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On the general solution of a quadratic functional equation and its Ulam stability in various abstract spaces

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Abstract

In this paper, we establish the general solution of a new quadratic functional equation $f\left(x-\frac{y+z}{2}\right)+f\left(x+\frac{y-z}{2}\right)+f(x+z)=3f(x)+\frac{1}{2}f(y)+\frac{3}{2}f(z)$. Next, the Ulam stability of this equation in a real normed space and a non-Archimedean space is studied, respectively.

Keywords: General solution, Ulam stability, Quadratic functional equation, Normed space, Non-Archimedean space.
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1. Introduction

At the current stage, the Ulam stability is gradually becoming one of the most active research topics in the theory of functional equations. The investigation of such stability problems of functional equations originated from a question of Ulam [22] concerning the stability of group homomorphisms, i.e.,

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Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ such that $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

Hyers [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Afterwards, the result of Hyers was generalized by Aoki [1] (For some historical comments regarding the work of Aoki, see [18]) for additive mappings and by Rassias [20] for linear mappings by considering an unbounded Cauchy difference. Later, Găvruta [6] replaced the unbounded Cauchy difference in the Rassas's Theorem and provided a further generalization by using a general control function. Since then, the Ulam stability problems of different types of functional equations in various spaces have been widely and extensively studied. For more detailed information, the reader can refer to [4, 12, 21].

Based on the existing research results, it is worth noting that the study of Ulam stability of different types of quadratic functional equations is an important branch. In recent years, there are many interesting results concerning these problems obtained by different authors [2, 3, 5, 8, 9, 10, 11, 13, 14, 15, 16, 17, 23].

In this paper, we construct the following functional equation:

$$f\left(x - \frac{y+z}{2}\right) + f\left(x + \frac{y-z}{2}\right) + f(x+z) = 3f(x) + \frac{1}{2}f(y) + \frac{3}{2}f(z) \tag{1.1}$$

It is easy to verify that the function $f(x) = x^2$ is a solution of Eq. (1.1) on $\mathbb{R}$. Naturally, we call the preceding equation a quadratic functional equation.

The main purpose of this paper is to establish the general solution of Eq. (1.1) on an Abelian group, and to investigate the Ulam stability of Eq. (1.1) in a real normed space and a non-Archimedean space, respectively.

2. General solution of Eq. (1.1) on an Abelian group

First of all, let us recall that a mapping $f : G \to F$ is said to be a *quadratic mapping* provided that it satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{2.1}$$

for all $x, y \in G$, where G, F are groups.

A mapping $S : G \times G \to F$ is called *biadditive* if and only if

$$S(x_1 + x_2, y) = S(x_1, y) + S(x_2, y) \text{ and } S(x, y_1 + y_2) = S(x, y_1) + S(x, y_2)$$

for all $x_1, x_2, x, y_1, y_2, y \in G$. In addition, if $S(x, y) = S(y, x)$ for all $x, y \in G$, then we say that S is *symmetric*.

As we all know, we have the following result concerning the general solution of Eq. (2.1).

Lemma 2.1 ([4]). *Let G be an Abelian group and F an Abelian group with the division by two. The mapping $f : G \to F$ is quadratic if and only if there exists a unique symmetric biadditive mapping $B : G \times G \to F$ such that*

$$f(x) = B(x, x)$$

for all $x \in G$.

In the following, we will establish the general solution of Eq. (1.1) on an Abelian group. The corresponding result shows that the solution of Eq. (1.1) is equivalent to the one of Eq. (2.1).

Theorem 2.2. *Let G and F be two Abelian groups with the division by two. A mapping $f : G \to F$ satisfies the functional equation (1.1) for all $x, y, z \in G$ if and only if it satisfies the functional equation (2.1), i.e., f is a quadratic mapping.*

Proof. Necessity: Putting $x = y = z = 0$ in (1.1), we get $f(0) = 0$. Replacing y, z by $x, -x$ in (1.1), respectively, we get

$$f(2x) - \frac{5}{2}f(x) - \frac{3}{2}f(-x) = 0 \tag{2.2}$$

for all $x \in G$. Similarly, replacing y, z by $-x, x$ in (1.1), respectively, we have

$$f(2x) - \frac{7}{2}f(x) - \frac{1}{2}f(-x) = 0 \tag{2.3}$$

for all $x \in G$. Subtracting the two resulting equations, we get $f(x) = f(-x)$ for all $x \in G$, i.e., f is an even mapping. Using the evenness of f, it follows from (2.2) or (2.3) that $f(2x) = 4f(x)$. Furthermore, we can obtain the functional equation (2.1) if we replace z by y in (1.1). That is to say, f is a quadratic mapping. *Sufficiency:* If f is a quadratic mapping, by Lemma 2.1, then there exists a unique symmetric biadditive mapping $B : G \times G \to F$ such that $f(x) = B(x, x)$ for all $x \in G$. Therefore, substituting it in the left-hand side of Eq. (1.1), we can obtain that

$$\begin{aligned}
&B\big(x - \frac{y+z}{2}, x - \frac{y+z}{2}\big) + B\big(x + \frac{y-z}{2}, x + \frac{y-z}{2}\big) + B(x+z, x+z) \\
&= B(x, x) - 2B\big(x, \frac{y+z}{2}\big) + B\big(\frac{y+z}{2}, \frac{y+z}{2}\big) + B(x, x) + 2B\big(x, \frac{y-z}{2}\big) \\
&\quad + B\big(\frac{y-z}{2}, \frac{y-z}{2}\big) + B(x, x) + 2B(x, z) + B(z, z) \\
&= 3B(x, x) + B(x, -y-z) + \frac{1}{4}B(y+z, y+z) + B(x, y-z) \\
&\quad + \frac{1}{4}B(y-z, y-z) + 2B(x, z) + B(z, z) \\
&= 3B(x, x) + B(x, -2z) + \frac{1}{2}B(y, y) + \frac{1}{2}B(z, z) + 2B(x, z) + B(z, z) \\
&= 3B(x, x) + \frac{1}{2}B(y, y) + \frac{3}{2}B(z, z),
\end{aligned}$$

which implies that the quadratic mapping $f(x) = B(x, x)$ satisfies Eq. (1.1). □

3. Stability of Eq. (1.1) in a real linear normed space

In this section, we shall prove the Hyers-Ulam stability of Eq. (1.1) in a real linear normed space. Throughout this section, X and Y will stand for a real vector space and a real Banach space, respectively. For notational convenience, we define the following operator

$$\begin{aligned}
D_q f(x, y, z) := f\left(x - \frac{y+z}{2}\right) + f\left(x + \frac{y-z}{2}\right) + f(x+z) \\
- 3f(x) - \frac{1}{2}f(y) - \frac{3}{2}f(z).
\end{aligned} \tag{3.1}$$

Theorem 3.1. *Let $j \in \{-1, 1\}$ be fixed and $\varphi : X \times X \times X \to [0, \infty)$ be a function such that*

$$\Phi^{(j)}(x, y, z) := \sum_{k=0}^{\infty} 4^{-jk}\varphi(2^{jk}x, 2^{jk}y, 2^{jk}z) < +\infty \tag{3.2}$$

for all $x, y, z \in X$. Assume that the mapping $f : X \to Y$ satisfies

$$\|D_q f(x, y, z)\| \le \varphi(x, y, z) \tag{3.3}$$

for all $x, y, z \in X$. Moreover, $f(0) = 0$ in the case $j = 1$. Then

$$Q(x) = \lim_{n\to\infty} 4^{jn} f(2^{-jn}x)$$

exists for each $x \in X$ and defines a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{4}\Phi^{(-j)}(x, x, x) \tag{3.4}$$

for all $x \in X$.

Proof. Using (3.2), we can easily know that $f(0) = 0$ in the case $j = -1$. Indeed, putting $x = y = z = 0$ in (3.3), we can obtain that $f(0) = 0$ because the condition $\Phi^{(-1)}(0, 0, 0) = \sum_{k=0}^{\infty} 4^k \varphi(0, 0, 0) < +\infty$ implies that $\varphi(0, 0, 0) = 0$.

Putting $y = z = x$ in (3.3), and since $f(0) = 0$, we can obtain that

$$\|4f(x) - f(2x)\| \leq \varphi(x, x, x) \tag{3.5}$$

for all $x \in X$. Replacing x by $2^{n-1}x$ and dividing by 4^n in (3.5), we get

$$\left\| \frac{1}{4^{n-1}} f(2^{n-1}x) - \frac{1}{4^n} f(2^n x) \right\| \leq \frac{1}{4^n}\varphi(2^{n-1}x, 2^{n-1}x, 2^{n-1}x) \tag{3.6}$$

for all $x \in X$ and $n \in \mathbb{N}$. Therefore, it follows from (3.5) and (3.6) that

$$\begin{aligned} \left\| f(x) - \frac{1}{4^n} f(2^n x) \right\| &\leq \sum_{i=1}^{n} \frac{\varphi(2^{i-1}x, 2^{i-1}x, 2^{i-1}x)}{4^i} \\ &= \sum_{i=0}^{n-1} \frac{\varphi(2^i x, 2^i x, 2^i x)}{4^{i+1}} \end{aligned} \tag{3.7}$$

for all $x \in X$ and $n \in \mathbb{N}$. Moreover, replacing x by $\frac{x}{2}$ in (3.5), we have

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq \varphi(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}) \tag{3.8}$$

for all $x \in X$. By using a similar argument, we can infer that

$$\left\| f(x) - 4^n f\left(\frac{x}{2^n}\right) \right\| \leq \sum_{i=0}^{n-1} 4^i \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}) \tag{3.9}$$

for all $x \in X$ and $n \in \mathbb{N}$. Here we claim that the sequence $\{4^{jn} f(2^{-jn}x)\}$ is a Cauchy sequence in the Banach space Y. Indeed, for all $m, n \in \mathbb{N}$, it follows from (3.7) and (3.9) that

$$\begin{aligned} &\|4^{j(n+m)} f(2^{-j(n+m)}x) - 4^{jm} f(2^{-jm}x)\| \\ =&4^{jm}\|4^{jn} f(2^{-j(n+m)}x) - f(2^{-jm}x)\| \\ \leq&\begin{cases} 4^{-m}\sum_{i=0}^{n-1} \frac{\varphi(2^{(i+m)}x, 2^{(i+m)}x, 2^{(i+m)}x)}{4^{i+1}}, & j = -1, \\ \frac{1}{4}\sum_{i=0}^{n-1} 4^{i+m+1}\varphi(\frac{x}{2^{i+m+1}}, \frac{x}{2^{i+m+1}}, \frac{x}{2^{i+m+1}}), & j = 1 \end{cases} \end{aligned}$$

for all $x \in X$. Obviously, in the case $j = -1$, the last expression tends to zero as $m \to \infty$. In the case $j = 1$, the condition (3.2) implies that the preceding expression also tends to zero as $m \to \infty$. Thus, we have proved that the sequence $\{4^{jn} f(2^{-jn}x)\}$ is Cauchy. According to the completeness of Y, we can define

$$Q(x) := \lim_{n\to\infty} 4^{jn} f(2^{-jn}x)$$

for all $x \in X$.

Now we show that Q satisfies Eq. (1.1). Replacing x, y, z by $2^{-jn}x, 2^{-jn}y, 2^{-jn}z$ in (3.3), respectively, and multiplying both sides by 4^{jn}, we can obtain that

$$\begin{aligned}4^{jn}\Big\| f\left(2^{-jn}\Big(x - \frac{y+z}{2}\Big)\right) &+ f\left(2^{-jn}\Big(x + \frac{y-z}{2}\Big)\right) + f\Big(2^{-jn}(x+z)\Big) \\ &- 3f(2^{-jn}x) - \frac{1}{2}f(2^{-jn}y) - \frac{3}{2}f(2^{-jn}z)\Big\| \\ \leq 4^{jn}\varphi(2^{-jn}x, 2^{-jn}y, 2^{-jn}z).\end{aligned}$$

Taking the limit in the preceding expression, by (3.2), it is easy to know that Q is a solution of Eq. (1.1) since the right-hand side tends to zero as $n \to \infty$.

By letting $n \to \infty$ in (3.7) and (3.9), we can infer that

$$\|f(x) - Q(x)\| \leq \begin{cases} \frac{1}{4}\Phi^{(1)}(x,x,x), & j = -1, \\ \frac{1}{4}\Phi^{(-1)}(x,x,x), & j = 1 \end{cases} \tag{3.10}$$

for all $x \in X$.

Next, we want to prove that Q is a unique such quadratic mapping. Assume that Q' is another quadratic function satisfying the inequality (3.4). Since $Q(2^{-jn}x) = 4^{-jn}Q(x)$, $Q'(2^{-jn}x) = 4^{-jn}Q'(x)$, it follows that

$$\begin{aligned}\|Q(x) - Q'(x)\| &= 4^{jn}\|Q(2^{-jn}x) - Q'(2^{-jn}x)\| \\ &\leq 4^{jn}(\|Q(2^{-jn}x) - f(2^{-jn}x)\| + \|f(2^{-jn}x) - Q'(2^{-jn}x)\|) \\ &\leq \frac{4^{jn}}{2}\Phi^{(-j)}(2^{-jn}x, 2^{-jn}x, 2^{-jn}x).\end{aligned}$$

From the condition (3.2), one can see that the last expression tends to zero as $n \to \infty$. Thus, we conclude that $Q(x) \equiv Q'(x)$. This completes the proof. □

Based on Theorem 3.1, we can obtain the following corollaries concerning the stability of Eq. (1.1).

Corollary 3.2. *Let $j \in \{-1, 1\}$ be fixed. Suppose that X is a real normed space and p, ϵ are real numbers with $\epsilon \geq 0$ and $p \neq 2$. Assume that the mapping $f : X \to Y$ satisfies the following inequality*

$$\|D_q f(x,y,z)\| \leq \epsilon(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$ ($X \setminus \{0\}$ if $p < 0$). Further, $f(0) = 0$ when $j = 1$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying the equality (1.1) and the following inequality

$$\|f(x) - Q(x)\| \leq \begin{cases} \frac{3\epsilon\|x\|^p}{4(1-2^{2-p})}, & j = -1,\ p > 2, \\ \frac{3\epsilon\|x\|^p}{4(1-2^{p-2})}, & j = 1,\ p < 2 \end{cases}$$

for all $x \in X$ ($X \setminus \{0\}$ if $p < 0$).

Proof. Letting $\varphi(x,y,z) = \epsilon(\|x\|^p + \|y\|^p + \|z\|^p)$ and then applying Theorem 3.1. we can obtain the desired result. □

Corollary 3.3. *Let $j \in \{-1, 1\}$ be fixed. Suppose that X is a real normed space and p, ϵ are real numbers with $\epsilon \geq 0$ and $p \neq \frac{2}{3}$. Assume that the mapping $f : X \to Y$ satisfies the following inequality*

$$\|D_q f(x,y,z)\| \leq \epsilon\|x\|^p\|y\|^p\|z\|^p$$

for all $x, y, z \in X$ ($X \setminus \{0\}$ if $p < 0$). Further, $f(0) = 0$ when $j = 1$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying the equality (1.1) and the following inequality

$$\|f(x) - Q(x)\| \leq \begin{cases} \frac{\epsilon\|x\|^{3p}}{4(1-2^{2-3p})}, & j = -1,\ p > \frac{2}{3}, \\ \frac{\epsilon\|x\|^p}{4(1-2^{3p-2})}, & j = 1,\ p < \frac{2}{3} \end{cases}$$

for all $x \in X$ ($X \setminus \{0\}$ if $p < 0$).

Proof. Letting $\varphi(x,y,z) = \epsilon\|x\|^p\|y\|^p\|z\|^p$ and then applying Theorem 3.1. we can obtain the desired result. □

Corollary 3.4. *Let $j \in \{-1,1\}$ be fixed. Suppose that X is a real normed space and p,q,r are positive real numbers with $p+q+r \neq 2$, ϵ a nonnegative real number. Assume that the mapping $f : X \to Y$ satisfies the following inequality*

$$\|D_q f(x,y,z)\| \leq \epsilon(\|x\|^p\|y\|^q\|z\|^r + \|x\|^{p+q+r} + \|y\|^{p+q+r} + \|z\|^{p+q+r})$$

for all $x,y,z \in X$. Further, $f(0)=0$ when $j=1$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying the equality (1.1) and the following inequality

$$\|f(x) - Q(x)\| \leq \begin{cases} \frac{\epsilon\|x\|^{p+q+r}}{1-2^{2-(p+q+r)}}, & j=-1,\ p+q+r>2, \\ \frac{\epsilon\|x\|^{p+q+r}}{1-2^{(p+q+r)-2}}, & j=1,\ p+q+r<2 \end{cases}$$

for all $x \in X$.

Proof. Letting $\varphi(x,y,z) = \epsilon(\|x\|^p\|y\|^q\|z\|^r + \|x\|^{p+q+r} + \|y\|^{p+q+r} + \|z\|^{p+q+r})$ and then applying Theorem 3.1. we can obtain the desired result. □

Especially, we can obtain the following Hyers-Ulam stability result of the quadratic functional equation (1.1).

Corollary 3.5. *Let X be a real normed space. For a given $\epsilon > 0$, if the mapping $f : X \to Y$ satisfies the following inequality*

$$\|D_q f(x,y,z)\| \leq \epsilon$$

for all $x,y,z \in X$ with $f(0)=0$, then there exists a unique quadratic mapping $Q : X \to Y$ satisfying the equality (1.1) and the following inequality

$$\|f(x) - Q(x)\| \leq \frac{\varepsilon}{3}$$

for all $x \in X$.

Furthermore, we have the following more general form of stability result concerning the Eq. (1.1) by using a similar method as shown in [8].

A function $H : [0,\infty) \times [0,\infty) \times [0,\infty) \to [0,\infty)$ is said to be *homogeneous of degree* p ($p > 0$) if it satisfies

$$H(tu,tv,tw) = t^p H(u,v,w)$$

for all $t,u,v,w \geq 0$.

For technical reasons, in the following, we denote by $\mathcal{H}_\varphi$ the set of all functions $H : [0,\infty) \times [0,\infty) \times [0,\infty) \to [0,\infty)$ that there exists $\varphi : [0,\infty) \to [0,\infty)$ such that

$$H(\lambda u, \lambda v, \lambda w) \leq \varphi(\lambda) H(u,v,w)$$

for all $u,v,w \geq 0$ and $\lambda > 0$, where φ satisfies the following properties:
(i) $\varphi(\lambda) > 0$ for all $\lambda > 0$;
(ii) $\varphi(2) \neq 4$;
(iii) $\varphi(2\lambda) = \varphi(2)\varphi(\lambda)$ for all $\lambda > 0$.

Obviously, if $p > 0$ and $p \neq 2$, then the homogeneous function H of degree p belongs to $\mathcal{H}_\varphi$ with $\varphi(t) = t^p$.

Corollary 3.6. *Let X be a real normed space. Assume that the mapping $f : X \to Y$ satisfies the following inequality*

$$\|D_q f(x, y, z)\| \le H(\|x\|, \|y\|, \|z\|)$$

for all $x, y, z \in X$ with $f(0) = 0$, where $H \in \mathcal{H}_\varphi$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying equality (1.1) and the following inequality

$$\|f(x) - Q(x)\| \le \frac{H(\|x\|, \|x\|, \|x\|)}{|\varphi(2) - 4|} \tag{3.11}$$

for all $x \in X$.

Proof. Setting $\varphi(x, y, z) = H(\|x\|, \|y\|, \|z\|)$. By Theorem 3.1, it suffices to verify that φ satisfies the condition (3.2). Therefore, we have

$$\begin{aligned}
\Phi^{(j)}(x, y, z) &= \sum_{k=0}^{\infty} 4^{-jk} \varphi(2^{jk}x, 2^{jk}y, 2^{jk}z) \\
&= \sum_{k=0}^{\infty} 4^{-jk} H(2^{jk}\|x\|, 2^{jk}\|y\|, 2^{jk}\|z\|) \\
&\le \sum_{k=0}^{\infty} 4^{-jk} \varphi(2^{jk}) H(\|x\|, \|y\|, \|z\|) \\
&= \sum_{k=0}^{\infty} 4^{-jk} \varphi(2)^{jk} H(\|x\|, \|y\|, \|z\|) \\
&= \sum_{k=0}^{\infty} \left(\frac{\varphi(2)}{4}\right)^{jk} H(\|x\|, \|y\|, \|z\|) \\
&= \begin{cases} \frac{4H(\|x\|,\|y\|,\|z\|)}{4-\varphi(2)}, & \text{when } j = 1,\ \varphi(2) < 4, \\ \frac{4H(\|x\|,\|y\|,\|z\|)}{\varphi(2)-4}, & \text{when } j = -1,\ \varphi(2) > 4. \end{cases}
\end{aligned}$$

Therefore, the inequality (3.11) holds true. □

Remark 3.7. In the preceding proof, the equality $\varphi(2^{jk}) = \varphi(2)^{jk}$ is applied. Indeed, when $j = 1$, it is obvious by the condition (iii). When $j = -1$, since $1 = \varphi(1) = \varphi(2)\varphi(\frac{1}{2})$, we know that $\varphi(2^{-1}) = \varphi(2)^{-1}$. Furthermore, we can infer that $\varphi(2^{-k}) = \varphi(2)^{-k}$ by the mathematical induction.

4. Stability of Eq. (1.1) in a non-Archimedean space

Here, we will establish the Hyers-Ulam stability of Eq. (1.1) in a non-Archimedean space. First of all, we recall some basic notions that are used in the following [18, 19].

A *non-Archimedean field* means that a field $\mathbb{K}$ endowed with a function (valuation) $|\cdot|_A : \mathbb{K} \to [0, \infty)$ which satisfies the following conditions:
(i) $|r|_A = 0$ if and only if $r = 0$;
(ii) $|rs|_A = |r|_A |s|_A$;
(iii) $|r + s|_A \le \max\{|r|_A, |s|_A\}$ for all $r, s \in \mathbb{K}$.

Remark 4.1. According to the above conditions, it is easy to know that $|1|_A = |-1|_A = 1$ and $|n|_A \le 1$ for all $n \in \mathbb{N}$.

Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|_A$. A function $\|\cdot\|_A : X \to \mathbb{R}$ is called a *non-Archimedean norm* (valuation) if it satisfies the following conditions:
(NA1) $\|x\|_A = 0$ if and only if $x = 0$;

(NA2) $\|rx\|_A = |r|_A\|x\|_A$ for all $r \in \mathbb{K}$ and $x \in X$;
(NA3) $\|x+y\|_A \leq \max\{\|x\|_A, \|y\|_A\}$ for all $x, y \in X$ (strong triangle inequality (ultrametric)).

Correspondingly, we call the pair $(X, \|\cdot\|_A)$ a *non-Archimedean space*. In addition, by (NA3), we can obtain that

$$\|x_n - x_m\|_A \leq \max\{\|x_{j+1} - x_j\|_A : m \leq j \leq n-1\}$$

for all $n, m \in \mathbb{N}$ with $n > m$. Thus, a sequence $\{x_n\}$ in a non-Archimedean space is *Cauchy* if and only if the sequence $\{x_{n+1} - x_n\}$ converges to zero. A sequence $\{x_n\}$ is said to be *convergent* if for any $\epsilon > 0$, there exist a natural number $n_0 \in \mathbb{N}$ and $x \in X$ such that $\|x_n - x\|_A \leq \epsilon$ for all $n \geq n_0$. Furthermore, a non-Archimedean space $(X, \|\cdot\|_A)$ is said to be *complete* if every Cauchy sequence in X is convergent.

Throughout this section, we assume that G is a group and $(X, \|\cdot\|_A)$ is a complete non-Archimedean space. Especially, in the following, when $j = -1$, it requires that G is a group with the division by two.

Theorem 4.2. *Let $j \in \{-1, 1\}$ be fixed and $\psi : G \times G \times G \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} |4|_A^{-jn} \psi(2^{jn}x, 2^{jn}y, 2^{jn}z) = 0 \tag{4.1}$$

and the limits

$$\Psi^{(j)}(x) := \begin{cases} \lim_{n\to\infty} \max\{|4|_A^{-jk} \psi(2^{jk}x, 2^{jk}x, 2^{jk}x) : 0 \leq k < n\}, \ j = 1, \\ \lim_{n\to\infty} \max\{|4|_A^{-jk} \psi(2^{jk}x, 2^{jk}x, 2^{jk}x) : 0 < k \leq n\}, \ j = -1, \end{cases} \tag{4.2}$$

exists for each $x \in G$. Suppose that the mapping $f : G \to X$ satisfies the inequality

$$\|D_q f(x, y, z)\|_A \leq \psi(x, y, z) \tag{4.3}$$

for all $x, y, z \in G$. Furthermore, $f(0) = 0$ in the case $j = -1$. Then there exists a quadratic mapping $Q : G \to X$ such that

$$\|f(x) - Q(x)\|_A \leq \frac{1}{|4|_A} \Psi^{(j)}(x) \tag{4.4}$$

for all $x \in G$. Moreover, if

$$\begin{cases} \lim_{m\to\infty} \lim_{n\to\infty} \max\{|4|_A^{-jk} \psi(2^{jk}x, 2^{jk}x, 2^{jk}x) : m \leq k < n+m\} = 0, \ j = 1, \\ \lim_{m\to\infty} \lim_{n\to\infty} \max\{|4|_A^{-jk} \psi(2^{jk}x, 2^{jk}x, 2^{jk}x) : m < k \leq n+m\} = 0, \ j = -1, \end{cases} \tag{4.5}$$

then Q is the unique quadratic mapping satisfying the equality (1.1).

Proof. By (4.1), it is easy to know that $f(0) = 0$ when $j = 1$. In fact, putting $x = y = z = 0$ in (4.3) we can obtain $f(0) = 0$ due to the condition (4.1) implies that $\lim_{n\to\infty} |4|_A^{-n} \psi(0, 0, 0) = 0$ and the fact $|4|_A^{-n} \geq 1$.

Using the same method as in the proof of Theorem 3.1, we get

$$\Big\| \frac{1}{4^{n-1}} f(2^{n-1}x) - \frac{1}{4^n} f(2^n x) \Big\|_A \leq \frac{1}{|4|_A^n} \varphi(2^{n-1}x, 2^{n-1}x, 2^{n-1}x) \tag{4.6}$$

for all $x \in G$ and $n \in \mathbb{N}$. Similarly, we can obtain that

$$\Big\| 4^n f(\frac{x}{2^n}) - 4^{n-1} f(\frac{x}{2^{n-1}}) \Big\|_A \leq |4|_A^{n-1} \varphi\Big(\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}\Big) \tag{4.7}$$

for all $x \in G$ and $n \in \mathbb{N}$. From (4.1), (4.6) and (4.7), it follows that the sequence $\{4^{-jn} f(2^{jn}x)\}$ is a Cauchy sequence in X. The completeness of X implies that the following expression

$$Q(x) := \lim_{n\to\infty} 4^{-jn} f(2^{jn}x)$$

is well defined. Now, we claim that

$$\|f(x)-4^{-jn}f(2^{jn}x)\|_A \le \begin{cases} \frac{1}{|4|_A^j}\max\{|4|_A^{-k}\psi(2^{jk}x,2^{jk}x,2^{jk}x):0\le k<n\}, & j=1,\\ |4|_A^j\max\{|4|_A^{k}\psi(2^{jk}x,2^{jk}x,2^{jk}x):0<k\le n\}, & j=-1\end{cases} \tag{4.8}$$

for all $n\in\mathbb{N}$ and $x\in G$. Substituting $n=1$ in (4.6) and (4.7), it is easy to verify that (4.8) is true for $n=1$. By mathematical induction, we can assume that the inequality (4.8) holds true for some $n\in\mathbb{N}$. Therefore, it follows from (4.6) and (4.7) that

$$\begin{aligned}&\|f(x)-4^{-j(n+1)}f(2^{j(n+1)}x)\|_A\\ &\le \max\{\|f(x)-4^{-jn}f(2^{jn}x)\|_A,\|4^{-jn}f(2^{jn}x)-4^{-j(n+1)}f(2^{j(n+1)}x)\|_A\}\\ &\le \begin{cases}\max\{\frac{1}{|4|_A^j}\max\{\frac{\psi(2^{jk}x,2^{jk}x,2^{jk}x)}{|4|_A^k}:0\le k<n\},\frac{1}{|4|_A^{n+1}}\varphi(2^nx,2^nx,2^nx)\}, & j=1,\\ \max\{|4|_A^j\max\{|4|_A^k\psi(2^{jk}x,2^{jk}x,2^{jk}x):0<k\le n\},|4|_A^n\varphi\Big(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}},\frac{x}{2^{n+1}}\},& j=-1\end{cases}\\ &=\begin{cases}\frac{1}{|4|_A^j}\max\{|4|_A^{-k}\psi(2^{jk}x,2^{jk}x,2^{jk}x):0\le k<n+1\}, & j=1,\\ |4|_A^j\max\{|4|_A^k\psi(2^{jk}x,2^{jk}x,2^{jk}x):0<k\le n+1\}, & j=-1,\end{cases}\end{aligned}$$

which proves the validity of the inequality (4.8) for $n+1$. By letting $n\to\infty$ in (4.8) and using (4.2), we get (4.4).

Replacing x and y by $2^{jn}x$ and $2^{jn}y$ in (4.3), respectively, we have

$$\begin{aligned}&\Big\|4^{-jn}f\left(2^{jn}\Big(x-\frac{y+z}{2}\Big)\right)+4^{-jn}f\left(2^{jn}\Big(x+\frac{y-z}{2}\Big)\right)+4^{-jn}f\Big(2^{jn}(x+z)\Big)\\ &-3\cdot 4^{-jn}f(2^{jn}x)-\frac{1}{2}\cdot 4^{-jn}f(2^{jn}y)-\frac{3}{2}\cdot 4^{-jn}f(2^{jn}z)\Big\|_A\\ &\le |4|_A^{-jn}\varphi(2^{jn}x,2^{jn}y,2^{jn}z).\end{aligned}$$

Taking the limit as $n\to\infty$ in the previous expression and using (4.1), we know that Q satisfies the equality (1.1).

To show the uniqueness of Q. Assume that Q' is another quadratic mapping satisfying (4.4). Then we can obtain

$$\begin{aligned}&\|Q(x)-Q'(x)\|_A\\ &=|4|_A^{-jm}\|Q(2^{jm}x)-Q'(2^{jm}x)\|_A\\ &\le |4|_A^{-jm}\max\{\|Q(2^{jm}x)-f(2^{jm}x)\|_A,\|f(2^{jm}x)-Q'(2^{jm}x)\|_A\}\\ &\le \frac{1}{|4|_A}\begin{cases}\lim_{n\to\infty}\max\{|4|_A^{-k}\psi(2^kx,2^kx,2^kx):m\le k<n+m\}, & j=1,\\ \lim_{n\to\infty}\max\{|4|_A^{k}\psi(2^{-k}x,2^{-k}x,2^{-k}x):m<k\le n+m\}, & j=-1\end{cases}\\ &=0.\end{aligned}$$

Taking the limit as $m\to\infty$ and using (4.5), we conclude that $Q(x)\equiv Q'(x)$. The proof of the theorem is now completed. □

Corollary 4.3. *Let $j\in\{-1,1\}$ be fixed and let $\rho:[0,\infty)\to[0,\infty)$ be a function such that*

$$\rho(|2^j|_At)\le[\rho(|2|_A)]^j\rho(t)$$

for all $t\ge 0$, and $j\cdot\rho(|2|_A)<j\cdot|4|_A$. Let X be a complete non-Archimedean space and let δ be a positive real number. Suppose that the mapping $f:G\to X$ satisfies the inequality

$$\|D_qf(x,y,z)\|_A\le\delta(\rho(\|x\|_A)+\rho(\|y\|_A)+\rho(\|z\|_A))$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : G \to X$ satisfying (1.1) such that

$$\|f(x) - Q(x)\|_A \leq \begin{cases} \frac{3\delta\rho(\|x\|_A)}{|4|_A}, & j = 1, \\ \frac{3\delta\rho(\|x\|_A)}{\rho(\|2\|_A)}, & j = -1 \end{cases}$$

for all $x \in G$.

Proof. Letting $\psi(x, y, z) = \delta(\rho(\|x\|_A) + \rho(\|y\|_A) + \rho(\|z\|_A))$. By Theorem 4.2, we have

$$\begin{aligned} &\lim_{n\to\infty} |4|_A^{-jn}\psi(2^{jn}x, 2^{jn}y, 2^{jn}z) \\ &= \lim_{n\to\infty} |4|_A^{-jn}\delta(\rho(|2^j|_A^n\|x\|_A) + \rho(|2^j|_A^n\|y\|_A) + \rho(|2^j|_A^n\|z\|_A)) \\ &\leq \lim_{n\to\infty} |4|_A^{-jn}\rho(|2|_A)^{jn}\psi(x, y, z) = 0 \end{aligned}$$

for all $x, y, z \in G$. Moreover, by (4.2), we can obtain

$$\begin{aligned} \Psi^{(j)}(x) &= \begin{cases} \lim_{n\to\infty} \max\{|4|_A^{-k}\psi(2^k x, 2^k x, 2^k x) : 0 \leq k < n\}, \ j = 1, \\ \lim_{n\to\infty} \max\{|4|_A^{k}\psi(2^{-k} x, 2^{-k} x, 2^{-k} x) : 0 < k \leq n\}, \ j = -1, \end{cases} \\ &= \begin{cases} \psi(x, x, x), & j = 1, \\ |4|_A\psi(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}), & j = -1. \end{cases} \end{aligned}$$

By (4.5), we get

$$\begin{aligned} &\begin{cases} \lim_{m\to\infty} \lim_{n\to\infty} \max\{|4|_A^{-k}\psi(2^k x, 2^k x, 2^k x) : m \leq k < n + m\}, \ j = 1, \\ \lim_{m\to\infty} \lim_{n\to\infty} \max\{|4|_A^{k}\psi(2^{-k} x, 2^{-k} x, 2^{-k} x) : m < k \leq n + m\}, \ j = -1, \end{cases} \\ &= \begin{cases} \lim_{m\to\infty} |4|_A^{-m}\psi(2^m x, 2^m x, 2^m x), & j = 1, \\ \lim_{m\to\infty} |4|_A^{m+1}\psi(2^{-(m+1)}x, 2^{-(m+1)}x, 2^{-(m+1)}x), & j = -1, \end{cases} \\ &= 0. \end{aligned}$$

Thus, all conditions of Theorem 4.2 are satisfied, and hence the corresponding conclusion holds true. □

Remark 4.4. The function $\rho(t) = t^{2p}$ $(0 < p \neq 1)$, $t \geq 0$ is an appropriate choice for the function ρ given in Corollary 4.3. Accurately speaking, when $j = 1$, we can choose $\rho(t) = t^{2p}$, $t \geq 0$, $p > 1$, when $j = -1$, we can choose $\rho(t) = t^{2p}$, $t \geq 0$, $0 < p < 1$, and with the further assumption that $|2|_A < 1$ under two cases.

Corollary 4.5. *Let $j \in \{-1, 1\}$ be fixed and let $\rho : [0, \infty) \to [0, \infty)$ a function such that*

$$\rho(|2^j|_A t) \leq [\rho(|2|_A)]^j \rho(t)$$

for all $t \geq 0$, and $j \cdot [\rho(|2|_A)]^3 < j \cdot |4|_A$. Let X be a complete non-Archimedean space and let δ be a positive real number. Suppose that the mapping $f : G \to X$ satisfies the inequality

$$\|D_q f(x, y, z)\|_A \leq \delta(\rho(\|x\|_A) \cdot \rho(\|y\|_A) \cdot \rho(\|z\|_A))$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : G \to X$ satisfying (1.1) such that

$$\|f(x) - Q(x)\|_A \leq \begin{cases} \frac{\delta[\rho(\|x\|_A)]^3}{|4|_A}, & j = 1, \\ \frac{\delta[\rho(\|x\|_A)]^3}{[\rho(\|2\|_A)]^3}, & j = -1 \end{cases}$$

for all $x \in G$.

Proof. Letting $\psi(x, y, z) = \delta(\rho(\|x\|_A) \cdot \rho(\|y\|_A) \cdot \rho(\|z\|_A))$. Using the same argument as in the proof of Corollary 4.3, we can easily obtain the required result. □

Remark 4.6. The function $\rho(t) = t^{\frac{2p}{3}}$ $(0 < p \neq 1)$, $t \geq 0$ is an appropriate choice for the function ρ given in Corollary 4.5. Precisely speaking, when $j = 1$, we can choose $\rho(t) = t^{2p}$, $t \geq 0$, $p > 1$, when $j = -1$, we can choose $\rho(t) = t^{2p}$, $t \geq 0$, $0 < p < 1$, and with the further assumption that $|2|_A < 1$ under two cases.

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