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Analytic and loop solutions for the K(2,2) equation (focusing branch)

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Abstract

In this paper, we study analytic and loop solutions of the K(2,2) equation(focusing branch), which is first proposed by Rosenau. The implicit analytic and loop solutions are obtained by using the dynamical system approach. Moreover, we investigate how the famous Rosenau-Hyman compactons can be recovered as limits of classical solitary wave solutions forming analytic homoclinic orbits for the reduced dynamical system by theoretical analysis and numerical simulation.

Keywords: Loop solution, peakon, compacton, solitary wave, K(2,2) equation.
2010 MSC: 35C08, 37K40.

1. Introduction and Preliminaries

It is well known that the study of nonlinear wave equations and their solutions are of great importance in many areas of physics. Classically, the solitary wave solutions of nonlinear wave equations are determined by analytic formulate (typically a $sech^2$ function or variants thereof) and serve as prototypical solutions that model physical localized waves. In the case of integrable systems, the solitary waves interact cleanly, and are known as solitons. For many examples, localized initial data ultimately break up into a finite collection of solitary wave solutions; this fact has been proved analytically for certain integrable equations such as the Korteweg-de Vries equation

$$u_t + uu_x + u_{xxx} = 0. \tag{1.1}$$

The appearance of non-analytic solitary wave solutions to new classes of nonlinear wave equations has vastly increased the menagerie of solutions appearing in model equations. The distinguishing feature of the systems

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admitting non-analytic solitary wave solutions is that, in contrast to the classical nonlinear wave equations, they all include a nonlinear dispersion term, meaning that the highest order derivatives (characterizing the dispersion relation) do not occur linearly in the system, but are typically multiplied by a function of the dependent variable.

The most important of the nonlinearly dispersive, integrable equation is the well-known Camassa-Holm equation [1]

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{1.2}$$

This equation has peaked solitary wave solution

$$u(x,t) = ce^{-|x-ct|}, \tag{1.3}$$

which have discontinuous first derivative at the wave peak in contrast to the smoothness of most previously known specious of solitary waves and thus are called peakons. Eq. (1.3) arise as models for shallow water waves. The peakons capture a characteristic of the traveling waves of greatest height-exact traveling solutions of the governing equations for water waves with a peak at their crest. Simper approximated shallow water models (like the classical Korteweg-de Vries equation) do not present traveling wave solutions with this feature. The peakons are to be understood as weak solutions in the sense of the papers [4].

In 1993, Rosenau and Hyman [11] introduced and studied a family of fully nonlinear dispersion Korteweg-de Vries equations(K(m,n)),

$$u_t \pm (u^m)_x + (u^n)_{xxx} = 0, \tag{1.4}$$

where both the convection term $(u^m)_x$ and the dispersion effect term $(u^n)_{xxx}$ are nonlinear. The $(+)$ case is known as the focusing branch and the $(-)$ case as the defocusing branch. These equations arise in the process of understanding the role of nonlinear dispersion in the formation of structures like liquid drops. Rosenau and Hyman derived a kind of solitary wave solutions called compactons for the K(2,2) equation(focusing branch)

$$u_t + (u^2)_x + (u^2)_{xxx} = 0. \tag{1.5}$$

The compactons have compact support. That is, they vanish identically outside a finite core region. The compactons were also found for other nonlinear dispersive equations [10, 12]. Mihaila et al. [8] studied the stability and dynamical properties of K(2,2) Rosenau-Hyman compactons.

Recently, Deng, Parkes and Cao [3] studied the K(2,2) equation(defocusing branch)

$$u_t - (u^2)_x + (u^2)_{xxx} = 0. \tag{1.6}$$

They obtained some new exact traveling wave solutions such as loop solitons, cuspons and periodic wave solutions by using the auxiliary elliptic equation method. In 1998, Vakhnenko and Parkes [13] found loop solution for the Vakhnenko equation. Moreover, the loop solutions were also found for the short-pulse equation by Parkes [9]. Li and Zhang [7] obtained the loop solutions for CH-DP equation by using bifurcation method.

Furthermore, some modified K(2,2) equations have also been studied by many authors. For example, Zhou and Tian [16] introduced the osmosis K(2,2) equation

$$u_t + (u^2)_x - (u^2)_{xxx} = 0, \tag{1.7}$$

where the negative coefficient of dispersion term denotes the contracting dispersion. They obtained some solitary wave solutions for Eq. (1.7). Chen and Li [2] studied the single peak solitary wave solutions for the osmosis K(2,2) equation under inhomogeneous boundary condition.

In this paper, we study the analytic and loop solutions for the K(2,2) equation (1.5) by using dynamical systems method. We also discuss how the famous Rosenau-Hyman compacton can be recovered as limits of classical solitary wave solutions forming analytic homoclinic orbits for the reduced dynamical system by theoretical analysis and numerical simulation.

The paper is organized as follows. In Subsection 2.1, we give analytic and loop solutions of the K(2,2) equation (1.5). In Subsection 2.2, we discuss the convergence of this solitary wave solutions.

2. Main results

2.1. Analytic and loop solutions

In this subsection, we study analytic and loop solutions. Substituting $u(x,t) = u(\xi)$ and $\xi = x - ct$ into Eq. (1.5), we have

$$-cu_\xi + (u^2)_\xi + (u^2)_{\xi\xi\xi} = 0. \tag{2.1}$$

Integrating (2.1) once, we obtain

$$2uu_{\xi\xi} = a + cu - u^2 - 2(u_\xi)^2, \tag{2.2}$$

where a is an integration constant. Let $y = u_\xi$ and $d\xi = 2u^2 d\zeta$, then we obtain the Hamiltonian vector field

$$\frac{dX(\zeta)}{d\zeta} = JH_X(X), \tag{2.3}$$

where $X = (u, y)^\top$,

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$H(u,y) = u^2y^2 + \frac{1}{4}u^4 - \frac{c}{3}u^3 - \frac{a}{2}u^2. \tag{2.4}$$

The system (2.3) has various dynamical behavior for different parametric conditions. It shows that there is only one fixed point for $a = -\frac{c^2}{4}$. When a increases and passes through $a = -\frac{c^2}{4}$, a homoclinic loop appears(see Fig. 1) and a saddle-node bifurcation occurs.

For the condition $-\frac{c^2}{4} < a < 0$, the system (2.3) has two equilibrium points $(u_1, 0)$ and $(u_2, 0)$, where

$$u_1 = \frac{1}{2}(c + \sqrt{c^2 + 4a}), \quad u_2 = \frac{1}{2}(c - \sqrt{c^2 + 4a}). \tag{2.5}$$

We assume that $c < 0$ throughout the paper, since there are similar results for $c > 0$. Let $h_1 = H(u_1, 0)$, then the homoclinic orbit connecting the equilibrium point $(u_1, 0)$ is determined by the algebraic equation

$$u^2y^2 + \frac{1}{4}u^4 - \frac{c}{3}u^3 - \frac{a}{2}u^2 - h_1 = 0. \tag{2.6}$$

The function $F(\phi) = 4h_1 + 2au^2 + \frac{4c}{3}u^3 - u^4$ has two real zeros α, β and a double zero u_1. This imply

$$F(u) = (u - u_1)^2(\alpha - u)(u - \beta). \tag{2.7}$$

The transformation $y = u_\xi$ and Eq. (2.6) allow us to use Leibnitz rule and conclude that

$$\int_\beta^u \frac{sds}{\sqrt{(s - \phi_1)^2(\alpha - s)(s - \beta)}} = \pm\frac{1}{2}\xi. \tag{2.8}$$

Corresponding to the homoclinic orbit, we obtain the implicit expression of analytic soliton solution

$$F_1(u) - \frac{u_1}{\sqrt{(\alpha - u_1)(u_1 - \beta)}} F_2(u) = K_1 \pm \frac{1}{2}\xi, \tag{2.9}$$

where

$$F_1(u) = \arctan\left(\frac{2u - \alpha - \beta}{2\sqrt{(\alpha - u)(u - \beta)}}\right),$$

$$F_2(u) = \ln\left|\frac{(\alpha+\beta)(u+u_1) - 2\alpha\beta - 2u_1u + 2\sqrt{(\alpha - u_1)(u_1 - \beta)(\alpha - u)(u - \beta)}}{u - u_1}\right|,$$

$$K_1 = -\frac{\pi}{2} - \frac{u_1}{\sqrt{(\alpha - u_1)(u_1 - \beta)}} \ln(\alpha - \beta).$$

On the other hand, the homoclinic tails and connecting orbit provide the loop solution. By using the transformation $y = u_\xi$ and Eq. (2.6) to do the integration, we have

$$\int_u^\alpha \frac{sds}{\sqrt{(s-u_1)^2(\alpha-s)(s-\beta)}} = \pm\frac{1}{2}\xi. \tag{2.10}$$

Thus we obtain implicit expression of loop solution

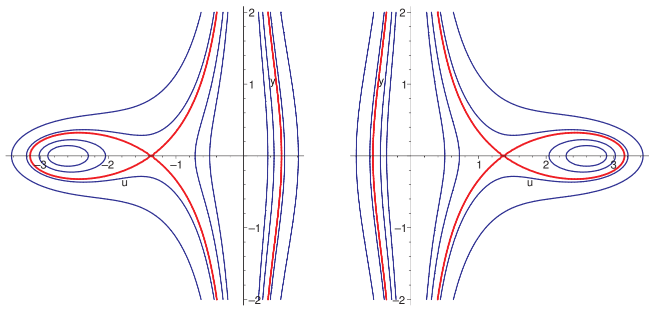
$$F_1(u) - \frac{u_1}{\sqrt{(\alpha-u_1)(u_1-\beta)}}F_2(u) = K_2 \pm \frac{1}{2}\xi, \tag{2.11}$$

where

$$K_2 = \frac{\pi}{2} - \frac{u_1}{\sqrt{(\alpha-u_1)(u_1-\beta)}}\ln(\alpha-\beta). \tag{2.12}$$

Remark 2.1. The loop solution, that is, the so-called loop soliton solution, is not one real soliton solution (see [5, 6, 15]).

Remark 2.2. We obtain the expression of the loop solution and the expression of the loop solution in [14] is different.



(1-1) $c < 0$ (1-2) $c > 0$

Fig. 1. Phase portraits of system (2.3) for $-\frac{c^2}{4} < a < 0$.

2.2. Convergence of analytic and loop solution

In this subsection, we investigate how the nonanalytic solitary wave solutions-compactons can be recovered as limits of classical solitary wave solutions forming analytic homoclinic orbits for the reduced dynamical system.

We let $C^k = C^k(R)$ denote the space of k times continuously differentiable functions defined on the real axis. The space of all infinitely differentiable functions with compact support in R is denoted by $C_c^\infty = C_c^\infty(R)$. The space $L^p = L^p(R)$ with $1 \le p \le \infty$ consists of all pth-power Lebesgue-integrable functions defined on the real line R with the usual modification if $p = \infty$. The standard norm of a function $f \in L^p$ will be denoted by $\parallel f \parallel_p$. The inner product of two functions f and g in L^2 is the integral

$$< f, g >= \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx, \tag{2.13}$$

where the overbar denotes complex conjugation. For any integer $k \ge 0$ and constant $p \ge 1$, the Sobolev space $W^{k,p} = W^{k,p}(R)$ consists of all tempered distributions f such that $f^{(m)} \in L^p$ for all $0 \le m \le k$. The space $W^{k,p}$ is usually denoted by H^k.

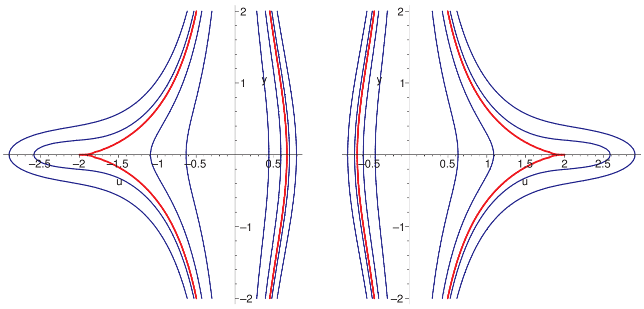
When $a \to -\frac{c^2}{4}$, the homoclinic orbit vanishes and three open curves appear(see Fig. 2). Consequently, the analytic solitary waves shrink to zero while loop waves converge to a limit loop solution. We do directly the integration

$$\int_u^{-\frac{c}{6}} \frac{sds}{\sqrt{(\frac{c}{2}-s)^3(s+\frac{c}{6})}} = \pm\frac{1}{2}\xi. \tag{2.14}$$

Then we obtain implicit expression of loop solution

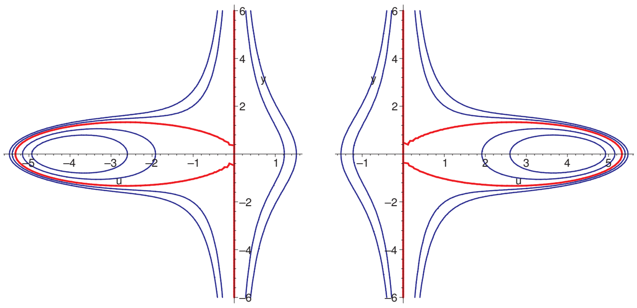
$$\arctan\left(\frac{\sqrt{3}(6u-c)}{3\sqrt{(c-2u)(c+6u)}}\right)+\frac{\sqrt{3(c-2u)(c+6u)}}{2(c+2u)}=\frac{\pi}{2}\pm\frac{1}{2}\xi, \tag{2.15}$$

The profiles of loop solitary waves are shown in Fig. 5(5-1) and Fig. 6(6-1).



(2-1) $c<0$ (2-2) $c>0$

Fig. 2. Phase portraits of system (2.3) for $4a+c^2=0$.



(3-1) $c<0$ (3-2) $c>0$

Fig. 3. Phase portraits of system (2.3) for $a=0$.

When $a=0$, the origin is a singular point for the dynamical system (2.3), which implies that the associated homoclinic orbit is traversed in finite time(see Fig. 3). Therefore, the corresponding travelling wave solution is no longer analytic, but has compact support. In the present case, the dynamical system can be integrated explicitly, leading to the compacton(see Fig. 4, Fig. 5(5-5) and Fig. 6(6-5))

$$\begin{cases} u(x,t)=\frac{4c}{3}\cos^2(\frac{x-ct}{4}), & |x-ct|\le 2\pi, \\ u(x,t)=0, & otherwise, \end{cases} \tag{2.16}$$

which is a weak solution of (1.5) in the following sense.

Definition 2.3. A solitary wave $u(\xi)$ with undisturbed depth $A=\lim_{|\xi|\to\infty}u(\xi)$ is a weak solution of the differential equation (2.1) if and only if $\varphi=u-A\in H^1$, and

$$<u^2-cu,g'>+<u^2,g'''>=0, \tag{2.17}$$

for any $g\in C_c^\infty(R)$.

We can see that there exists an analytic soliton solution $A+\varphi(\xi)$ for each $A\in(\frac{c}{2},0)$ if $-\frac{c^2}{4}<a<0$ from Subsection 2.1. Let $\varepsilon=A$, then solitary wave solutions satisfy the differential equation

$$4(u_A+\varepsilon)^2(u_A')^2=u_A^2(B_1-u_A)(u_A-B_2), \tag{2.18}$$

where

$$B_1=\frac{1}{3}(2c-6\varepsilon+\sqrt{4c^2-6c\varepsilon+6\varepsilon^2}) \tag{2.19}$$

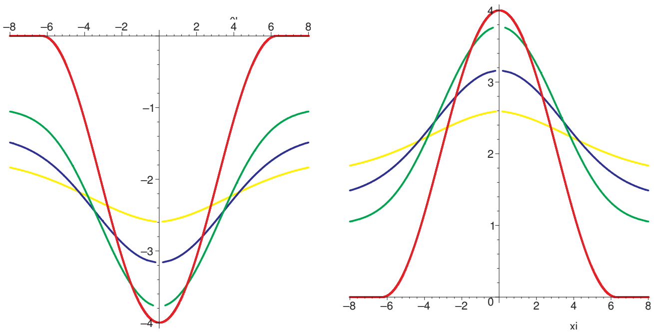
and

$$B_2 = \frac{1}{3}(2c - 6\varepsilon - \sqrt{4c^2 - 6c\varepsilon + 6\varepsilon^2}). \tag{2.20}$$

Using the inequality $B_2 < u_A(\xi) < A$ valid for all $\xi \in R$, one may show that sequences of functions φ'_ε and φ''_ε are uniformly bounded on the real axis. Therefore, the Ascoli-Arzelà Theorem shows that, as $\varepsilon \to 0$, there exist subsequences of the families φ_ε and φ'_ε, without loss of generality still denoted by φ_ε and φ'_ε, which are uniformly convergent to a function u and its derivative u', respectively, on any compact set of R. Here we are relying on the fact that each φ_ε is an even function, since φ_ε is symmetric with respect to its elevation and translation invariant. Taking the limit on both sides of (2.18) as $\varepsilon \to 0$ leads to the equation

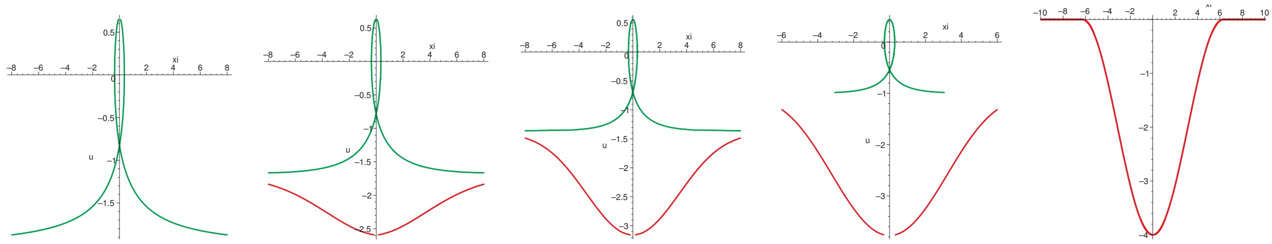
$$4u_A^2(u'_A)^2 = -u_A^3(u_A - \frac{4c}{3}). \tag{2.21}$$

Since $\lim_{\varepsilon\to 0} \max \varphi(\xi) = u_1 < 0$ and each φ_ε is even, monotone on each side of the origin and exponentially decaying to zero at infinity, the limiting function φ_ε is a nontrivial solution of (2.21). Therefore, as an even and monotone decreasing function on the positive real axis, $\varphi_0 = u$ is the compacton solution (2.16).



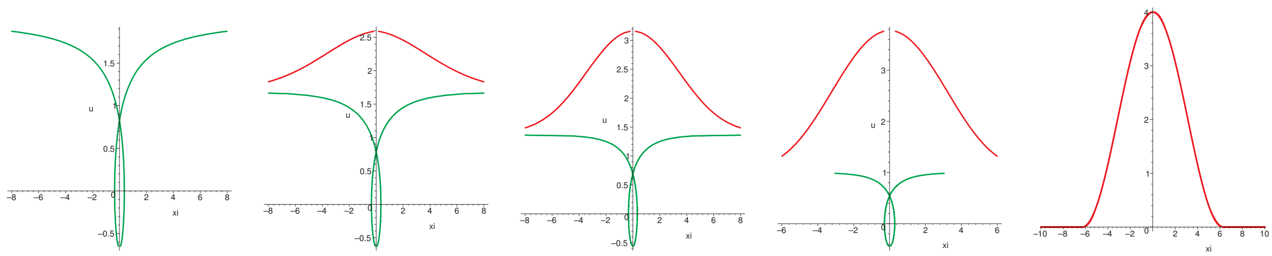
(4-1) $c < 0$ (4-2) $c > 0$

Fig. 4. Analytic solitons and compactons of Eq. (1.5).



(5-1) (5-2) (5-3) (5-4) (5-5)

Fig. 5. Convergence of analytic solitons and loop solitons of Eq. (1.5) for $c < 0$.



(6-1) (6-2) (6-3) (6-4) (6-5)

Fig. 6. Convergence of analytic solitons and loop solitons of Eq. (1.5) for $c > 0$.

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