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New results of positive solutions for second-order nonlinear three-point integral boundary value problems

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Abstract

In this paper, we investigate the existence of positive solutions for second-order nonlinear three-point integral boundary value problems. By using the Leray-Schauder fixed point theorem, some sufficient conditions for the existence of positive solutions are obtained, which improve the results of literature Tariboon and Sitthiwirattham [J. Tariboon, T. Sitthiwirattham, Boundary Value Problems, **2010** (2010), 1–11].

Keywords: Positive solution, nonlinear three-point integral boundary value problems, Leray-Schauder fixed point theorem
2010 MSC: 34B15

1. Introduction

As the extensive applications of boundary value problems for differential equations in physics, biology and engineering sciences, the solvability of boundary value problems has received great attention from many authors and become a very hot research topic. The study of the existence of solutions of multipoint boundary value problems for linear second-order differential equations was initiated by Il'in and Moiseev [3, 4]. Since then, Gupta [8] studied three-point boundary value problems for nonlinear second-order differential equations. Recently, all sorts of multipoint boundary value problems for nonlinear differential equations have been studied by many authors. We refer the readers to [1, 2, 5, 6, 9, 10, 11, 12] and the references therein.

In 2010, Tariboon and Sitthiwirattham [10] studied the existence of positive solutions of the following nonlinear three-point integral boundary value problems

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$$u'' + a(t)f(u) = 0, \quad t \in (0,\ 1), \tag{1.1}$$

$$u(0) = 0,\ \alpha \int_0^\eta u(s)ds = u(1), \tag{1.2}$$

where $0 < \eta < 1$, $0 < \alpha < \frac{2}{\eta^2}$.
Assume that:
(H_1) $f \in C([0,+\infty),[0,+\infty))$;
(H_2) $a \in C([0,1],[0,+\infty))$ and there exists $t_0 \in [\eta,\ 1]$, such that $a(t_0) > 0$.
Let $\quad f_0 = \lim\limits_{u\to 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim\limits_{u\to\infty} \frac{f(u)}{u}$.

The literature [10] studied the existence of positive solutions of boundary value problems (1.1)-(1.2) by using Krasnoselskii fixed point theorem, they obtained the following results:
Theorem A1. *Assume* $(H_1),(H_2)$ *hold. If* $f_0 = 0,\ f_\infty = \infty$ *(superlinear), then the boundary value problem* (1.1)-(1.2) *has at least one positive solution.*
Theorem A2. *Assume* $(H_1),(H_2)$ *hold. If* $f_0 = \infty,\ f_\infty = 0$ *(sublinear), then the boundary value problem* (1.1)-(1.2) *has at least one positive solution.*

In this paper, we study the existence of positive solutions of the boundary value problem (1.1)-(1.2) by applying Leray-Schauder fixed point theorem, our results improve the above Theorem A1 and Theorem A2.

2. Preliminaries

Consider boundary value problem

$$u'' + p(t) = 0, \quad t \in (0,\ 1)\ , \tag{2.1}$$

$$u(0) = 0,\ \alpha \int_0^\eta u(s)ds = u(1). \tag{2.2}$$

Lemma 2.1. ([10]) *Let* $\alpha\eta^2 \neq 2, p(t) \in C[0,1]$, *then the problem* (2.1)-(2.2) *has a unique solution*

$$u(t) = -\int_0^t (t-s)p(s)ds - \frac{\alpha t}{2-\alpha\eta^2}\int_0^\eta (\eta-s)^2 p(s)ds + \frac{2t}{2-\alpha\eta^2}\int_0^1 (1-s)p(s)ds.$$

Lemma 2.2. ([10]) *Let* $0 < \alpha < \frac{2}{\eta^2}$. *If* $p(t) \in C[0,1]$ *and* $p(t) \geq 0$, *then the unique solution* $u(t)$ *of the problem* (2.1)-(2.2) *satisfies* $u(t) \geq 0, \quad t \in [0,\ 1]$.

Lemma 2.3. ([10]) *Let* $0 < \alpha < \frac{2}{\eta^2}$. *If* $p(t) \in C[0,1]$ *and* $p(t) \geq 0$, *then the unique solution* $u(t)$ *of the problem* (2.1)-(2.2) *satisfies* $\inf\limits_{t\in[\eta,\ 1]} u(t) \geq \gamma \| u\|$, *where* $\gamma = \min\{\frac{\alpha\eta^2}{2},\ \frac{\alpha\eta(1-\eta)}{2-\alpha\eta^2},\ \eta\}$.

For any $y(t) \in C[0,1]$, consider boundary value problem

$$u'' + a(t)f(y(t)) = 0, \quad t \in (0,\ 1), \tag{2.3}$$

$$u(0) = 0,\ \alpha \int_0^\eta u(s)ds = u(1). \tag{2.4}$$

By Lemma 2.1, we know the problem (2.3)-(2.4) has unique solution

$$u(t) = -\int_0^t (t-s)a(s)f(y(s))ds - \frac{\alpha t}{2-\alpha\eta^2}\int_0^\eta (\eta-s)^2 a(s)f(y(s))ds + \frac{2t}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)f(y(s))ds.$$

Define operator

$$Ty(t) = -\int_0^t (t-s)a(s)f(y(s))ds - \frac{\alpha t}{2-\alpha\eta^2}\int_0^\eta (\eta-s)^2 a(s)f(y(s))ds + \frac{2t}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)f(y(s))ds.$$

Obviously, $y(t)$ is the solution of the boundary value problem (1.1)-(1.2) if and only if $y(t)$ is the fixed point of operator T.

Lemma 2.4. ([7])(Leray-Schauder) *Let Ω be the convex subset of Banach space X, $0 \in \Omega$, $\Phi : \Omega \to \Omega$ be completely continuous operator. Then, either (i) Φ has at least one fixed point in Ω; or (ii) the set $\{x \in \Omega | x = \lambda\Phi x, 0 < \lambda < 1\}$ is unbounded.*

3. Main results

In this paper, we obtain new results of positive solutions for nonlinear three-point integral boundary value problem (1.1) and (1.2).
Let

$$X = C[0,1], \ \ \beta = \int_0^1 (1-s)a(s)ds.$$

Theorem 3.1. *Assume $(H_1),(H_2)$ hold. If $f_0 = 0$, then the boundary value problem (1.1)-(1.2) has at least one positive solution.*

Proof. Choose $\varepsilon > 0$ and $\varepsilon \leq \frac{2-\alpha\eta^2}{2\beta}$. By $f_0 = 0$, we know there exists constant $B > 0$, such that $f(y) < \varepsilon y$ for $0 < y \leq B$.
Let

$$\Omega = \left\{ y \mid \ y \in C[0,1], y \geq 0, \|y\| \leq B, \inf_{t\in[\eta,\,1]} y(t) \geq \gamma \|y\| \right\},$$

then Ω is the convex subset of X.
For $y \in \Omega$, by Lemmas 2.2 and 2.3, we know $\quad Ty(t) \geq 0$ and $\inf_{t\in[\eta,\,1]} Ty(t) \geq \gamma \| Ty\|$.
On the other hand,

$$\begin{aligned} Ty(t) &\leq \frac{2t}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)f(y(s))ds \leq \frac{2t}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)\varepsilon y(s)ds \\ &\leq \|y\| \frac{2\varepsilon}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)ds \leq \|y\| \leq B. \end{aligned}$$

Thus, $\| Ty\| \leq B$. Hence, $T\Omega \subset \Omega$.
It is easy to check that $T : \Omega \to \Omega$ is completely continuous.
For $y \in \Omega$ and $y = \lambda Ty, 0 < \lambda < 1$, we have $y(t) = \lambda Ty(t) < Ty(t) \leq B$, which implies $\| y\| \leq B$. So, $\{y \in \Omega | y = \lambda Ty, 0 < \lambda < 1\}$ is bounded. By Lemma 2.4, we know the operator T has at least one fixed point in Ω. Thus the boundary value problem (1.1)-(1.2) has at least one positive solution. The proof is complete. □

Remark 3.2. The condition of Theorem 1 is weaker than that of Theorem A1 in [10], the condition $f_\infty = \infty$ is unnecessary.

Theorem 3.3. *Assume $(H_1),(H_2)$ hold. If $f_\infty = 0$, then the boundary value problem (1.1)-(1.2) has at least one positive solution.*

Proof. Choose $\varepsilon > 0$ and $\varepsilon \leq \frac{2-\alpha\eta^2}{4\beta}$. By $f_\infty = 0$, we know there exists constant $N > 0$, such that $f(y) < \varepsilon y$ for $y > N$.
Select

$$B \geq N + 1 + \frac{4\beta}{2-\alpha\eta^2} \max_{0\leq y\leq N} f(y).$$

Let

$$\Omega = \left\{ y \mid \ y \in C[0,1], y \geq 0, \|y\| \leq B, \inf_{t\in[\eta,\, 1]} y(t) \geq \gamma \, \|y\| \right\},$$

then Ω is the convex subset of X.

For $y \in \Omega$, by Lemmas 2.2 and 2.3, we know $\quad Ty(t) \geq 0 \quad$ and $\inf_{t\in[\eta,\, 1]} Ty(t) \geq \gamma \| \, Ty\|$.

On the other hand,

$$\begin{aligned}
Ty(t) &\leq \frac{2t}{2-\alpha\eta^2} \int_0^1 (1-s)a(s)f(y(s))ds \leq \frac{2}{2-\alpha\eta^2} \int_0^1 (1-s)a(s)f(y(s))ds \\
&= \frac{2}{2-\alpha\eta^2} \left(\int_{J_1=\{s\in[0,1],y(s)>N\}} (1-s)a(s)f(y(s))ds + \int_{J_2=\{s\in[0,1],y(s)\leq N\}} (1-s)a(s)f(y(s))ds \right) \\
&\leq \frac{2}{2-\alpha\eta^2} \int_0^1 (1-s)a(s)\varepsilon y(s)ds + \frac{2}{2-\alpha\eta^2} \int_0^1 (1-s)a(s)ds \cdot \max_{0\leq y\leq N} f(y) \\
&\leq \frac{2\varepsilon}{2-\alpha\eta^2} \| \, y\| \int_0^1 (1-s)a(s)ds + \frac{2}{2-\alpha\eta^2} \int_0^1 (1-s)a(s)ds \cdot \max_{0\leq y\leq N} f(y) \\
&\leq \frac{2\varepsilon}{2-\alpha\eta^2} B \int_0^1 (1-s)a(s)ds + \frac{2}{2-\alpha\eta^2} \int_0^1 (1-s)a(s)ds \cdot \max_{0\leq y\leq N} f(y) \\
&= \frac{2\varepsilon}{2-\alpha\eta^2} B\beta + \frac{2}{2-\alpha\eta^2} \beta \cdot \max_{0\leq y\leq N} f(y) \\
&\leq \frac{1}{2}B + \frac{1}{2}B = B.
\end{aligned}$$

Thus, $\| \, Ty\| \leq B$. Hence, $T\Omega \subset \Omega$.
It is easy to check that $T : \Omega \to \Omega$ is completely continuous.
For $y \in \Omega$ and $y = \lambda Ty, 0 < \lambda < 1$, we have $y(t) = \lambda Ty(t) < Ty(t) \leq B$, which implies $\| \, y\| \leq B$. So, $\{y \in \Omega | y = \lambda Ty, 0 < \lambda < 1\}$ is bounded. By Lemma 2.4, we know the operator T has at least one fixed point in Ω. Thus the boundary value problem (1.1)-(1.2) has at least one positive solution. The proof is complete. □

Remark 3.4. The condition of Theorem 3.3 is weaker than that of Theorem A2 in [10], the condition $f_0 = \infty$ is unnecessary.

Theorem 3.5. *Assume $(H_1), (H_2)$ hold. If there exists constant $\rho_1 > 0$, such that $f(y) \leq \frac{(2-\alpha\eta^2)\rho_1}{2\beta}$ for $0 < y \leq \rho_1$, then the boundary value problem* (1.1)-(1.2) *has at least one positive solution.*

Proof. Let $\Omega = \left\{ y \mid \ y \in C[0,1], y \geq 0, \| \, y\| \leq \rho_1, \inf_{t\in[\eta,\, 1]} y(t) \geq \gamma \| \, y\| \right\}$, then Ω is the convex subset of X.
For $y \in \Omega$, by Lemmas 2.2 and 2.3, we know

$$Ty(t) \geq 0 \qquad \text{and} \qquad \inf_{t\in[\eta,\, 1]} Ty(t) \geq \gamma \| \, Ty\|.$$

On the other hand,

$$Ty(t) \leq \frac{2t}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)f(y(s))ds \leq \frac{2}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)\frac{(2-\alpha\eta^2)\rho_1}{2\beta}ds = \rho_1.$$

Thus, $\| Ty\| \leq \rho_1$. Hence, $T\Omega \subset \Omega$. It is easy to check that $T:\Omega \to \Omega$ is completely continuous.
For $y \in \Omega$ and $y = \lambda Ty, 0 < \lambda < 1$, we have $y(t) = \lambda Ty(t) < Ty(t) \leq \rho_1$, which implies $\| y\| \leq \rho_1$. So, $\{y \in \Omega | y = \lambda Ty, 0 < \lambda < 1\}$ is bounded. By Lemma 2.4, we know the operator T has at least one fixed point in Ω. Thus the boundary value problem (1.1)-(1.2) has at least one positive solution. The proof is complete. □

Theorem 3.6. *Assume $(H_1), (H_2)$ hold. If there exists constant $\rho_2 > 0$, such that $f(y) \leq \frac{(2-\alpha\eta^2)\rho_2}{2\beta}$ for $y \geq \rho_2$, then the boundary value problem* (1.1)-(1.2) *has at least one positive solution.*

Proof. Choose

$$d > 1 + \rho_2 + \frac{2\beta}{2-\alpha\eta^2}\max_{0\leq y\leq\rho_2} f(y).$$

Let

$$\Omega = \left\{ y \mid \ y \in C[0,1], y \geq 0, \|y\| \leq d, \inf_{t\in[\eta,\,1]} y(t) \geq \gamma\,\|y\| \right\},$$

then Ω is the convex subset of X.
For $y \in \Omega$, by Lemmas 2.2 and 2.3, we know $\quad Ty(t) \geq 0\;$ and $\inf_{t\in[\eta,\,1]} Ty(t) \geq \gamma\| Ty\|$.

On the other hand,

$$\begin{aligned} Ty(t) &\leq \frac{2t}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)f(y(s))ds \leq \frac{2}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)f(y(s))ds \\ &= \frac{2}{2-\alpha\eta^2}\left(\int_{J_1=\{s\in[0,1],y(s)>\rho_2\}}(1-s)a(s)f(y(s))ds + \int_{J_2=\{s\in[0,1],y(s)\leq\rho_2\}}(1-s)a(s)f(y(s))ds\right) \\ &\leq \frac{2}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)\frac{(2-\alpha\eta^2)\rho_2}{2\beta}ds + \frac{2}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)ds\cdot\max_{0\leq y\leq\rho_2} f(y) \\ &= \rho_2 + \frac{2\beta}{2-\alpha\eta^2}\max_{0\leq y\leq\rho_2} f(y) < d. \end{aligned}$$

Thus, $\| Ty\| \leq d$. Hence, $T\Omega \subset \Omega$.
It is easy to check that $T:\Omega \to \Omega$ is completely continuous.
For $y \in \Omega$ and $y = \lambda Ty, 0 < \lambda < 1$, we have $y(t) = \lambda Ty(t) < Ty(t) \leq d$, which implies $\| y\| \leq d$. So, $\{y \in \Omega | y = \lambda Ty, 0 < \lambda < 1\}$ is bounded. By Lemma 2.4, we know the operator T has at least one fixed point in Ω. Thus the boundary value problem (1.1)-(1.2) has at least one positive solution. The proof is complete. □

4. Example

Consider second-order nonlinear three-point integral boundary value problem:

$$u'' + a(t)\frac{u}{1+u^n} = 0, \quad t \in (0,\ 1)\ , \tag{4.1}$$

$$u(0) = 0,\ \alpha\int_0^\eta u(s)ds = u(1), \tag{1.2}$$

where $0 < \eta < 1$, $0 < \alpha < \frac{2}{\eta^2}$, $f(u) = \frac{u}{1+u^n}$, $(n > 0)$.
Obviously, $f_\infty = \lim\limits_{u\to\infty} \frac{f(u)}{u} = \lim\limits_{u\to\infty} \frac{1}{1+u^n} = 0$. Thus, by Theorem 3.3 of this paper, we know the boundary value problem (4.1)-(1.2) has at least one positive solution.

Remark 4.1. It is easy to know $\lim\limits_{u\to 0^+} \frac{f(u)}{u} = \lim\limits_{u\to 0^+} \frac{1}{1+u^n} = 1$, $f_0 \neq \infty$, which does not satisfy the condition $f_0 = \infty$ of Theorem A2 in [10], so Theorem A2 in [10] can not judge the existence of positive solutions for the boundary value problem (4.1)-(1.2). However, by Theorem 3.3 of this paper, we know the boundary value problem (4.1)-(1.2) has at least one positive solution.

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References

[1] Z. J. Du, W. G. Ge, X. J. Lin, *Existence of solutions for a class of third-order nonlinear boundary value problems*, J. Math. Anal. Appl., **294** (2004), 104–112. 1

[2] Y. Q. Feng, S. Y. Liu, *Solvability of a third-order two-point boundary value problem*, Appl. Math. Letters, **18** (2005), 1034–1040. 1

[3] V. A. Il'in, E. I. Moiseev, *Nonlocal boundary value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects*, Differential Equations, **23**(1987), 803–810. 1

[4] V. A. Il'in, E. I. Moiseev, *Nonlocal boundary value problem of the first kind for a Sturm-Liouville operator*, Differential Equations, **23** (1987), 979–987. 1

[5] J. R. Graef, C. Qian, B. Yang, *A three point boundary value problem for nonlinear fourth order differential equations*, J. Math. Anal. Appl., **287** (2003), 217–233. 1

[6] J. R. Graef, B. Yang, *Positive solutions to a multi-point higher order boundary value problem*, J. Math. Anal. Appl., **316** (2006), 409–421. 1

[7] A. Granas, J. Dugundji, *Fixed point theory*, Springer-Verlag, New York, (2003). 2.4

[8] C. P. Gupta, *Solvability of three-point nonlinear boundary value problem for a second order ordinary differential equation*, J. Math. Anal. Appl., **168** (1992), 540–551. 1

[9] B. Liu, *Positive solutions of a nonlinear four-point boundary value problem*, Appl. Math. Comput., **155** (2004), 179–203. 1

[10] J. Tariboon, T. Sitthiwirattham, *Positive solutions of a nonlinear three-point integral boundary value problem,* Boundary Value Problems, **2010** (2010), 1–11. 1, 1, 2.1, 2.2, 2.3, 3.2, 3.4, 4.1

[11] J. R. L. Webb, *Positive solutions of some three-point boundary value problems via fixed point index theory*, Nonlinear Anal., **47** (2001), 4319–4332. 1

[12] Y. M. Zhou, Y. Xu, *Positive solutions of three-point boundary value problems for systems of nonlinear second order ordinary differential equations*, J. Math. Anal. Appl., **320** (2006), 578–590. 1