d), F(b,c,d,a), F(c,d,a,b), F(d,a,b,c))$ *is comparable to* $(F(x,y,z,w), F(y,z,w,x), F(z,w,x,y), F(w,x,y,z))$ *and* $(F(u,v,r,l), F(v,r,l,u), F(r,l,u,v), F(l,u,v,r))$. *Then* F *and* g *have a unique quadruple common fixed point* (x,y,z,w) *such that* $x = gx = F(x,y,z,w)$, $y = gy = F(y,z,w,x)$, $z = gz = F(z,w,x,y)$, *and* $w = gw = F(w,x,y,z)$.

Proof. From Theorem 2.1, the set of coupled coincidences is non-empty. We shall show that if (x,y,z,w) and (u,v,r,l) are quadruple coincidence points of F and g, that is,

$$\begin{aligned} F(x,y,z,w) &= gx, \ \ F(u,v,r,l) = gu, \\ F(y,z,w,x) &= gy, \ \ F(v,r,l,u) = gv, \\ F(z,w,x,y) &= gz, \ \ F(r,l,u,v) = gr, \\ F(w,x,y,z) &= gw, \ \ F(l,u,v,r) = gl. \end{aligned}$$

Next, we illustrate that (gx, gy, gz, gw) and (gu, gv, gr, gl) are equal. By assumption, there exists $(a,b,c,d) \in X^4$ such that $(F(a,b,c,d), F(b,c,d,a), F(c,d,a,b), F(d,a,b,c))$ is comparable to $(F(x,y,z,w), F(y,z,w,x), F(z,w,x,y), F(w,x,y,z))$ and $(F(u,v,r,l), F(v,r,l,u), F(r,l,u,v), F(l,u,v,r))$.

We define the sequence (ga_n), (gb_n), (gc_n), and (gd_n) such that $a_0 = a$, $b_0 = b$, $c_0 = c$, $d_0 = d$ and

$$\begin{aligned} ga_n &= F(a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1}), \ \ gb_n = F(b_{n-1}, c_{n-1}, d_{n-1}, a_{n-1}), \\ gc_n &= F(c_{n-1}, d_{n-1}, a_{n-1}, b_{n-1}), \ \ gd_n = F(d_{n-1}, a_{n-1}, b_{n-1}, c_{n-1}) \end{aligned} \tag{2.26}$$

for all $n \in \mathbf{N}$. Further, set $x_0 = x$, $y_0 = y$, $z_0 = z$, $w_0 = w$ and $u_0 = u$, $v_0 = v$, $r_0 = r$, $l_0 = l$ and in the same way define the sequences (gx_n), (gy_n), (gz_n), (gw_n) and (gu_n), (gv_n), (gr_n), (gl_n). Then it is easy to see that

$$\begin{aligned} gx_1 &= F(x,y,z,w), \ \ gu_1 = F(u,v,r,l), \\ gy_1 &= F(y,z,w,x), \ \ gv_1 = F(v,r,l,u), \\ gz_1 &= F(z,w,x,y), \ \ gr_1 = F(r,l,u,v), \\ gw_1 &= F(w,x,y,z), \ \ gl_1 = F(l,u,v,r). \end{aligned} \tag{2.27}$$

Since $(F(x,y,z,w), F(y,z,w,x), F(z,w,x,y), F(w,x,y,z)) = (gx_1, gy_1, gz_1, gw_1) = (gx, gy, gz, gw)$ is comparable to $(F(a,b,c,d), F(b,c,d,a), F(c,d,a,b), F(d,a,b,c)) = (ga_1, gb_1, gc_1, gd_1)$, then it is easy to show $(gx, gy, gz, gw) \succeq (ga_n, gb_n, gc_n, gd_n)$. Recursively, we get that

$$(gx, gy, gz, gw) \succeq (ga_n, gb_n, gc_n, gd_n) \ \text{ for all } \ n \in \mathbf{N}. \tag{2.28}$$

It can conclude that $gx \succeq ga_n$, $gy \preceq gb_n$, $gz \succeq gc_n$, $gw \preceq gd_n$. By (2.27), (2.28) and (2.1) , we can get

$$\begin{aligned}
&\varphi\Big(\frac{G(gx,gx,ga_{n+1})+G(gb_{n+1},gy,gy)+G(gz,gz,gc_{n+1})+G(gd_{n+1},gw,gw)}{4}\Big)\\
&=\varphi\Big(\frac{1}{4}\big[G(F(x,y,z,w),F(x,y,z,w),F(a_n,b_n,c_n,d_n))+G(F(b_n,c_n,d_n,a_n),F(y,z,w,x),F(y,z,w,x))\\
&\quad+G(F(z,w,x,y),F(z,w,x,y),F(c_n,d_n,a_n,b_n))+G(F(d_n,a_n,b_n,c_n),F(w,x,y,z),F(w,x,y,z))\big]\Big)\\
&\le\varphi\Big(\frac{G(gx,gx,ga_n)+G(gy,gy,gb_n)+G(gz,gz,gc_n)+G(gw,gw,gd_n)}{4}\Big)\\
&\quad-\psi\Big(\frac{G(gx,gx,ga_n)+G(gy,gy,gb_n)+G(gz,gz,gc_n)+G(gw,gw,gd_n)}{4}\Big).
\end{aligned} \tag{2.29}$$

Thus,

$$\begin{aligned}
&\varphi\Big(\frac{G(gx,gx,ga_{n+1})+G(gb_{n+1},gy,gy)+G(gz,gz,gc_{n+1})+G(gd_{n+1},gw,gw)}{4}\Big)\\
&\le\varphi\Big(\frac{G(gx,gx,ga_n)+G(gy,gy,gb_n)+G(gz,gz,gc_n)+G(gw,gw,gd_n)}{4}\Big).
\end{aligned}$$

From the property of φ, we have

$$\begin{aligned}
&\frac{G(gx,gx,ga_{n+1})+G(gb_{n+1},gy,gy)+G(gz,gz,gc_{n+1})+G(gd_{n+1},gw,gw)}{4}\\
&\le\frac{G(gx,gx,ga_n)+G(gy,gy,gb_n)+G(gz,gz,gc_n)+G(gw,gw,gd_n)}{4}.
\end{aligned}$$

Hence, using $(G4)$ of Definition 1.1, we know that the sequence $\{\frac{1}{4}[G(ga_n,gx,gx)+G(gb_n,gy,gy)+G(gc_n,gz,gz)+G(gd_n,gw,gw)]\}$ is decreasing. Therefore, there exists $\alpha > 0$ such that

$$\lim_{n\to\infty}\frac{G(ga_n,gx,gx)+G(gb_n,gy,gy)+G(gc_n,gz,gz)+G(gd_n,gw,gw)}{4}=\alpha.$$

We shall show that $\alpha = 0$. Suppose to the contrary $\alpha > 0$. Taking the limit as $n \to \infty$ in (2.29), then we can get

$$\begin{aligned}
\varphi(\alpha)&\le\varphi(\alpha)-\lim_{n\to\infty}\psi\Big(\frac{G(ga_n,gx,gx)+G(gb_n,gy,gy)+G(gc_n,gz,gz)+G(gd_n,gw,gw)}{4}\Big)\\
&<\varphi(\alpha),
\end{aligned}$$

which is a contraction. Thus $\alpha = 0$, that is,

$$\lim_{n\to\infty}\frac{G(ga_n,gx,gx)+G(gb_n,gy,gy)+G(gc_n,gz,gz)+G(gd_n,gw,gw)}{4}=0.$$

This yields that

$$\begin{aligned}
&\lim_{n\to\infty}G(ga_n,gx,gx)=0,\ \ \lim_{n\to\infty}G(gb_n,gy,gy)=0,\\
&\lim_{n\to\infty}G(gc_n,gz,gz)=0,\ \ \lim_{n\to\infty}G(gd_n,gw,gw)=0.
\end{aligned}$$

Analogously, we can conclude that

$$\begin{aligned}
&\lim_{n\to\infty}G(ga_n,gu,gu)=0,\ \ \lim_{n\to\infty}G(gb_n,gv,gv)=0,\\
&\lim_{n\to\infty}G(gc_n,gr,gr)=0,\ \ \lim_{n\to\infty}G(gd_n,gl,gl)=0.
\end{aligned}$$

By the uniqueness of the limit, we can get $(gx, gy, gz, gw) = (gu, gv, gr, gl)$. Since $gx = F(x, y, z, w)$, $gy = F(y, z, w, x)$, $gz = F(z, w, x, y)$, and $gz = F(z, w, x, y)$, by commutativity of F and g, we have

$$\begin{aligned}
gx^*&=g(gx)=gF(x,y,z,w)=F(gx,gy,gz,gw),\\
gy^*&=g(gy)=gF(y,z,w,x)=F(gy,gz,gw,gx),\\
gz^*&=g(gz)=gF(z,w,x,w)=F(gz,gw,gx,gy),\\
gw^*&=g(gw)=gF(w,x,y,z)=F(gw,gx,gy,gz),
\end{aligned}$$

where $gx = x^*$, $gy = y^*$, $gz = z^*$, and $gw = w^*$. Thus, (x^*, y^*, z^*, w^*) is a quadruple coincidence point of F and g. Consequently, (gx^*, gy^*, gz^*, gz^*) and (gx, gy, gz, gw) are equal. We deduce

$$gx^* = gx = x^*,\ \ gy^* = gy = y^*,\ \ gz^* = gz = z^*,\ \ gw^* = gw = w^*.$$

Therefore, (x^*, y^*, z^*, w^*) is a quadruple common fixed point of F and g. To prove the uniqueness, assume that (p, q, i, j) is another quadruple common fixed point. Then, it is clearly that $p = gp = gx^* = x^*$, $q = gq = gy^* = y^*$, and $i = gi = gz^* = z^*$, $j = gj = gw^* = w^*$. The proof is completed. □

Next, we give an example to illustrate that Theorem 2.1 is an extension of Theorem 1.19.

Example 2.5. Let $X = \mathbf{R}$ and $(X, \preceq)$ be a partially ordered set with the natural ordering of real numbers. Let $G(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in X$. Then (X, G) is a complete G-metric space. Let the mapping $g : X \to X$ be defined by

$$g(x) = x \ \ for \ \ all \ \ x \in X,$$

and let the mapping $F : X^4 \to X$ be defined by

$$F(x, y, z, w) = \frac{x - 2y + z - 2w}{8}$$

for all $x, y, z, w \in X$. Then F satisfies the mixed g-monotone property and F commutes with g. Now, we suppose that (1.1) holds, that is, there exists $\varphi \in \Phi$ and $\psi \in \Psi$ such that (1.1) holds. This means that

$$\begin{aligned}
\varphi\big(G(F(x,y,z,w), F(u,v,s,t), F(a,b,c,d))\big) &= \varphi\Big(G(\frac{x-2y+z-2w}{8}, \frac{u-2v+s-2t}{8}, \\
&\qquad \frac{a-2b+c-2d}{8})\Big) \\
&= \varphi\Big(|\frac{x-2y+z-2w}{8} - \frac{u-2v+s-2t}{8}| \\
&\qquad +|\frac{u-2v+s-2t}{8} - \frac{a-2b+c-2d}{8}| \\
&\qquad +|\frac{a-2b+c-2d}{8} - \frac{x-2y+z-2w}{8}|\Big) \\
&\le \frac{1}{4}\varphi\big((|x-u| + |u-a| + |a-x|) + (|y-v| + |v-b| \\
&\qquad +|b-y|) + (|z-s| + |s-c| + |c-z|) + (|w-t| \\
&\qquad +|t-d| + |d-w|)\big) \\
&\qquad -\psi\big(\frac{1}{4}[(|x-u| + |u-a| + |a-x|) + (|y-v| + |v-b| \\
&\qquad +|b-y|) + (|z-s| + |s-c| + |c-z|) + (|w-t| \\
&\qquad +|t-d| + |d-w|)]\big)
\end{aligned}$$

for all $gx \geq gu \geq ga$, $gy \leq gv \leq gb$, $gz \geq gs \geq gs$ and $gw \leq gs \leq gd$. Take $gx = gu = ga$, $gy = gv = gb$, $gz = gs = gc$ and $gw \neq gt \neq gd$ in the previous inequality and denote $r = \frac{1}{4}[|w - t| + |t - d| + |d - w|]$. We get

$$\varphi(r) \le \frac{1}{4}\varphi(4r) - \psi(r), \ \ r > 0.$$

On the other hand, by (iii_φ), we have $\frac{1}{4}\varphi(4r) \le \varphi(r)$ and therefore, we deduce that, for all $r > 0$, $\psi(r) \le 0$, that is, $\psi(r) = 0$, which contradicts (i_ψ). This shows that F and g do not satisfy (1.1).

Now, we prove that (2.1) holds. Indeed, since we have

$$\begin{aligned}G(F(x,y,z,w),F(u,v,s,t),F(a,b,c,d)) &= G(\frac{x-2y+z-2w}{8},\frac{u-2v+s-2t}{8},\frac{a-2b+c-2d}{8})\\ &=|\frac{x-2y+z-2w}{8}-\frac{u-2v+s-2t}{8}|\\ &\quad+|\frac{u-2v+s-2t}{8}-\frac{a-2b+c-2d}{8}|\\ &\quad+|\frac{a-2b+c-2d}{8}-\frac{x-2y+z-2w}{8}|\\ &\le \frac{1}{8}|x-u|+\frac{1}{4}|y-v|+\frac{1}{8}|z-s|+\frac{1}{4}|w-t|\\ &\quad+\frac{1}{8}|u-a|+\frac{1}{4}|v-b|+\frac{1}{8}|s-c|+\frac{1}{4}|t-d|\\ &\quad+\frac{1}{8}|a-x|+\frac{1}{4}|b-y|+\frac{1}{8}|c-z|+\frac{1}{4}|d-w|.\end{aligned} \tag{2.30}$$

Similarly, we can achieve the following inequalities as follows:

$$\begin{aligned}G(F(y,z,w,x),F(v,s,t,u),F(b,c,d,a)) &\le \frac{1}{8}|y-v|+\frac{1}{4}|z-s|+\frac{1}{8}|w-t|+\frac{1}{4}|x-u|\\ &\quad+\frac{1}{8}|v-b|+\frac{1}{4}|s-c|+\frac{1}{8}|t-d|+\frac{1}{4}|u-a|\\ &\quad+\frac{1}{8}|b-y|+\frac{1}{4}|c-z|+\frac{1}{8}|b-w|+\frac{1}{4}|a-x|,\end{aligned} \tag{2.31}$$

$$\begin{aligned}G(F(z,w,x,y),F(s,t,u,v),F(c,d,a,b)) &\le \frac{1}{8}|z-s|+\frac{1}{4}|w-t|+\frac{1}{8}|x-u|+\frac{1}{4}|y-v|\\ &\quad+\frac{1}{8}|s-c|+\frac{1}{4}|t-d|+\frac{1}{8}|u-a|+\frac{1}{4}|v-b|\\ &\quad+\frac{1}{8}|c-z|+\frac{1}{4}|d-w|+\frac{1}{8}|a-x|+\frac{1}{4}|b-y|,\end{aligned} \tag{2.32}$$

and

$$\begin{aligned}G(F(w,x,y,z),F(t,u,v,s),F(d,a,b,c)) &\le \frac{1}{8}|w-t|+\frac{1}{4}|x-u|+\frac{1}{8}|y-v|+\frac{1}{4}|z-s|\\ &\quad+\frac{1}{8}|t-d|+\frac{1}{4}|u-a|+\frac{1}{8}|v-b|+\frac{1}{4}|s-c|\\ &\quad+\frac{1}{8}|d-w|+\frac{1}{4}|a-x|+\frac{1}{8}|b-y|+\frac{1}{4}|c-z|.\end{aligned} \tag{2.33}$$

Combined with (2.30)-(2,33), we can get

$$\begin{aligned}&\frac{1}{4}\big[G(F(x,y,z,w),F(u,v,s,t),F(a,b,c,d))+G(F(y,z,w,x),F(v,s,t,u),F(b,c,d,a))\\ &\qquad G(F(z,w,x,y),F(s,t,u,v),F(c,d,a,b))+G(F(w,x,y,z),F(t,u,v,s),F(d,a,b,c))\big]\\ &\le \frac{1}{4}\times\frac{6}{8}\big[|x-u|+|w-t|+|z-s|+|w-t|+|u-a|+|v-b|+|s-c|+|t-d|\\ &\qquad+|a-x|+|b-y|+|c-z|+|d-w|\big].\\ &=\frac{3}{16}\big[|x-u|+|w-t|+|z-s|+|w-t|+|u-a|+|v-b|+|s-c|+|t-d|\\ &\qquad+|a-x|+|b-y|+|c-z|+|d-w|\big].\end{aligned} \tag{2.34}$$

On the other hand, from (2.1), we have

$$\begin{aligned}&\frac{1}{4}\big[G(gx,gu,ga)+G(gy,gv,gb)+G(gz,gs,gc)+G(gw,gt,gd)\big]\\ &=\frac{1}{4}\big[G(x,u,a)+G(y,v,b)+G(z,s,c)+G(w,t,d)\big]\\ &=\frac{1}{4}\big[|x-u|+|w-t|+|z-s|+|w-t|+|u-a|+|v-b|+|s-c|+|t-d|\\ &\qquad+|a-x|+|b-y|+|c-z|+|d-w|\big].\end{aligned} \tag{2.35}$$

By (2.34) and (2.35), If we take $\varphi(t) = \frac{1}{2}t$ and $\psi(t) = \frac{1}{8}t$, then (2.1) holds with noting that $x_0 = -2$, $y_0 = 3$, $z_0 = -2$ and $w_0 = 3$. So by our Theorem 2.1 we obtain that F and g have a quadruple coupled fixed point (0,0,0,0) but Theorem 1.1 does not apply to F in this example. Hence, our results generalize and extend Theorem 1.19.

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