



Tripled fixed point theorems in cone metric spaces under F -invariant set and c -distance

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Abstract

The concept of cone metric spaces has been introduced recently as a generalization of metric spaces. The aim of this paper is to give the definitions of F -invariant sets denoted by M in case of $M \in X^6$ in cone and ordered cone version. we also establish some tripled fixed point theorems in cone metric spaces under the concept of an F -invariant set for mappings $F : X^3 \rightarrow X$ and c -distance on the one hand, and in partially ordered cone metric spaces under the same concepts on the other hand. The present theorems expand and generalize several well-known comparable results in literature in cone metric spaces and ordered cone metric spaces, respectively. An interesting example is given to support our results. ©2015 All rights reserved.

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1. Introduction and Preliminaries

In the mid-20th century, K -metric and K -normed spaces were introduced see [24, 41, 43] by using an ordered Banach space instead of the set of real numbers, as the codomain for a metric. In 2007, Huang and Zhang [21] introduced the concept of cone metric spaces as a generalization of metric spaces and defined convergent and Cauchy sequences in the terms of interior points of the underlying cone. They also proved some fixed point theorems in such spaces in the same work. Afterwards, many articles in proving fixed point theorems in cone metric spaces have appeared, for more information see [1, 4, 22, 32].

In [8], Bhaskar and Lakshmikantham have introduced the concept of mixed monotone property and proved fixed point in partially ordered metric spaces. Then, they have evidenced coupled fixed point theorems for mappings that satisfy mixed monotone property and applied their theorems to produce some applications

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in the problems of existence and uniqueness of solution for a periodic boundary value problem. Because of their important results, a lot of articles in that topic have been dedicated to improve and generalize fixed point theorems using mixed monotone property, see [2, 12, 20, 36, 37, 39]. In [28], Lakshmikantham and Ćirić have introduced the concept of mixed g -monotone property and proved coupled coincidence and common coupled fixed point theorems for mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ for such nonlinear contractive mappings in partially ordered complete metric spaces. In 2009, Sabetghadam *et al.* [33] considered the definition of coupled fixed point for a mapping in complete cone metric space and proved some coupled fixed point theorems. For more results about the study of the coupled fixed point and common coupled fixed point, see for example [3, 25, 31, 35].

In 2011, Berinde *et al.* [7], introduced the definition of mixed monotone property and the definition of tripled fixed point for mapping $F : X \times X \times X \rightarrow X$ and proved tripled fixed point theorems for contractive type mappings having that property in partially ordered metric spaces. Furthermore, Borcut *et al.* [10] and Borcut [9] have introduced the concept of a tripled coincidence point for a pair of nonlinear contractive mappings $F : X^3 \rightarrow X$ and $g : X \rightarrow X$ for some general classes of contractive type mappings, ([10] have generalized the results of [28]). Subsequently, Karapinar [26], has proved some new tripled fixed point theorems by using a generalization of the results of Luong and Thuang [29]. After that, Borcut *et al.* [11], have presented new results of the existence and uniqueness of tripled fixed points for nonlinear mappings in partially ordered complete metric spaces that extend the results in the previous works.

In 2010, Samet and Vetro [34] have established some new coupled fixed point theorems in complete metric space and have introduced the definition of fixed point of N -order of the mapping $F : X^N \rightarrow X$ and the definition of F -invariant subset of X^{2N} in complete metric spaces. Note that, Berinde *et al.* [7] defined differently the notion of a tripled fixed point in the case of ordered sets in order to keep true the mixed monotone property. On the other hand, Aydi *et al.* [6] and (Murthy and Rashmi [30]) have studied common tripled fixed point theorem for W -compatible (w -compatible) mappings in abstract metric spaces (ordered cone metric spaces), respectively. Very recently, Abusalim and Noorani [5], have established some new tripled coincidence point and common tripled fixed point theorems in cone metric spaces.

Recently in 2011, Cho *et al.* [14] introduced a new fantastic idea that known as c -distance in cone metric space as a popularization of w -distance of Kada *et al.* [23]. They also have proved some fixed point theorems in partially ordered cone metric spaces using that concept. Subsequently, Sintunavarat *et al.* [38] have studied fixed point and common fixed point theorems for generalized contraction mappings using c -distance, too. After that, a large number of articles have appeared in studying fixed point, common fixed point, coupled fixed point, common coupled fixed point, tripled fixed point and tripled coincidence point theorems in cone metric spaces under c -distance idea. The reader may see [5, 15, 17, 18, 19, 42].

In 2012, new coupled fixed point theorems under contraction mappings by using the concept of mixed monotone property and c -distance in partially ordered cone metric spaces have been established by Cho *et al.* [13] as the followings:

Theorem 1.1. *Let (X, \sqsubseteq) be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c -distance on X and $F : X \times X \rightarrow X$ be a continuous function having the mixed monotone property such that*

$$q(F(x, y), F(x^*, y^*)) \preceq \frac{k}{2}(q(x, x^*) + q(y, y^*))$$

for some $k \in [0, 1)$ and all $x, y, x^*, y^* \in X$ with

$$(x \sqsubseteq x^*) \wedge (y \supseteq y^*) \quad \text{or} \quad (x \supseteq x^*) \wedge (y \sqsubseteq y^*).$$

If there exist $x_0, y_0 \in X$ such that

$$x_0 \sqsubseteq F(x_0, y_0) \quad \text{and} \quad F(y_0, x_0) \sqsubseteq y_0,$$

then F has a coupled fixed point (u, v) . Moreover, we have $q(u, u) = \theta$ and $q(v, v) = \theta$.

After that, Sintunavarat *et al.* [40] have weakened the condition of mixed monotone property in results of Cho *et al.* [13] by using the concept of F -invariant set that have been introduced by [34], but in cone version as below:

Theorem 1.2. *Let (X, d) be a complete cone metric space. Let q be a c -distance on X , M be a nonempty subset of X^4 and $F : X \times X \rightarrow X$ be a continuous function such that*

$$q(F(x, y), F(x^*, y^*)) \preceq \frac{k}{2}(q(x, x^*) + q(y, y^*))$$

for some $k \in [0, 1)$ and all $x, y, x^*, y^* \in X$ with

$$(x, y, x^*, y^*) \in M \quad \text{or} \quad (x^*, y^*, x, y) \in M.$$

If M is F -invariant and there exist $x_0, y_0 \in X$ such that

$$(F(x_0, y_0), F(y_0, x_0), x_0, y_0) \in M,$$

then F has a coupled fixed point (u, v) . Moreover, if $(u, v, u, v) \in M$, then $q(u, u) = \theta$ and $q(v, v) = \theta$.

Very recently, Karapinar *et al.* [27], have evidenced some coupled fixed point theorems in cone metric spaces by using the concept of an F -invariant set. Further, they have given an example that is not applied in the results of both Sintunavarat *et al.* [40] and Cho *et al.* [13], but can be applied into their results. They also have applied their results in partially ordered cone metric spaces and consider an application to solve some integral equations.

Theorem 1.3 ([13]). *Let (X, d) be a complete cone metric space. Let q be a c -distance on X , M be a nonempty subset of X^4 and $F : X \times X \rightarrow X$ be a continuous function such that*

$$q(F(x, y), F(x^*, y^*)) + q(F(y, x), F(y^*, x^*)) \preceq k(q(x, x^*) + q(y, y^*))$$

for some $k \in [0, 1)$ and all $x, y, x^*, y^* \in X$ with

$$(x, y, x^*, y^*) \in M \quad \text{or} \quad (x^*, y^*, x, y) \in M.$$

If M is F -invariant and there exist $x_0, y_0 \in X$ such that

$$(F(x_0, y_0), F(y_0, x_0), y_0) \in M,$$

then F has a coupled fixed point (u, v) . Furthermore, if $(u, v, u, v) \in M$, then $q(u, u) = \theta$ and $q(v, v) = \theta$.

Now, we recall some important definitions that we need in our results.

Let E be a real Banach space and θ denote to the zero element in E . A cone P is a subset of E such that:

1. P is nonempty set closed and $P \neq \{\theta\}$,
2. If a, b are nonnegative real numbers and $x, y \in P$ then $ax + by \in P$,
3. $x \in P$ and $-x \in P$ implies $x = \theta$.

For any cone $P \subset E$, the partial ordering \preceq with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. The notation of \prec stand for $x \preceq y$ but $x \neq y$. Also, we used $x \ll y$ to indicate that $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of P . A cone P is called normal if there exists a number K such that

$$\theta \preceq x \preceq y \implies \|x\| \leq K\|y\|,$$

for all $x, y \in E$. The least positive number K satisfying the above condition is called the normal constant of P .

Definition 1.4 ([21]). Let X be a nonempty set and E be a real Banach space equipped with the partial ordering \preceq with respect to the cone P . Suppose that the mapping $d : X \times X \rightarrow E$ satisfies the following condition:

1. $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$ for all $x, y \in X$,
3. $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.5 ([21]). Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

1. for all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x) \ll c$ for all $n > N$ then x_n is said to be convergent and x is the limit of $\{x_n\}$. We denote this by $x_n \rightarrow x$.
2. for all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all $n, m > N$ then $\{x_n\}$ is called a Cauchy sequence in X .
3. a cone metric space (X, d) is called complete if every Cauchy sequence in X is convergent.

Lemma 1.6. ([22]).

1. If E is a real Banach space with a cone P and $a \preceq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.
2. If $c \in \text{int}P$, $\theta \preceq a_n$ and $a_n \rightarrow \theta$, then there exists a positive integer N such that $a_n \ll c$ for all $n \geq N$.

Next we give the notation of c -distance on a cone metric space which is a generalization of ω -distance of Kada *et al.* [23] with some properties.

Definition 1.7 ([14]). Let (X, d) be a cone metric space. A function $q : X \times X \rightarrow E$ is called a c -distance on X if the following conditions hold:

- (q1) $\theta \preceq q(x, y)$ for all $x, y \in X$,
- (q2) $q(x, y) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$,
- (q3) for each $x \in X$ and $n \geq 1$, if $q(x, y_n) \preceq u$ for some $u = u_x \in P$, then $q(x, y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$,
- (q4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Example 1.8 ([14]). Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$. Let $X = [0, \infty)$ and define a mapping $d : X \times X \rightarrow E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q : X \times X \rightarrow E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a c -distance on X .

Example 1.9 ([17, 16]). Let $E = \mathbb{R}^2$ and $P = \{(x, y) \in E : x, y \geq 0\}$. Let $X = [0, 1]$ and define a mapping $d : X \times X \rightarrow E$ by $d(x, y) = (|x - y|, |x - y|)$ for all $x, y \in X$. Then (X, d) is a complete cone metric space. Define a mapping $q : X \times X \rightarrow E$ by $q(x, y) = (y, y)$ for all $x, y \in X$. Then q is a c -distance on X .

Lemma 1.10 ([14]). Let (X, d) be a cone metric space and q is a c -distance on X . Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$. Suppose that u_n is a sequence in P converging to θ . Then the following hold:

1. If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $y = z$.
2. If $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $\{y_n\}$ converges to z .
3. If $q(x_n, x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X .
4. If $q(y, x_n) \preceq u_n$, then $\{x_n\}$ is a Cauchy sequence in X .

Remark 1.11 ([14]). 1. $q(x, y) = q(y, x)$ does not necessarily for all $x, y \in X$.

2. $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

Definition 1.12 ([8]). An element $(x, y) \in X^2$ is said to be a coupled fixed point of the mapping $F : X^2 \rightarrow X$ if $F(x, y) = x$ and $F(y, x) = y$.

The following definition has been taken from [34]

Definition 1.13. Let X be a non-empty set and $F : X^N \rightarrow X$ be a given mapping ($N \geq 2$). An element $(x_1, x_2, \dots, x_N) \in X^N$ is said to be a fixed point of N -order of the mapping F if

$$\begin{cases} x_1 = F(x_1, x_2, \dots, x_N), \\ x_2 = F(x_2, x_3, \dots, x_N, x_1), \\ \vdots \\ x_N = F(x_N, x_1, \dots, x_{N-1}). \end{cases}$$

If $N = 3$, then we have the following definition:

Definition 1.14. An element $(x, y, z) \in X^3$ is said to be a tripled fixed point of the mapping $F : X^3 \rightarrow X$ if $F(x, y, z) = x$, $F(y, z, x) = y$ and $F(z, x, y) = z$.

Berinde *et al.* [7], have defined differently the notion of a tripled fixed point for mapping $F : X \times X \times X \rightarrow X$ in the case of ordered sets in order to keep true the mixed monotone property as below.

Definition 1.15 ([7]). An element $(x, y, z) \in X^3$ is said to be a tripled fixed point of the mapping $F : X^3 \rightarrow X$ if $F(x, y, z) = x$, $F(y, x, y) = y$ and $F(z, y, x) = z$.

For the following definitions, (X, \sqsubseteq) denotes a partially ordered set. By $x \sqsubseteq y$, we mean $x \sqsubseteq y$ but $x \neq y$. A mapping $f : X \rightarrow X$ is said to be non-decreasing (non-increasing) if for all $x, y \in X$, $x \sqsubseteq y$ implies $f(x) \sqsubseteq f(y)$ ($f(y) \sqsubseteq f(x)$), respectively.

Definition 1.16 ([7]). Let (X, \sqsubseteq) be a partially ordered set and $F : X^3 \rightarrow X$. We say that F has the mixed monotone property if $F(x, y, z)$ is non-decreasing in x, z and is non-increasing in y , that is, for any $x, y, z \in X$,

$$x_1, x_2 \in X, x_1 \sqsubseteq x_2 \quad \Rightarrow \quad F(x_1, y, z) \sqsubseteq F(x_2, y, z),$$

$$y_1, y_2 \in X, y_1 \sqsubseteq y_2 \quad \Rightarrow \quad F(x, y_1, z) \supseteq F(x, y_2, z),$$

and

$$z_1, z_2 \in X, z_1 \sqsubseteq z_2 \quad \Rightarrow \quad F(x, y, z_1) \sqsubseteq F(x, y, z_2).$$

2. Main results

In this section, we evidence some tripled fixed point theorems under the concept of c -distance by using the idea of F -invariant in cone metric spaces and apply our results in partially ordered cone metric spaces. Foremost, we give the definition of an F -invariant set in cone version.

Definition 2.1. Let (X, d) be a cone metric space and $F : X^3 \rightarrow X$ be a given mapping. Let M be a nonempty subset of X^6 . We say that M is an F -invariant subset of X^6 if and only if for all $x, y, z, w, e, s \in X$, we have

$$F_1 (x, y, z, w, e, s) \in M \Leftrightarrow (s, e, w, z, y, x) \in M;$$

$$F_2 (x, y, z, w, e, s) \in M \Rightarrow (F(x, y, z), F(y, z, x), F(z, x, y), F(w, e, s), F(e, s, w), F(s, w, e)) \in M.$$

We obtain that the set $M = X^6$ is trivially F -invariant.

Example 2.2. Let (X, d) be a cone metric space endowed with a partial order \sqsubseteq . Let $F : X^3 \rightarrow X$ be a mapping satisfying the mixed monotone property; that is, for all $x, y, z \in X$, we have

$$x_1, x_2 \in X, \quad x_1 \sqsubseteq x_2 \quad \Rightarrow \quad F(x_1, y, z) \sqsubseteq F(x_2, y, z),$$

$$y_1, y_2 \in X, \quad y_1 \sqsubseteq y_2 \quad \Rightarrow \quad F(x, y_1, z) \sqsupseteq F(x, y_2, z),$$

and

$$z_1, z_2 \in X, \quad z_1 \sqsubseteq z_2 \quad \Rightarrow \quad F(x, y, z_1) \sqsubseteq F(x, y, z_2).$$

Define the subset $M \subseteq X^6$ by

$$M = \{(a, b, c, d, e, s) : d \sqsubseteq a, b \sqsubseteq e, s \sqsubseteq c\}.$$

Then, M is F -invariant of X^6

Now, we prove some tripled fixed point theorems on cone metric space under c -distance using the concept of F -invariant.

Theorem 2.3. Let (X, d) be a complete cone metric space and q be a c -distance on X . Let M be a nonempty subset of X^6 and $F : X^3 \rightarrow X$ be a function such that

$$\begin{aligned} q(F(x, y, z), F(x^*, y^*, z^*)) + q(F(y, z, x), F(y^*, z^*, x^*)) + q(F(z, x, y), F(z^*, x^*, y^*)) \\ \leq k(q(x, x^*) + q(y, y^*) + q(z, z^*)) \end{aligned} \tag{2.1}$$

for some $k \in [0, 1)$ and all $x, y, z, x^*, y^*, z^* \in X$ with

$$(x, y, z, x^*, y^*, z^*) \in M \quad \text{or} \quad (x^*, y^*, z^*, x, y, z) \in M.$$

If M is F -invariant and there exist $x_0, y_0, z_0 \in X$ such that

$$(F(x_0, y_0, z_0), F(y_0, z_0, x_0), F(z_0, x_0, y_0), x_0, y_0, z_0) \in M,$$

then F has a tripled fixed point (u, v, w) . Furthermore, if $(u, v, w, u, v, w) \in M$, then $q(u, u) = \theta$, $q(v, v) = \theta$ and $q(w, w) = \theta$.

Proof. Since $F(X \times X \times X) \subseteq X$, we can construct three sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X such that

$$\begin{aligned} x_n = F(x_{n-1}, y_{n-1}, z_{n-1}), \quad y_n = F(y_{n-1}, z_{n-1}, x_{n-1}) \quad \text{and} \\ z_n = F(z_{n-1}, x_{n-1}, y_{n-1}). \end{aligned} \tag{2.2}$$

for all $n \in \mathbb{N}$. Since

$$(F(x_0, y_0, z_0), F(y_0, z_0, x_0), F(z_0, x_0, y_0), x_0, y_0, z_0) = (x_1, y_1, z_1, x_0, y_0, z_0) \in M,$$

and M is an F -invariant set, we get

$$(F(x_1, y_1, z_1), F(y_1, z_1, x_1), F(z_1, x_1, y_1), F(x_0, y_0, z_0), F(y_0, z_0, x_0), F(z_0, x_0, y_0)) = (x_2, y_2, z_2, x_1, y_1, z_1) \in M.$$

Again, using the fact that M is an F -invariant set, we have

$$(F(x_2, y_2, z_2), F(y_2, z_2, x_2), F(z_2, x_2, y_2), F(x_1, y_1, z_1), F(y_1, z_1, x_1), F(z_1, x_1, y_1)) = (x_3, y_3, z_3, x_2, y_2, z_2) \in M.$$

By repeating the argument similar to the above, we get

$$\begin{aligned} & (F(x_{n-1}, y_{n-1}z_{n-1}), F(y_{n-1}, z_{n-1}, x_{n-1}), F(z_{n-1}, x_{n-1}, y_{n-1}), x_{n-1}, y_{n-1}, z_{n-1}) \\ & = (x_n, y_n, z_n, x_{n-1}, y_{n-1}, z_{n-1}) \in M. \end{aligned}$$

for all $n \in \mathbb{N}$. From (2.1), we have

$$\begin{aligned} q(x_n, x_{n+1}) + q(y_n, y_{n+1}) + q(z_n, z_{n+1}) & = q(F(x_{n-1}, y_{n-1}z_{n-1}), F(x_n, y_n, z_n)) \\ & \quad + q(F(y_{n-1}, z_{n-1}, x_{n-1}), F(y_n, z_n, x_n)) \\ & \quad + q(F(z_{n-1}, x_{n-1}, y_{n-1}), F(z_n, x_n, y_n)) \\ & \preceq k(q(x_{n-1}, x_n) + q(y_{n-1}, y_n) + q(z_{n-1}, z_n)). \end{aligned} \quad (2.3)$$

We repeat the above process for n -times, we get

$$q(x_n, x_{n+1}) + q(y_n, y_{n+1}) + q(z_n, z_{n+1}) \preceq k^n(q(x_0, x_1) + q(y_0, y_1) + q(z_0, z_1)). \quad (2.4)$$

Put $q_n = q(x_n, x_{n+1}) + q(y_n, y_{n+1}) + q(z_n, z_{n+1})$. Then, from (2.4) we have

$$q_n \preceq k^n q_0 \quad (2.5)$$

Let $m, n \in \mathbb{N}$ with $m > n$. Then we have

$$q(x_n, x_m) \preceq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \dots + q(x_{m-1}, x_m),$$

$$q(y_n, y_m) \preceq q(y_n, y_{n+1}) + q(y_{n+1}, y_{n+2}) + \dots + q(y_{m-1}, y_m),$$

and

$$q(z_n, z_m) \preceq q(z_n, z_{n+1}) + q(z_{n+1}, z_{n+2}) + \dots + q(z_{m-1}, z_m).$$

Then we have,

$$\begin{aligned} q(x_n, x_m) + q(y_n, y_m) + q(z_n, z_m) & \preceq q_n + q_{n+1} + \dots + q_{m-1} \\ & \preceq k^n q_0 + k^{n+1} q_0 + \dots + k^{m-1} q_0 \\ & = (k^n + k^{n+1} + \dots + k^{m-1}) q_0 \\ & = k^n (1 + k + k^2 + \dots + k^{m-1-n}) q_0 \\ & \preceq \frac{k^n}{1-k} q_0. \end{aligned} \quad (2.6)$$

From (2.6) we have

$$q(x_n, x_m) \preceq \frac{k^n}{1-k} q_0 \rightarrow \theta \quad \text{as } n \rightarrow \infty,$$

$$q(y_n, y_m) \preceq \frac{k^n}{1-k} q_0 \rightarrow \theta \quad \text{as } n \rightarrow \infty,$$

and

$$q(z_n, z_m) \preceq \frac{k^n}{1-k} q_0 \rightarrow \theta \quad \text{as } n \rightarrow \infty,$$

Thus, Lemma 1.10 (3) shows that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are Cauchy sequences in X . Since X is complete, there exists u, v and $w \in X$ such that $x_n \rightarrow u, y_n \rightarrow v$ and $z_n \rightarrow w$ as $n \rightarrow \infty$. By q3 in Definition 1.7 we have:

$$q(x_n, u) \preceq \frac{k^n}{1-k} q_0, \quad (2.7)$$

$$q(y_n, v) \preceq \frac{k^n}{1-k} q_0, \quad (2.8)$$

and

$$q(z_n, w) \preceq \frac{k^n}{1-k} q_0. \quad (2.9)$$

On the other hand, we can get

$$\begin{aligned} q(x_n, F(u, v, w)) + q(y_n, F(v, w, u)) + q(z_n, F(w, u, v)) &= q(F(x_{n-1}, y_{n-1}, z_{n-1}), F(u, v, w)) \\ &\quad + q(F(y_{n-1}, z_{n-1}, x_{n-1}), F(v, w, u)) \\ &\quad + q(F(z_{n-1}, x_{n-1}, y_{n-1}), F(w, u, v)) \\ &\preceq k(q(x_{n-1}, u) + q(y_{n-1}, v) + q(z_{n-1}, w)) \\ &\preceq k\left(\frac{k^{n-1}}{1-k} q_0 + \frac{k^{n-1}}{1-k} q_0 + \frac{k^{n-1}}{1-k} q_0\right) \\ &= \frac{3k^n}{1-k} q_0. \end{aligned} \quad (2.10)$$

Therefor

$$q(x_n, F(u, v, w)) \preceq \frac{3k^n}{1-k} q_0, \quad (2.11)$$

$$q(y_n, F(v, w, u)) \preceq \frac{3k^n}{1-k} q_0, \quad (2.12)$$

and

$$q(z_n, F(w, u, v)) \preceq \frac{3k^n}{1-k} q_0. \quad (2.13)$$

Also, from (2.7), we have

$$q(x_n, u) \preceq \frac{k^n}{1-k} q_0 \preceq \frac{3k^n}{1-k} q_0. \quad (2.14)$$

By Lemma 1.10 (1), (2.11) and (2.14), we have $u = F(u, v, w)$. By the same way we have $v = F(v, w, u)$ and $w = F(w, u, v)$. Therefor, (u, v, w) is a tripled fixed point of F .

Finally, we assume that $(u, v, w, u, v, w) \in M$. By (2.1) we have

$$\begin{aligned} q(u, u) + q(v, v) + q(w, w) &= q(F(u, v, w), F(u, v, w)) + q(F(v, w, u), F(v, w, u)) \\ &\quad + q(F(w, u, v), F(w, u, v)) \\ &\preceq k(q(u, u) + q(v, v) + q(w, w)). \end{aligned}$$

Since $0 \leq k < 1$, by lemma 1.10 (1), we have $q(u, u) + q(v, v) + q(w, w) = \theta$. But $q(u, u) \succeq \theta$, $q(v, v) \succeq \theta$ and $q(w, w) \succeq \theta$.

Hence, $q(u, u) = \theta$, $q(v, v) = \theta$ and $q(w, w) = \theta$. □

Theorem 2.4. *In addition to the hypotheses of Theorem 2.3, suppose that for any three elements x, y and $z \in X$, we have*

$$(x, y, z, x, y, z) \in M \quad \text{or} \quad (y, z, x, y, z, x) \in M \quad \text{or} \quad (z, x, y, z, x, y) \in M.$$

Then the tripled fixed point has the form (u, u, u) , where $u \in X$.

Proof. As in the the proof of Theorem 2.3, there exists a tripled fixed point $(u, v, w) \in X^3$. Therefore

$$u = F(u, v, w) \quad v = F(v, w, u) \quad w = F(w, u, v).$$

Moreover, $q(u, u) = 0$, $q(v, v) = 0$ and $q(w, w) = 0$ if $(u, v, w, u, v, w) \in M$.

From the additional hypothesis, we have

$$(u, v, w, u, v, w) \in M \quad \text{or} \quad (v, w, u, v, w, u) \in M \quad \text{or} \quad (w, u, v, w, u, v) \in M. \quad (2.15)$$

By (2.1), we get

$$\begin{aligned} q(u, v) + q(w, u) + q(v, w) &= q(F(u, v, w), F(v, w, u)) \\ &\quad + q(F(w, u, v), F(u, v, w)) \\ &\quad + q(F(v, w, u), F(w, u, v)) \\ &\preceq k(q(u, v) + q(v, w) + q(w, u)), \end{aligned} \tag{2.16}$$

Since M is an F -invariant set then, $(w, v, u, w, v, u) \in M$. By applying the contractive condition, we have

$$\begin{aligned} q(w, v) + q(u, w) + q(v, u) &= q(F(w, u, v), F(v, w, u)) \\ &\quad + q(F(u, v, w), F(w, u, v)) \\ &\quad + q(F(v, w, u), F(u, v, w)) \\ &\preceq k(q(w, v) + q(u, w) + q(v, u)), \end{aligned} \tag{2.17}$$

Since $0 \leq k < 1$, we get from (2.16) that $q(u, v) + q(w, u) + q(v, w) = \theta$. Therefore, $q(u, v) = \theta$, $q(w, u) = \theta$ and $q(v, w) = \theta$. We also have $q(u, u) = \theta$, $q(w, w) = \theta$ and $q(v, v) = \theta$. Let $u_n = \theta$ and $x_n = u$. Then

$$q(x_n, u) \preceq u_n$$

and

$$q(x_n, v) \preceq u_n$$

From Lemma 1.10 (1), we have $u = v$. By the same way we have $u = w$ and $w = v$. By using the same way for the other arrangement in (2.15) we have the same results. Therefore, the tripled fixed point of F has the form (u, u, u) . \square

Theorem 2.5. *Let (X, d) be a complete cone metric space and q be a c -distance on X . Let M be a nonempty subset of X^6 and $F : X^3 \rightarrow X$ be a continuous function such that*

$$\begin{aligned} q(F(x, y, z), F(x^*, y^*, z^*)) + q(F(y, z, x), F(y^*, z^*, x^*)) + q(F(z, x, y), F(z^*, x^*, y^*)) \\ \preceq k(q(x, x^*) + q(y, y^*) + q(z, z^*)) \end{aligned} \tag{2.18}$$

for some $k \in [0, 1)$ and all $x, y, z, x^*, y^*, z^* \in X$ with

$$(x, y, z, x^*, y^*, z^*) \in M \quad \text{or} \quad (x^*, y^*, z^*, x, y, z) \in M.$$

Also, suppose that

i) there exist $x_0, y_0, z_0 \in X$ such that

$$(F(x_0, y_0, z_0), F(y_0, z_0, x_0), F(z_0, x_0, y_0), x_0, y_0, z_0) \in M,$$

ii) three sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ with $(x_{n+1}, y_{n+1}, z_{n+1}, x_n, y_n, z_n) \in M$ for all $n \in \mathbb{N}$ and $x_n \rightarrow x$, $y_n \rightarrow y$ and $z_n \rightarrow z$, then $(x, y, z, x_n, y_n, z_n) \in M$ for all $n \in \mathbb{N}$.

If M is F -invariant set, then F has a tripled fixed point. Furthermore, if $(u, v, w, u, v, w) \in M$, then $q(u, u) = \theta$, $q(v, v) = \theta$ and $q(w, w) = \theta$.

Proof. As in the proof of Theorem 2.3, we can construct three Cauchy sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X such that

$$(x_n, y_n, z_n, x_{n-1}, y_{n-1}, z_{n-1}) \in M$$

for all $n \in \mathbb{N}$. Moreover, we have from the assumption $x_n \rightarrow u, y_n \rightarrow v$ and $z_n \rightarrow w$ where $u, v, w \in X$. Therefore, by the assumption, we have $(u, v, w, x_n, y_n, z_n) \in M$. Since F is continuous, taking $n \rightarrow \infty$ in (2.2), we get

$$\begin{aligned} \lim_{n \rightarrow \infty} x_{n+1} &= \lim_{n \rightarrow \infty} F(x_n, y_n, z_n) = F(\lim_{n \rightarrow \infty} x_n, \lim_{n \rightarrow \infty} y_n, \lim_{n \rightarrow \infty} z_n) = F(u, v, w), \\ \lim_{n \rightarrow \infty} y_{n+1} &= \lim_{n \rightarrow \infty} F(y_n, z_n, x_n) = F(\lim_{n \rightarrow \infty} y_n, \lim_{n \rightarrow \infty} z_n, \lim_{n \rightarrow \infty} x_n) = F(v, w, u), \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} z_{n+1} = \lim_{n \rightarrow \infty} F(z_n, x_n, y_n) = F(\lim_{n \rightarrow \infty} z_n, \lim_{n \rightarrow \infty} x_n, \lim_{n \rightarrow \infty} y_n) = F(w, u, v).$$

By the uniqueness of the limits, we have $u = F(u, v, w), v = F(v, w, u)$ and $w = F(w, u, v)$. Therefore, (u, v, w) is a tripled fixed point of F . The proof of $q(u, u) = \theta, q(v, v) = \theta$ and $q(w, w) = \theta$ is the same as the proof in Theorem 2.3. □

Now, we introduce the definition of an F -invariant set in ordered case to keep both the mixed monotone property and the definition of tripled fixed point in ordered case true

Definition 2.6. Let (X, d) be a cone metric space and $F : X^3 \rightarrow X$ be a given mapping. Let M be a nonempty subset of X^6 . We say that M is an F -invariant subset of X^6 if and only if for all $x, y, z, w, e, s \in X$, we have

$$F_1 \quad (x, y, z, w, e, s) \in M \Leftrightarrow (s, e, w, z, y, x) \in M;$$

$$F_2 \quad (x, y, z, w, e, s) \in M \Rightarrow (F(x, y, z), F(y, x, y), F(z, y, x), F(w, e, s), F(e, w, e), F(s, e, w)) \in M.$$

By applying (2.3) and (2.4) in a partially ordered cone metric spaces, we get the following corollaries, respectively.

Corollary 2.7. Let (X, \sqsubseteq) be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c -distance on X and $F : X^3 \rightarrow X$ be a function having the mixed monotone property such that

$$\begin{aligned} q(F(x, y, z), F(x^*, y^*, z^*)) + q(F(y, x, y), F(y^*, x^*, y^*)) + q(F(z, y, x), F(z^*, y^*, x^*)) \\ \leq k(q(x, x^*) + q(y, y^*) + q(z, z^*)) \end{aligned} \tag{2.19}$$

for some $k \in [0, 1)$ and all $x, y, z, x^*, y^*, z^* \in X$ with

$$(x \sqsubseteq x^*) \wedge (y \sqsupseteq y^*) \wedge (z \sqsubseteq z^*) \quad \text{or} \quad (x \sqsupseteq x^*) \wedge (y \sqsubseteq y^*) \wedge (z \sqsupseteq z^*).$$

If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \sqsubseteq F(x_0, y_0, z_0), \quad F(y_0, x_0, y_0) \sqsubseteq y_0 \quad \text{and} \quad z_0 \sqsubseteq F(z_0, y_0, x_0),$$

then F has a tripled fixed point (u, v, w) . Furthermore, we have $q(u, u) = \theta, q(v, v) = \theta$ and $q(w, w) = \theta$.

Proof. Let $M = \{(a, b, c, d, e, f) : a \sqsupseteq d, b \sqsubseteq e, c \sqsupseteq f\} \subseteq X^6$. We obtain that M is an F -invariant set. By (2.19), we have

$$\begin{aligned} q(F(x, y, z), F(x^*, y^*, z^*)) + q(F(y, z, x), F(y^*, z^*, x^*)) + q(F(z, x, y), F(z^*, x^*, y^*)) \\ \leq k(q(x, x^*) + q(y, y^*) + q(z, z^*)) \end{aligned} \tag{2.20}$$

for some $k \in [0, 1)$ and all $x, y, z, x^*, y^*, z^* \in X$ with $(x, y, z, x^*, y^*, z^*) \in M$ or $(x^*, y^*, z^*, x, y, z) \in M$. Now, all the hypotheses of Theorem 2.3 hold. Thus, F has a tripled fixed point. □

Corollary 2.8. *In addition to the hypotheses of Corollary 2.7, suppose that x, y and $z \in X$ are comparable, then the tripled fixed point has the form (u, u, u) , where $u \in X$.*

Proof. This result is obtained by (2.4). □

Example 2.9. Consider Example 1.9. Define a mapping $F : X^3 \rightarrow X$ by $F(x, y, z) = \frac{3x+2y+z}{12}$ for all $(x, y, z) \in X^3$. Let $M = X^6$, Then M is an F -invariant set. Assume that $x, y, z, x^*, y^*, z^* \in X$ with

$$(x, y, z, x^*, y^*, z^*) \in M \quad \text{or} \quad (x^*, y^*, z^*, x, y, z) \in M.$$

Since M is F -invariant, then there exist $x_0, y_0, z_0 \in X$ such that

$$(F(x_0, y_0, z_0), F(y_0, z_0, x_0), F(z_0, x_0, y_0), x_0, y_0, z_0) \in M,$$

Now, applying the contractive condition we have

$$\begin{aligned} & q(F(x, y, z), F(x^*, y^*, z^*)) + q(F(y, z, x), F(y^*, z^*, x^*)) + q(F(z, x, y), F(z^*, x^*, y^*)) \\ &= (F(x^*, y^*, z^*), F(x^*, y^*, z^*)) + (F(y^*, z^*, x^*), F(y^*, z^*, x^*)) \\ & \quad + (F(z^*, x^*, y^*), F(z^*, x^*, y^*)) \\ &= \left(\frac{3x^* + 2y^* + z^*}{12}, \frac{3x^* + 2y^* + z^*}{12} \right) + \left(\frac{3y^* + 2z^* + x^*}{12}, \frac{3y^* + 2z^* + x^*}{12} \right) \\ & \quad + \left(\frac{3z^* + 2x^* + y^*}{12}, \frac{3z^* + 2x^* + y^*}{12} \right) \\ &= \frac{1}{2}(x^*, x^*) + (y^*, y^*) + (z^*, z^*) \\ &\leq k(q(x, x^*) + q(y, y^*) + q(z, z^*)), \end{aligned}$$

where $k = \frac{2}{3} \in [0, 1)$. Hence, all the conditions of Theorems 2.3 are satisfied. Therefore, F has a tripled fixed point. It is easy to say that $(0, 0, 0)$ is the tripled fixed point of F .

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