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Positive solutions for Caputo fractional differential equations involving integral boundary conditions

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Abstract

In this work we study integral boundary value problem involving Caputo differentiation

$$\begin{cases} {}^cD_t^q u(t) = f(t, u(t)), 0 < t < 1, \\ \alpha u(0) - \beta u(1) = \int_0^1 h(t)u(t)\mathrm{d}t, \gamma u'(0) - \delta u'(1) = \int_0^1 g(t)u(t)\mathrm{d}t, \end{cases}$$

where $\alpha, \beta, \gamma, \delta$ are constants with $\alpha > \beta > 0$, $\gamma > \delta > 0$, $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R})$, $g, h \in C([0,1], \mathbb{R}^+)$ and ${}^cD_t^q$ is the standard Caputo fractional derivative of fractional order $q(1 < q < 2)$. By using some fixed point theorems we prove the existence of positive solutions.

Keywords: Caputo fractional boundary value problem, fixed point theorem, positive solution.
2010 MSC: 34B15, 34B25.

1. Introduction

In recent years, fractional differential equations have been widely used in diffusion and transport theory, chaos and turbulence, viscoelastic mechanics, non-newtonian fluid mechanics etc. It has received highly attention and becomes one of the hottest issues in the international research field. For instance, Westerlund [14] utilized fractional differential equations to depict the transmission of electromagnetic wave, the one dimensional model is

$$\mu_0\varepsilon_0\frac{\partial^2 E(x,t)}{\partial t^2} + \mu_0\varepsilon_0 x_{00} D_t^\nu E(x,t) + \frac{\partial^2 E(x,t)}{\partial t^2} = 0,$$

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where $\mu_0, \varepsilon_0, x_0$ are constants, ${}_0D_t^{\nu}E(x,t) = \frac{\partial^{\nu}E(x,t)}{\partial t^{\nu}}$ is a fractional derivative.

As an excellent tool, fixed point method is used for investigating nonlinear boundary value problems and there are a lot of papers devoted to this direction. We refer the reader to some papers involving fractional differential equations [1, 2, 4, 5, 11, 12, 13, 15, 16, 17, 18] and the references therein. For example, Leggett-Williams fixed point theorem is used to study the existence of multiple positive solutions for some integral boundary value problems [4, 11, 16]. However, all these works were done under assumption that the nonlinear term is nonnegative. Therefore, it is natural to discuss the existence of positive solutions while the nonlinear term is sign-changing, for instance, see [15, 17, 18].

In [17] the author obtained the existence of positive solutions for the coupled integral boundary value problem for systems of nonlinear fractional q-difference equations

$$\begin{aligned} &D_q^{\alpha}u(t) + \lambda f(t, u(t), v(t)) = 0,\ D_q^{\beta}v(t) + \lambda g(t, u(t), v(t)) = 0,\ t \in (0,1),\ \lambda > 0, \\ &D_q^{j}u(0) = D_q^{j}v(0) = 0, 0 \le j \le n-2, u(1) = \mu \int_0^1 v(s)d_qs, v(1) = \nu \int_0^1 u(s)d_qs. \end{aligned} \tag{1.1}$$

Under the semipositone nonlinearities, by applying the nonlinear alternative of Leray-Schauder type and Krasnoselskii's fixed point theorems, several existence theorems for (1.1) had been established.

Motivated by the above works, we investigate the existence of positive solutions for integral boundary value problems involving Caputo differentiation

$$\begin{cases} {}^cD_t^q u(t) = f(t, u(t)), 0 < t < 1, \\ \alpha u(0) - \beta u(1) = \int_0^1 h(t)u(t)\mathrm{d}t, \gamma u'(0) - \delta u'(1) = \int_0^1 g(t)u(t)\mathrm{d}t, \end{cases} \tag{1.2}$$

where $\alpha, \beta, \gamma, \delta$ are real constants with $\alpha > \beta > 0$, $\gamma > \delta > 0$, $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R})$, $g, h \in C([0,1], \mathbb{R}^+)$ and ${}^cD_t^q$ is the standard Caputo fractional derivative of fractional order $q(1 < q < 2)$. We consider the two cases:

(1) The nonlinearity is asymptotically linear at infinity, maybe it is negative and unbounded.
(2) The nonlinearity is bounded from below, including sign-changing.

2. Preliminaries

We first offer some basic definitions and facts used throughout this paper. For more details, see [7, 9, 10].

Definition 2.1. For a function f given on the interval $[a, b]$, the Caputo derivative of fractional order q is defined as

$${}^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s)\mathrm{d}s, \quad n = [q] + 1,$$

where $[q]$ denotes the integer part of q.

Definition 2.2. The Riemann-Liouville fractional integral of order q for a function f is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)\mathrm{d}s, \quad q > 0,$$

provided that such integral exists.

Lemma 2.3. *Let $q > 0$. Then the differential equation ${}^cD^q u(t) = 0$ has solutions*

$$u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n$, $n = [q] + 1$.

Lemma 2.4. *Let $q > 0$. Then*

$$I^q({}^cD^q u)(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n$, $n = [q] + 1$.

Lemma 2.5. *Let $q \in (1, 2)$ and $y \in C[0, 1]$. Then boundary value problem*

$$\begin{cases} {}^cD_t^q u(t) = y(t), 0 < t < 1, \\ \alpha u(0) - \beta u(1) = 0, \gamma u'(0) - \delta u'(1) = 0, \end{cases}$$

has a unique solution u in the form

$$u(t) = \int_0^1 G(t, s) y(s) \mathrm{d}s,$$

where

$$G(t, s) = \begin{cases} \frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{\beta(1-s)^{q-1}}{(\alpha-\beta)\Gamma(q)} + \frac{\beta\delta(q-1)(1-s)^{q-2}}{(\alpha-\beta)(\gamma-\delta)\Gamma(q)} + \frac{\delta(q-1)t(1-s)^{q-2}}{(\gamma-\delta)\Gamma(q)}, & 0 \le s \le t \le 1, \\ \frac{\beta(1-s)^{q-1}}{(\alpha-\beta)\Gamma(q)} + \frac{\beta\delta(q-1)(1-s)^{q-2}}{(\alpha-\beta)(\gamma-\delta)\Gamma(q)} + \frac{\delta(q-1)t(1-s)^{q-2}}{(\gamma-\delta)\Gamma(q)}, & 0 \le t \le s \le 1. \end{cases}$$

Proof. By Definitions 2.1, 2.2 and Lemmas 2.3, 2.4, we have

$$u(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} y(s) \mathrm{d}s + c_1 + c_2 t \tag{2.1}$$

for $c_1, c_2 \in \mathbb{R}$. Then

$$u(0) = c_1, \quad u(1) = \frac{1}{\Gamma(q)} \int_0^1 (1 - s)^{q-1} y(s) \mathrm{d}s + c_1 + c_2,$$
$$u'(0) = c_2, \quad u'(1) = \frac{q - 1}{\Gamma(q)} \int_0^1 (1 - s)^{q-2} y(s) \mathrm{d}s + c_2.$$

In view of the boundary conditions $\alpha u(0) - \beta u(1) = 0, \gamma u'(0) - \delta u'(1) = 0$, we obtain

$$c_1 = \frac{\beta}{(\alpha - \beta)\Gamma(q)} \int_0^1 (1 - s)^{q-1} y(s) \mathrm{d}s + \frac{\beta\delta(q - 1)}{(\alpha - \beta)(\gamma - \delta)\Gamma(q)} \int_0^1 (1 - s)^{q-2} y(s) \mathrm{d}s,$$
$$c_2 = \frac{\delta(q - 1)}{(\gamma - \delta)\Gamma(q)} \int_0^1 (1 - s)^{q-2} y(s) \mathrm{d}s.$$

Substituting c_1, c_2 into the equation (2.1), we find

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} y(s) \mathrm{d}s + \frac{\delta(q - 1)t}{(\gamma - \delta)\Gamma(q)} \int_0^1 (1 - s)^{q-2} y(s) \mathrm{d}s \\ &\quad + \frac{\beta}{(\alpha - \beta)\Gamma(q)} \int_0^1 (1 - s)^{q-1} y(s) \mathrm{d}s + \frac{\beta\delta(q - 1)}{(\alpha - \beta)(\gamma - \delta)\Gamma(q)} \int_0^1 (1 - s)^{q-2} y(s) \mathrm{d}s \\ &= \int_0^1 G(t, s) y(s) \mathrm{d}s. \end{aligned}$$

This completes the proof. □

Lemma 2.6. *Let $q \in (1, 2)$. Then boundary value problem*

$$\begin{cases} {}^cD_t^q u(t) = 0, 0 < t < 1, \\ \alpha u(0) - \beta u(1) = \int_0^1 h(t) u(t) \mathrm{d}t, \gamma u'(0) - \delta u'(1) = \int_0^1 g(t) u(t) \mathrm{d}t \end{cases}$$

can be expressed in the form

$$u(t) = \frac{1}{\alpha-\beta}\int_0^1 h(t)u(t)\mathrm{d}t + \phi(t)\int_0^1 g(t)u(t)\mathrm{d}t,$$

where $\phi(t) := \frac{\beta+(\alpha-\beta)t}{(\alpha-\beta)(\gamma-\delta)}$ *for* $t \in [0,1]$.

Proof. By Lemma 2.3 we see

$$u(t) = c_3 + c_4 t, \text{ where } c_3, c_4 \in \mathbb{R}.$$

Consequently,

$$(\alpha-\beta)c_3 - \beta c_4 = \int_0^1 h(t)u(t)\mathrm{d}t, \ (\gamma-\delta)c_4 = \int_0^1 g(t)u(t)\mathrm{d}t.$$

Hence

$$u(t) = \frac{t}{\gamma-\delta}\int_0^1 g(t)u(t)\mathrm{d}t + \frac{1}{\alpha-\beta}\int_0^1 h(t)u(t)\mathrm{d}t + \frac{\beta}{(\alpha-\beta)(\gamma-\delta)}\int_0^1 g(t)u(t)\mathrm{d}t.$$

This completes the proof. □

Let $q \in (1,2)$ and $y \in C[0,1]$. Then from Lemmas 2.5, 2.6 we can obtain the boundary value problem

$$\begin{cases} {}^cD_t^q u(t) = y(t), 0 < t < 1, \\ \alpha u(0) - \beta u(1) = \int_0^1 h(t)u(t)\mathrm{d}t, \gamma u'(0) - \delta u'(1) = \int_0^1 g(t)u(t)\mathrm{d}t \end{cases} \tag{2.2}$$

is equivalent to

$$u(t) = \int_0^1 G(t,s)y(s)\mathrm{d}s + \frac{1}{\alpha-\beta}\int_0^1 h(t)u(t)\mathrm{d}t + \phi(t)\int_0^1 g(t)u(t)\mathrm{d}t. \tag{2.3}$$

Throughout this paper we always assume that the following condition holds:
(H1) $\kappa = \kappa_1\kappa_4 - \kappa_2\kappa_3 > 0, \kappa_1 \geq 0, \kappa_4 \geq 0$, where

$$\kappa_1 = 1 - \frac{1}{\alpha-\beta}\int_0^1 h(t)\mathrm{d}t, \quad \kappa_2 = \int_0^1 h(t)\phi(t)\mathrm{d}t,$$
$$\kappa_3 = \frac{1}{\alpha-\beta}\int_0^1 g(t)\mathrm{d}t, \ \kappa_4 = 1 - \int_0^1 g(t)\phi(t)\mathrm{d}t.$$

Lemma 2.7. *Suppose (H1) holds. Then* (2.3) *is equivalent to*

$$u(t) = \int_0^1 H(t,s)y(s)\mathrm{d}s,$$

where

$$\begin{aligned} H(t,s) = G(t,s) &+ \frac{1}{\kappa(\alpha-\beta)}\left[\kappa_4\int_0^1 h(t)G(t,s)\mathrm{d}t + \kappa_2\int_0^1 g(t)G(t,s)\mathrm{d}t\right] \\ &+ \frac{\phi(t)}{\kappa}\left[\kappa_3\int_0^1 h(t)G(t,s)\mathrm{d}t + \kappa_1\int_0^1 g(t)G(t,s)\mathrm{d}t\right]. \end{aligned}$$

Proof. Multiplying $h(t)$ on both sides of (2.3) and integrating over $[0,1]$, we find

$$\begin{aligned} &\int_0^1 h(t)u(t)\mathrm{d}t \\ &= \int_0^1 h(t)\int_0^1 G(t,s)y(s)\mathrm{d}s\mathrm{d}t + \frac{1}{\alpha-\beta}\int_0^1 h(t)\mathrm{d}t\int_0^1 h(t)u(t)\mathrm{d}t + \int_0^1 h(t)\phi(t)\mathrm{d}t\int_0^1 g(t)u(t)\mathrm{d}t. \end{aligned}$$

Similarly,

$$\begin{aligned}&\int_0^1 g(t)u(t)\mathrm{d}t\\ &= \int_0^1 g(t)\int_0^1 G(t,s)y(s)\mathrm{d}s\mathrm{d}t + \frac{1}{\alpha-\beta}\int_0^1 g(t)\mathrm{d}t\int_0^1 h(t)u(t)\mathrm{d}t + \int_0^1 g(t)\phi(t)\mathrm{d}t\int_0^1 g(t)u(t)\mathrm{d}t.\end{aligned}$$

Consequently, we get

$$\begin{bmatrix} \kappa_1 & -\kappa_2 \\ -\kappa_3 & \kappa_4 \end{bmatrix}\begin{bmatrix} \int_0^1 h(t)u(t)\mathrm{d}t \\ \int_0^1 g(t)u(t)\mathrm{d}t \end{bmatrix} = \begin{bmatrix} \int_0^1 h(t)\int_0^1 G(t,s)y(s)\mathrm{d}s\mathrm{d}t \\ \int_0^1 g(t)\int_0^1 G(t,s)y(s)\mathrm{d}s\mathrm{d}t \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} \int_0^1 h(t)u(t)\mathrm{d}t \\ \int_0^1 g(t)u(t)\mathrm{d}t \end{bmatrix} = \frac{1}{\kappa}\begin{bmatrix} \kappa_4 & \kappa_2 \\ \kappa_3 & \kappa_1 \end{bmatrix}\begin{bmatrix} \int_0^1 h(t)\int_0^1 G(t,s)y(s)\mathrm{d}s\mathrm{d}t \\ \int_0^1 g(t)\int_0^1 G(t,s)y(s)\mathrm{d}s\mathrm{d}t \end{bmatrix}.$$

As a result, we have

$$\begin{aligned}u(t) &= \int_0^1 G(t,s)y(s)\mathrm{d}s\\ &+ \frac{1}{\kappa(\alpha-\beta)}\left[\kappa_4\int_0^1 h(t)\int_0^1 G(t,s)y(s)\mathrm{d}s\mathrm{d}t + \kappa_2\int_0^1 g(t)\int_0^1 G(t,s)y(s)\mathrm{d}s\mathrm{d}t\right]\\ &+ \frac{\phi(t)}{\kappa}\left[\kappa_3\int_0^1 h(t)\int_0^1 G(t,s)y(s)\mathrm{d}s\mathrm{d}t + \kappa_1\int_0^1 g(t)\int_0^1 G(t,s)y(s)\mathrm{d}s\mathrm{d}t\right].\end{aligned}$$

This completes the proof. □

Lemma 2.8. *Let*

$$\mathcal{K}_1 := 1 + \frac{1}{\kappa(\alpha-\beta)}\left(\kappa_4\int_0^1 h(t)\mathrm{d}t + \kappa_2\int_0^1 g(t)\mathrm{d}t\right) + \frac{\phi(0)}{\kappa}\left(\kappa_3\int_0^1 h(t)\mathrm{d}t + \kappa_1\int_0^1 g(t)\mathrm{d}t\right),$$

$$\mathcal{K}_2 := 1 + \frac{1}{\kappa(\alpha-\beta)}\left(\kappa_4\int_0^1 h(t)\mathrm{d}t + \kappa_2\int_0^1 g(t)\mathrm{d}t\right) + \frac{\phi(1)}{\kappa}\left(\kappa_3\int_0^1 h(t)\mathrm{d}t + \kappa_1\int_0^1 g(t)\mathrm{d}t\right).$$

Then the following inequalities hold:

$$\mathcal{K}_1\mathcal{M}(s) \le H(t,s) \le \frac{\alpha}{\beta}\mathcal{K}_2\mathcal{M}(s),\ \forall t\in[0,1], s\in(0,1).$$

Proof. Let

$$g_1(t,s) = \frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{\beta(1-s)^{q-1}}{(\alpha-\beta)\Gamma(q)} + \frac{\beta\delta(q-1)(1-s)^{q-2}}{(\alpha-\beta)(\gamma-\delta)\Gamma(q)} + \frac{\delta(q-1)t(1-s)^{q-2}}{(\gamma-\delta)\Gamma(q)},\ 0\le s\le t\le 1,$$

and

$$g_2(t,s) = \frac{\beta(1-s)^{q-1}}{(\alpha-\beta)\Gamma(q)} + \frac{\beta\delta(q-1)(1-s)^{q-2}}{(\alpha-\beta)(\gamma-\delta)\Gamma(q)} + \frac{\delta(q-1)t(1-s)^{q-2}}{(\gamma-\delta)\Gamma(q)},\ 0\le t\le s\le 1.$$

For given $s\in(0,1)$, g_1, g_2 are increasing with respect to t for $t\in[0,1]$. Hence,

$$\begin{aligned}\min_{t\in[0,1]} G(t,s) &= \min\{\min_{t\in[s,1]} g_1(t,s), \min_{t\in[0,s]} g_2(t,s)\} = \min\{g_1(s,s), g_2(0,s)\} = g_2(0,s)\\ &= \frac{\beta(1-s)^{q-1}}{(\alpha-\beta)\Gamma(q)} + \frac{\beta\delta(q-1)(1-s)^{q-2}}{(\alpha-\beta)(\gamma-\delta)\Gamma(q)} := \mathcal{M}(s),\end{aligned}$$

$$\begin{aligned}\max_{t\in[0,1]} G(t,s) &= \max\{\max_{t\in[s,1]} g_1(t,s), \max_{t\in[0,s]} g_2(t,s)\} = \max\{g_1(1,s), g_2(s,s)\} = g_1(1,s)\\ &= \frac{(1-s)^{q-1}}{\Gamma(q)} + \frac{\beta(1-s)^{q-1}}{(\alpha-\beta)\Gamma(q)} + \frac{\beta\delta(q-1)(1-s)^{q-2}}{(\alpha-\beta)(\gamma-\delta)\Gamma(q)} + \frac{\delta(q-1)(1-s)^{q-2}}{(\gamma-\delta)\Gamma(q)} = \frac{\alpha}{\beta}\mathcal{M}(s).\end{aligned}$$

Therefore,

$$\mathcal{M}(s) \le G(t,s) \le \frac{\alpha}{\beta}\mathcal{M}(s),\ \forall t\in[0,1],\ s\in(0,1).$$

This yields

$$\begin{aligned}\mathcal{K}_1\mathcal{M}(s) &\le G(t,s) + \frac{1}{\kappa(\alpha-\beta)}\left[\kappa_4\int_0^1 h(t)G(t,s)\mathrm{d}t + \kappa_2\int_0^1 g(t)G(t,s)\mathrm{d}t\right]\\ &\quad + \frac{\phi(t)}{\kappa}\left[\kappa_3\int_0^1 h(t)G(t,s)\mathrm{d}t + \kappa_1\int_0^1 g(t)G(t,s)\mathrm{d}t\right]\\ &\le \frac{\alpha}{\beta}\mathcal{K}_2\mathcal{M}(s).\end{aligned}$$

This completes the proof. □

Let $E := C[0,1]$, $\|u\| := \max_{t\in[0,1]}|u(t)|$, $P := \{u\in E : u(t)\ge 0, \forall t\in[0,1]\}$. Then $(E, \|\cdot\|)$ becomes a real Banach space and P is a cone on E.

Definition 2.9. Given a cone P in a real Banach space E, a functional $\alpha : P \to [0,\infty)$ is said to be nonnegative continuous concave on P, provided $\alpha(tx+(1-t)y) \ge t\alpha(x) + (1-t)\alpha(y)$, for all $x, y \in P$ with $t\in[0,1]$.

Let $a, b, r > 0$ be constants and α as defined above, we denote $P_r = \{y\in P : \|y\| < r\}$, $P\{\alpha, a, b\} = \{y \in P : \alpha(y) \ge a, \|y\| \le b\}$.

Lemma 2.10. *(Leggett-Williams fixed point theorem, see [3, 8]) Assume E is a real Banach space, $P\subset E$ is a cone. Let $A : \overline{P}_c \to \overline{P}_c$ be completely continuous and α be a nonnegative continuous concave functional on P such that $\alpha(y) \le \|y\|$, for $y\in\overline{P}_c$. Suppose that there exist $0 < a < b < d \le c$ such that*
(1) $\{y \in P(\alpha,\ b,\ d) |\ \alpha(y) > b\} \ne \emptyset$ and $\alpha(Ay) > b$, for all $y\in P(\alpha,\ b,\ d)$,
(2) $\|Ay\| < a$, for all $\|y\| \le a$,
(3) $\alpha(Ay) > b$ for all $y \in P(\alpha, b, c)$ with $\|Ay\| > d$.
Then A has at least three fixed points y_1, y_2, y_3 satisfying

$$\|y_1\| < a, \quad b < \alpha(y_2), \quad \|y_3\| > a, \quad \alpha(y_3) < b.$$

Lemma 2.11. *(see [6]) Let E be a Banach space, and $A : E \to E$ be a completely continuous operator. Assume that $T : E\to E$ is a bounded linear operator such that 1 is not an eigenvalue of T and*

$$\lim_{\|u\|\to\infty}\frac{\|Au - Tu\|}{\|u\|} = 0.$$

Then A has a fixed point in E.

Define $A : E \to E$

$$(Au)(t) = \int_0^1 H(t,s)f(s,u(s))\mathrm{d}s.$$

Then, by Lemmas 2.5, 2.6 and 2.7, the existence of solutions for (1.2) is equivalent to the existence of fixed points for the operator A. Furthermore, in view of the continuity H and f, we can adopt the Ascoli-Arzela theorem to prove A is a completely continuous operator.

3. Main results

For convenience, we set

$$\xi=\int_0^1\frac{\alpha}{\beta}\mathcal{K}_2\mathcal{M}(s)\mathrm{d}s=\frac{\alpha}{\beta}\mathcal{K}_2\left(\frac{\beta}{q(\alpha-\beta)\Gamma(q)}+\frac{\beta\delta}{(\alpha-\beta)(\gamma-\delta)\Gamma(q)}\right),\quad \theta=\frac{\mathcal{K}_1\beta}{\mathcal{K}_2\alpha},$$

$$D_1=\frac{\alpha^2\mathcal{K}_2^2}{\mathcal{K}_1\beta^2}\left(\frac{\beta}{q(\alpha-\beta)\Gamma(q)}+\frac{\beta\delta}{(\alpha-\beta)(\gamma-\delta)\Gamma(q)}\right),$$

$$l=\mathcal{K}_1\int_0^1\mathcal{M}(s)\mathrm{d}s=\mathcal{K}_1\left(\frac{\beta}{q(\alpha-\beta)\Gamma(q)}+\frac{\beta\delta}{(\alpha-\beta)(\gamma-\delta)\Gamma(q)}\right).$$

Theorem 3.1. *Let (H1) hold true. Moreover, suppose that*
(H2) $f(t,0)\not\equiv 0$ *for all* $t\in[0,1]$,
(H3) $\lim_{u\to+\infty}\frac{f(t,u)}{u}=\lambda,$ *uniformly for* $t\in[0,1]$, *where* $|\lambda|<\xi^{-1}$.
Then (1.2) *has a positive solution.*

Proof. Define $T:P\to P$

$$(Tu)(t)=\lambda\int_0^1 H(t,s)u(s)\mathrm{d}s. \tag{3.1}$$

Clearly, T is a bounded linear operator, and by Lemmas 2.5, 2.6 and 2.7 we know that (3.1) is equivalent to

$$\begin{cases}{}^cD_t^q u(t)=\lambda u(t), 0<t<1,\\ \alpha u(0)-\beta u(1)=\int_0^1 h(t)u(t)\mathrm{d}t, \gamma u'(0)-\delta u'(1)=\int_0^1 g(t)u(t)\mathrm{d}t.\end{cases} \tag{3.2}$$

Next we show 1 is not an eigenvalue of T. We divide two cases.

Case 1. $\lambda=0$.

This implies ${}^cD_t^q u(t)=0$, and by Lemma 2.6 we get

$$u(t)=\frac{1}{\alpha-\beta}\int_0^1 h(t)u(t)\mathrm{d}t+\phi(t)\int_0^1 g(t)u(t)\mathrm{d}t.$$

Multiplying $h(t),g(t)$ on both sides and integrating over $[0,1]$, we find

$$\int_0^1 h(t)u(t)\mathrm{d}t=\frac{1}{\alpha-\beta}\int_0^1 h(t)\mathrm{d}t\int_0^1 h(t)u(t)\mathrm{d}t+\int_0^1 h(t)\phi(t)\mathrm{d}t\int_0^1 g(t)u(t)\mathrm{d}t,$$

$$\int_0^1 g(t)u(t)\mathrm{d}t=\frac{1}{\alpha-\beta}\int_0^1 g(t)\mathrm{d}t\int_0^1 h(t)u(t)\mathrm{d}t+\int_0^1 g(t)\phi(t)\mathrm{d}t\int_0^1 g(t)u(t)\mathrm{d}t.$$

Consequently,

$$\begin{bmatrix}\kappa_1 & -\kappa_2\\ -\kappa_3 & \kappa_4\end{bmatrix}\begin{bmatrix}\int_0^1 h(t)u(t)\mathrm{d}t\\ \int_0^1 g(t)u(t)\mathrm{d}t\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

This, together with (H1), yields

$$\int_0^1 h(t)u(t)\mathrm{d}t=0,\qquad \int_0^1 g(t)u(t)\mathrm{d}t=0.$$

Also, $u(t)\equiv 0$, $t\in[0,1]$ for the fact that $g,h\geq 0$ and $g,h\not\equiv 0$. This contradicts to the definition of eigenvalue and eigenfunction.

Case 2. $\lambda\neq 0$.

We assume that 1 is an eigenvalue of T, i.e., $Tu = u$. So,

$$\begin{aligned}\|u\| = \|Tu\| &= |\lambda| \max_{t\in[0,1]} \int_0^1 H(t,s)u(s)\mathrm{d}s \le |\lambda| \int_0^1 \frac{\alpha}{\beta}\mathcal{K}_2\mathcal{M}(s)\mathrm{d}s\|u\| \\ &= |\lambda|\frac{\alpha}{\beta}\mathcal{K}_2\left(\frac{\beta}{q(\alpha-\beta)\Gamma(q)} + \frac{\beta\delta}{(\alpha-\beta)(\gamma-\delta)\Gamma(q)}\right)\|u\| < \|u\|.\end{aligned}$$

This is impossible.

Above all, 1 is not an eigenvalue of T, as required.

On the other hand, by (H3), for all $\varepsilon > 0$, there exists $M_1 > 0$ such that $|f(t,u) - \lambda u| \le \varepsilon u$, for $t \in [0,1]$, $u \ge M_1$. Moreover, if $u \le M_1$, then $|f(t,u) - \lambda u|$ is bounded for all $t \in [0,1]$. Consequently, there exists $\zeta > 0$ such that

$$|f(t,u) - \lambda u| \le \varepsilon u + \zeta, \text{ for } t \in [0,1], \ u \in \mathbb{R}^+.$$

Hence

$$\begin{aligned}\|Au - Tu\| &= \max_{t\in[0,1]} \left|\int_0^1 H(t,s)(f(s,u(s)) - \lambda u(s))\mathrm{d}s\right| \\ &\le \max_{t\in[0,1]} \int_0^1 H(t,s)|(f(s,u(s)) - \lambda u(s))|\mathrm{d}s \\ &\le \max_{t\in[0,1]} \int_0^1 H(t,s)\mathrm{d}s(\varepsilon\|u\| + \zeta),\end{aligned}$$

and

$$\lim_{\|u\|\to\infty} \frac{\|Au - Tu\|}{\|u\|} \le \lim_{\|u\|\to\infty} \frac{\max_{t\in[0,1]}\int_0^1 H(t,s)\mathrm{d}s(\varepsilon\|u\| + \zeta)}{\|u\|} = 0.$$

So, A has a fixed point in E. Note that 0 is not a fixed point of A, and thus A has a positive fixed point, i.e., (1.2) has a positive solution. This completes the proof. □

Theorem 3.2. *Let (H1) hold true. Moreover, suppose that*
(H4) There exists $M > 0$ such that $f(t,u) \ge -M$ for $(t,u) \in [0,1] \times \mathbb{R}^+$,
There exist positive constants e, b, c, N with $MD_1 < e < e + MD_1\theta < b < \theta^2 c$, $\frac{1}{\theta} < N < \frac{cl}{b\xi}$ such that
(H5) $f(t,u) < \frac{e}{\xi} - M$ for $t \in [0,1]$, $0 \le u \le e$,
(H6) $f(t,u) \ge \frac{b}{l}N - M$ for $t \in [0,1]$, $b - MD_1\theta \le u \le \frac{b}{\theta^2}$,
(H7) $f(t,u) \le \frac{c}{\xi} - M$ for $t \in [0,1]$, $0 \le u \le c$.
Then (1.2) has at least two positive solutions.

Proof. Let ω be a solution of

$$\begin{cases} {}^cD_t^q u(t) = 1, 0 < t < 1, \\ \alpha u(0) - \beta u(1) = \int_0^1 h(t)u(t)\mathrm{d}t, \gamma u'(0) - \delta u'(1) = \int_0^1 g(t)u(t)\mathrm{d}t, \end{cases}$$

and $z = M\omega$. By Lemma 2.7 we have

$$\begin{aligned}z(t) = M\omega(t) &= M\int_0^1 H(t,s)\mathrm{d}s \le M\int_0^1 \frac{\alpha}{\beta}\mathcal{K}_2\mathcal{M}(s)\mathrm{d}s \\ &= M\frac{\alpha}{\beta}\mathcal{K}_2\left(\frac{\beta}{q(\alpha-\beta)\Gamma(q)} + \frac{\beta\delta}{(\alpha-\beta)(\gamma-\delta)\Gamma(q)}\right) \\ &= MD_1\theta < e\theta.\end{aligned}$$

We easily obtain that (1.2) has a positive solution u if and only if $u+z=\widetilde{u}$ is a solution of the boundary value problem

$$\begin{cases} {}^cD_t^q u(t)=\widetilde{f}(t,u(t)-z(t)), 0<t<1, \\ \alpha u(0)-\beta u(1)=\int_0^1 h(t)u(t)\mathrm{d}t, \gamma u'(0)-\delta u'(1)=\int_0^1 g(t)u(t)\mathrm{d}t, \end{cases} \tag{3.3}$$

and $\widetilde{u}\geq z$ for $t\in(0,1)$, where $\widetilde{f}:[0,1]\times\mathbb{R}^+\to\mathbb{R}^+$ is defined by

$$\widetilde{f}(t,y)=\begin{cases} f(t,y)+M, & (t,y)\in[0,1]\times[0,+\infty), \\ f(t,0)+M, & (t,y)\in[0,1]\times(-\infty,0). \end{cases}$$

For $u\in P$, we define

$$Tu(t)=\int_0^1 H(t,s)\widetilde{f}(s,u(s)-z(s))\mathrm{d}s.$$

Next we check $T(P)\subseteq P_0$, where $P_0=\{u\in P:\min_{t\in[0,1]}u(t)\geq\theta\|u\|\}$, $\theta=\frac{\mathcal{K}_1\beta}{\mathcal{K}_2\alpha}$. Indeed, for $u\in P$, Lemma 2.8 implies

$$\int_0^1 \mathcal{K}_1\mathcal{M}(s)\widetilde{f}(s,u(s)-z(s))\mathrm{d}s\leq Tu(t)\leq\int_0^1\frac{\alpha}{\beta}\mathcal{K}_2\mathcal{M}(s)\widetilde{f}(s,u(s)-z(s))\mathrm{d}s.$$

Hence,

$$Tu(t)\geq\frac{\mathcal{K}_1\beta}{\mathcal{K}_2\alpha}\int_0^1\frac{\alpha}{\beta}\mathcal{K}_2\mathcal{M}(s)\widetilde{f}(s,u(s)-z(s))\mathrm{d}s\geq\theta\|Tu\|.$$

In what follows, we show that all the conditions of Lemma 2.10 are satisfied. We first define the nonnegative, continuous concave functional $\alpha:P\to[0,\infty)$ by $\alpha(u)=\min_{t\in[0,1]}|u(t)|$. For each $u\in P$, it is easy to see $\alpha(u)\leq\|u\|$. We prove that $T(\overline{P}_c)\subseteq\overline{P}_c$. Let $u\in\overline{P}_c$. Then

(i) if $u(t)\geq z(t)$, we have $0\leq u(t)-z(t)\leq u(t)\leq c$ and $\widetilde{f}(t,u(t)-z(t))=f(t,u(t)-z(t))+M\geq0$. By (H7) we have $\widetilde{f}(t,u(t)-z(t))\leq\frac{c}{\xi}$.

(ii) if $u(t)<z(t)$, we have $u(t)-z(t)<0$ and $\widetilde{f}(t,u(t)-z(t))=f(t,0)+M\geq0$. By (H7) we have $\widetilde{f}(t,u(t)-z(t))\leq\frac{c}{\xi}$.

Therefore, we have proved that, if $u\in\overline{P}_c$, then $\widetilde{f}(t,u(t)-z(t))\leq\frac{c}{\xi}$ for $t\in[0,1]$. Then,

$$\|Tu\|=\max_{t\in[0,1]}\int_0^1 H(t,s)\widetilde{f}(s,u(s)-z(s))\mathrm{d}s\leq\int_0^1\frac{\alpha}{\beta}\mathcal{K}_2\mathcal{M}(s)\mathrm{d}s\frac{c}{\xi}=c.$$

Therefore, we have $T(\overline{P}_c)\subseteq\overline{P}_c$. Especially, if $u\in\overline{P}_e$, then (H5) yields $\widetilde{f}(t,u(t)-z(t))\leq\frac{e}{\xi}$ for $t\in[0,1]$. So, we have $T:\overline{P}_e\to P_e$, i.e., the assumption (2) of Lemma 2.10 holds.

To verify condition (1) of Lemma 2.10, let $u(t)=\frac{b}{\theta^2}$, then $u\in P$, $\alpha(u)=b/\theta^2>b$, i.e., $\{u\in P(\alpha,b,\frac{b}{\theta^2}):\alpha(u)>b\}\neq\emptyset$. Moreover, if $u\in P(\alpha,b,\frac{b}{\theta^2})$, then $\alpha(u)\geq b$, and $b\leq\|u\|\leq\frac{b}{\theta^2}$. Thus, $0<b-MD_1\theta\leq u(t)-z(t)\leq u(t)\leq\frac{b}{\theta^2}, t\in[0,1]$. From (H6) we obtain $\widetilde{f}(t,u(t)-z(t))\geq\frac{b}{l}N$ for $t\in[0,1]$. By the definition of α, we have

$$\begin{aligned}\alpha(Tu)&=\min_{t\in[0,1]}Tu(t)\geq\theta\|Tu\|\geq\theta\max_{t\in[0,1]}\int_0^1 H(t,s)\widetilde{f}(s,u(s)-z(s))\mathrm{d}s\geq\theta\frac{b}{l}N\int_0^1\mathcal{K}_1\mathcal{M}(s)\mathrm{d}s\\&=\theta Nb>b.\end{aligned}$$

Therefore, condition (1) of Lemma 2.10 is satisfied with $d=b/\theta^2$.

Finally, we show condition (3) of Lemma 2.10 is satisfied. For this we choose $u\in P(\alpha,b,c)$ with $\|Tu\|>b/\theta^2$. Then we have $\alpha(Tu)=\min_{t\in[0,1]}Tu(t)\geq\theta\|Tu\|\geq\frac{b}{\theta}>b$. Hence, condition (3) of Lemma 2.10 holds with $\|Tu\|>b/\theta^2$.

From now on, all the hypotheses of Lemma 2.10 are satisfied. Hence T has at least three positive fixed points $\widetilde{u}_1$, $\widetilde{u}_2$ and $\widetilde{u}_3$ such that

$$\|\widetilde{u}_1\| < e, \quad b < \alpha(\widetilde{u}_2), \quad \|\widetilde{u}_3\| > e, \quad \alpha(\widetilde{u}_3) < b.$$

Furthermore, $\widetilde{u}_i = u_i + z$ $(i = 1, 2, 3)$ are solutions of (3.3). Moreover,

$$\widetilde{u}_2(t) \geq \theta\|\widetilde{u}_2\| \geq \theta\alpha(\widetilde{u}_2) > \theta b > \theta M D_1 \geq z(t), \quad t \in [0,1],$$
$$\widetilde{u}_3(t) \geq \theta\|\widetilde{u}_3\| > \theta e > \theta M D_1 \geq z(t), \quad t \in [0,1].$$

So $u_2 = \widetilde{u}_2 - z$, $u_3 = \widetilde{u}_3 - z$ are two positive solutions of (1.2). This completes the proof. □

4. Examples

We now present two simple examples to explain our results. Let $q = 1.5$, $\alpha = \gamma = 2$, $\beta = \delta = 1$, $h(t) = t^3$, $g(t) = t^2$. Then $\phi(t) = 1 + t, \kappa_1 = \frac{3}{4}, \kappa_2 = \frac{9}{20}, \kappa_3 = \frac{1}{3}, \kappa_4 = \frac{5}{12}, \kappa = \frac{13}{80}$, $\mathcal{K}_1 = \frac{60}{13}$, $\mathcal{K}_2 = \frac{20}{3}$, $\int_0^1 \mathcal{M}(s)\mathrm{d}s = \frac{10}{3\sqrt{\pi}}$, $\xi = \frac{400}{9\sqrt{\pi}} \approx 25.08$, $\xi^{-1} \approx 0.04$, $\theta = \frac{9}{26}$, $D_1 = \frac{10400}{81\sqrt{\pi}} \approx 72.44$, $l = \frac{200}{13\sqrt{\pi}} \approx 8.68$.

Example 4.1. Let $f(t,u) = \lambda u + \rho u^\sigma - e^t + \eta$, $\forall t \in [0,1], u \in \mathbb{R}^+$, where $\sigma \in (0,1)$, $|\lambda| < 0.04$, $\rho \neq 0$, $\eta \neq 0$. Then (H2) and (H3) hold, by Theorem 3.1, (1.2) has at least one positive solution.

Example 4.2. We choose $M = 0.01$, $e = 0.85$, $b = 2$, $N = 4$, $c = 30$, and

$$\varphi(u) = \begin{cases} 0.01u, & 0 \leq u \leq 1, \\ -0.08u^2 + 1.54u - 1.45, & 1 \leq u \leq 1.9, \\ 1.1872, & 1.9 \leq u < +\infty. \end{cases}$$

Then $\varphi \in C(\mathbb{R}^+, \mathbb{R}^+)$. Furthermore, let $f(t,u) = \varphi(u) - 0.01$ for all $t \in [0,1]$, $u \in \mathbb{R}^+$. Then $MD_1 = \frac{104}{81\sqrt{\pi}} \approx 0.72$, $MD_1\theta = \frac{4}{9\sqrt{\pi}} \approx 0.25$, $\theta^2 c = \frac{1215}{338} \approx 3.59$, $\theta^{-1} \approx 2.89$ and $\frac{cl}{b\xi} = \frac{135}{26} \approx 5.19$. Clearly, $MD_1 < e < e + MD_1\theta < b < \theta^2 c$, $\theta^{-1} < N < \frac{cl}{b\xi}$ and $f(t,u) \geq -0.01 = -M$. On the other hand,

(i) $f(t,u) < 0.02 < \frac{e}{\xi} - M = \frac{153\sqrt{\pi}}{8000} - 0.01 \approx 0.024$ for $t \in [0,1]$, $0 \leq u \leq 0.85$,

(ii) $f(t,u) > f(1.74) \approx 0.98 > \frac{b}{l}N - M = \frac{13\sqrt{\pi}}{25} - 0.01 \approx 0.91$ for $t \in [0,1]$, $1.74 < 2 - \frac{4}{9\sqrt{\pi}} \leq u \leq \frac{1325}{81} \approx 16.69$,

(iii) $f(t,u) \leq 1.1772 < 1.186 < \frac{c}{\xi} - M = \frac{27\sqrt{\pi}}{40} - 0.01$ for $t \in [0,1]$, $0 \leq u \leq 30$.

Hence, (H4)-(H7) are satisfied, by Theorem 3.2, (1.2) has at least two positive solutions.

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